ADVANCED ENGINEERING MATHEMATICS

FIFTH EDITION

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9.9 The Numerical Solution of Partial Differential Equations

The work of this chapter has made it abundantly clear that there are many interesting and important problems involving partial differential equations for which exact solutions (usually in the form of infinite series) can be found. It is equally true, however, that applied scientists are encountering more and more problems in partial differential equations which cannot be solved exactly; and it is the purpose of this section to present an introductory account of the numerical methods by which approximate solutions to such equations can be obtained. In doing this, we will find that solution procedures differ somewhat according as the equation to be solved is elliptic, parabolic, or hyperbolic; and we shall discuss Laplace's equation, the one-dimensional heat equation, and the one-dimensional wave equation as respective prototypes.

As in the case of ordinary differential equations which must be solved numerically, the objective here is to obtain approximate values for the solution on a suitable set of points. Usually this means that one seeks the values of the solution at the points of a rectangular grid, or lattice,† extending over some portion of the domain of the solution. Fig. 9.20 illustrates several possibilities. Fig. 9.20*a* shows a rectangular grid covering a rectangular region in the xy plane in which we might be attempting to solve Laplace's equation. Fig. 9.20*b* shows a rectangular grid superposed on a

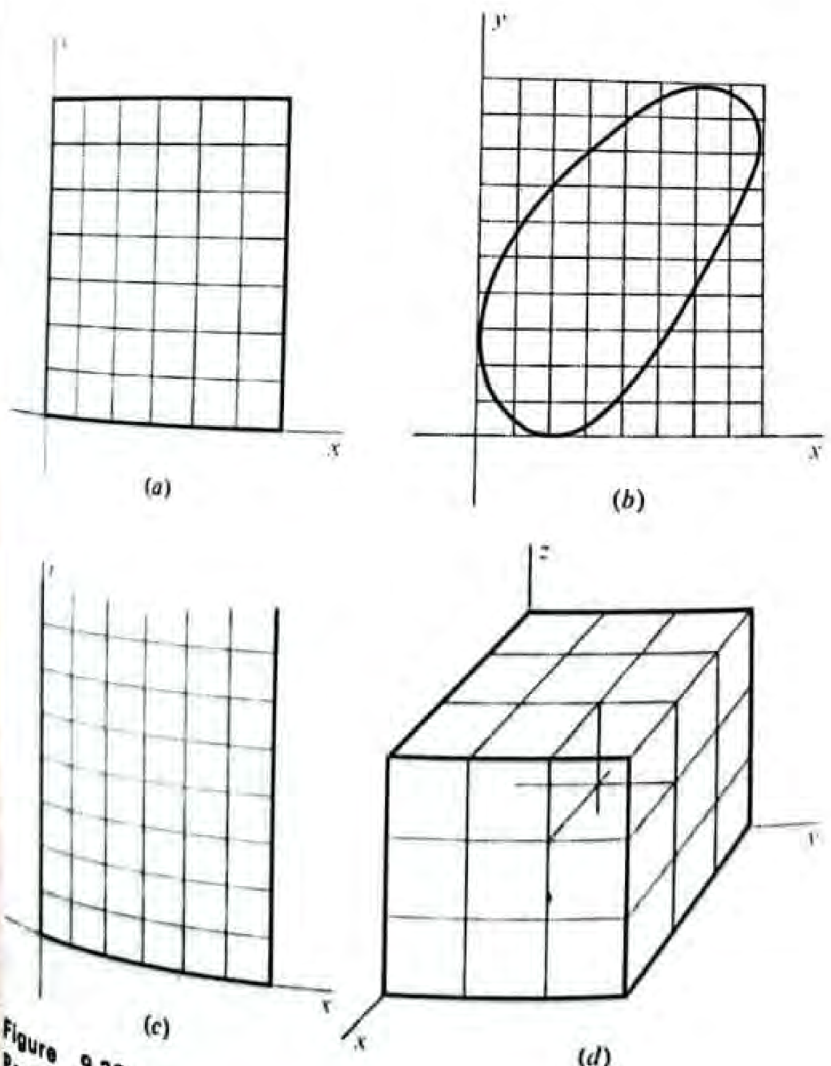


Figure 9.20
Rectangular lattices superimposed on various regions.

† The terms ... solution are sought.

nonrectangular region in which we might be attempting to solve Laplace's equation. Fig. 9.20c show a rectangular grid superposed on an infinite rectangular region in the xt plane representing the domain over which we might be attempting to solve either the one-dimensional heat equation or the one-dimensional wave equation. Finally, Fig. 9.20d suggests a rectangular grid superposed on a rectangular region in space in which we might be attempting to solve a problem involving Laplace's equation in three dimensions.

In Figs. 9.20a, c, and d, the outermost points of the lattice all fall on the boundary of the region, where the values of the solution are given as data of the problem. This is not the case for the irregular region shown in Fig. 9.20b, however, and the general formulas which we shall soon develop must be modified for those lattice points in the region which are adjacent to, but not actually on, the boundary.

The fundamental idea on which the numerical solution of partial differential equations is based is this: Each of the partial derivatives which appears in the equation is replaced by a finite-difference approximation. When these differences are evaluated at each of the mesh points, the result is a set of simultaneous equations which can be solved either directly or by various iterative procedures. Of course, if the number of lattice points is even moderately large, hand solution becomes prohibitively time-consuming, and high-speed computers must be used.

Specifically, in a plane grid, if the coordinates of the mesh points (named neutrally for the moment) are $p_i = p_0 + ih$ and $q_j = q_0 + jk$, then from the usual difference quotient approximation to the first derivative we have

$$(1a)\qquad \left.\frac{\partial f}{\partial p}\right|_{p_i,q_j} = \frac{f(p_{i+1},q_j) - f(p_i,q_j)}{h} = \frac{f_{i+1,j} - f_{i,j}}{h}$$

and, similarly,

$$(1b)\qquad \left.\frac{\partial f}{\partial q}\right|_{p_i,q_j} = \frac{f_{i,j+1} - f_{i,j}}{k}$$

Furthermore, if we differentiate Stirling's interpolation formula [Eq. (18), Sec. 5.2] twice, set $r = 1$, neglect all differences beyond the second, and identify p_i with x_0, we have

$$(2a)\qquad \left.\frac{\partial^2 f}{\partial p^2}\right|_{p_i,q_j} = \frac{f(p_{i+1},q_j) - 2f(p_i,q_j) + f(p_{i-1},q_j)}{h^2} = \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{h^2}$$

and, similarly

$$(2b)\qquad \left.\frac{\partial^2 f}{\partial q^2}\right|_{p_i,q_j} = \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{k^2}$$

We now turn our attention to the application of these general ideas to the solution of elliptic, parabolic, and hyperbolic equations.

CASE 1. ELLIPTIC EQUATIONS (LAPLACE'S EQUATION IN TWO DIMENSIONS) Using Eqs. (2) to approximate each of the partial derivatives in the two-dimensional form of Laplace's equation, namely,

$$\partial^2 f \quad + \quad \partial^2 f$$

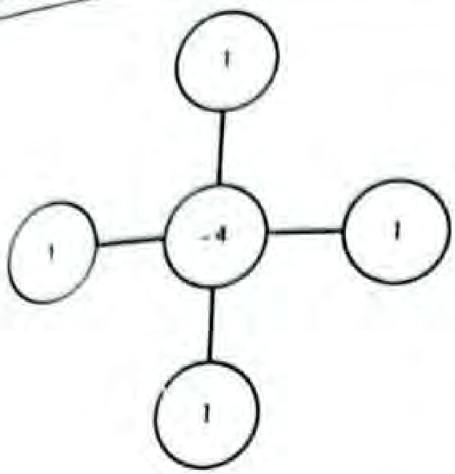


Figure 9.21
The finite difference "star" which approximates

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

we obtain, as a difference equation approximating the actual equation,

$$\frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{h^2} + \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{k^2} = 0$$

or, making the natural and convenient assumption that $h = k$ and solving for $f_{i,j}$,

$$f_{i,j} = \frac{f_{i+1,j} + f_{i,j+1} + f_{i-1,j} + f_{i,j-1}}{4} \tag{3}$$

This asserts that *the value of f at any mesh point is equal to the average of the values of f at the four adjacent mesh points*, as shown in Fig. 9.21. The configuration shown in Fig. 9.21 is often called a **star**.

If Eq. (3) is evaluated at each of the mesh points which are not boundary points, where the value of the solution f is initially given, the result is a system of simultaneous linear equations in the unknown functional values, f_{ij}. The number of equations is, of course, just equal to the number of mesh points at which the value of f is to be calculated; and (at least for rectangular regions) it can be shown that this system of equations always has a unique nontrivial solution.† In important problems, where high accuracy is required, the number of mesh points, and hence the number of equations, may be anywhere from several hundred to several thousand. Each equation is very simple, however, for none can contain more than five of the unknown functional values. Large systems of linear equations of this simple structure have been extensively studied, and efficient computer programs are available for their solution.

To illustrate the formulation and solution of such a system, let us attempt to approximate the steady-state temperature distribution in the square region shown in Fig. 9.22, using the grid obtained by dividing each edge into four equal parts. The unknowns in this problem are the temperatures at the nine points of the grid which are not boundary points and at which the temperature is not determined by the given boundary conditions.

At the outset, we note that from symmetry $f_{11} = f_{31}$, $f_{12} = f_{32}$, and $f_{13} = f_{33}$, so that our problem actually involves only six equations in the six unknowns f_{11},

† See, for instance, E. Isaacson and H. B. Keller, "Analysis of Numerical Methods," pp. 447–448, Wiley, New York, 1966.

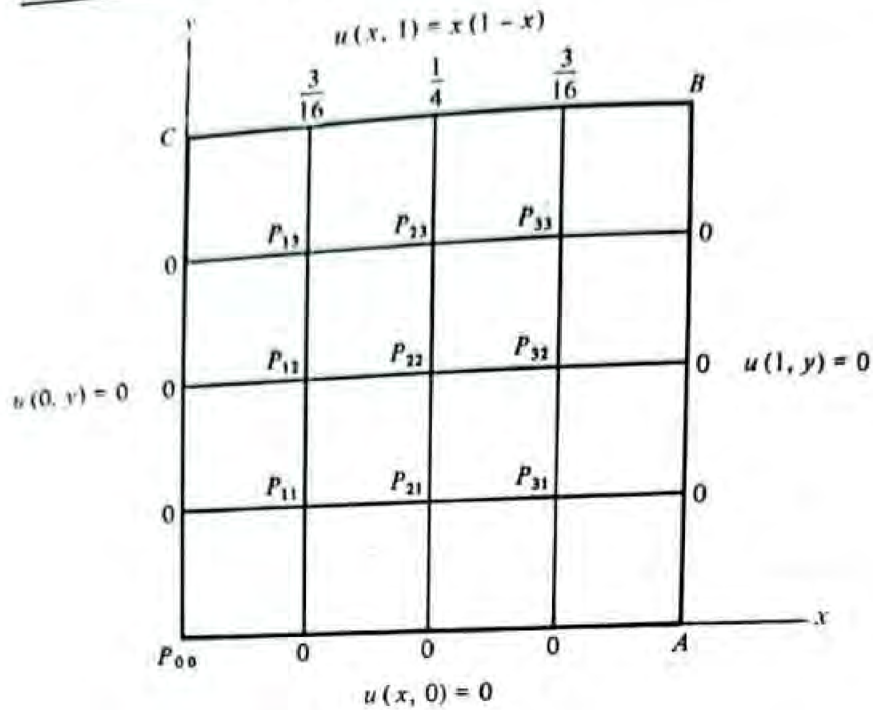


Figure 9.22
A typical lattice used in the approximate solution of Laplace's equation in the unit square.

$f_{12}, f_{13}, f_{21}, f_{22}$, and f_{23}. Applying formula (3) at each of the six mesh points P_{11}, $P_{12}, P_{13}, P_{21}, P_{22}, P_{23}$ and taking into account the symmetries we have just noted and the known values of f on the boundary, we have at P_{11}

$$4f_{11} - f_{01} - f_{10} - f_{21} - f_{12} = 0$$

or, noting that by hypothesis $f_{01} = f_{10} = 0$,

(4) $$4f_{11} - f_{21} - f_{12} = 0$$

Similarly, at $P_{12}, P_{13}, P_{21}, P_{22}, P_{23}$ we have, respectively,

(5) $$4f_{12} - f_{11} - f_{22} - f_{13} = 0$$

(6) $$4f_{13} - f_{12} - f_{23} = \tfrac{3}{16}$$

(7) $$4f_{21} - 2f_{11} - f_{22} = 0$$

(8) $$4f_{22} - f_{21} - 2f_{12} - f_{23} = 0$$

(9) $$4f_{23} - f_{22} - 2f_{13} = \tfrac{1}{4}$$

Using Eqs. (4), (5), and (6) to eliminate f_{21}, f_{22}, and f_{23} from Eqs. (7), (8), and (9), we obtain the system

$$\begin{aligned} 15f_{11} - 8f_{12} + f_{13} &= 0 \\ -8f_{11} + 16f_{12} - 8f_{13} &= -\tfrac{3}{16} \\ f_{11} - 8f_{12} + 15f_{13} &= 1 \end{aligned}$$

from which we find at once that

$$f_{11} \doteq 0.0151 \doteq f_{31} \qquad f_{12} \doteq 0.0391 \doteq f_{32} \qquad f_{13} \doteq 0.0865 \doteq f_{33}$$

and from these

$$f_{21} \doteq 0.0212 \qquad f_{22} \doteq 0.0547 \qquad f_{23} \doteq 0.1194$$

The correct values, as determined from the series solution obtained by the method of separation of variables, are

$$f_{11} = f_{31} \doteq 0.0137 \qquad f_{12} = f_{32} \doteq 0.0364 \qquad f_{13} = f_{33} \doteq 0.0833$$
$$f_{21} \doteq 0.0194 \qquad f_{22} \doteq 0.0513 \qquad f_{23} \doteq 0.1159$$

Problems in which the normal derivative, rather than the function itself, is specified along a part, or all, of the boundary can also be handled by the method we have been discussing. Suppose, for instance, that along the edge AB of the region shown in Fig. 9.22 the value of the normal derivative is required to be λ times the value of the function. Then approximating the derivative by its difference quotient, the boundary condition

$$\frac{\partial f}{\partial n} = \lambda f$$

that is,

$$\frac{\partial f}{\partial x} = \lambda f$$

becomes

$$\frac{f_{4j} - f_{3j}}{h} = \lambda f_{4j} \qquad \text{or} \qquad f_{4j} = \frac{f_{3j}}{1 - \lambda h}$$

Now, applying Eq. (3) at P_{33}, for example, we have

$$4f_{33} - f_{43} - f_{34} - f_{23} - f_{32} = 0$$

or

$$4f_{33} - f_{33}/(1 - \lambda h) - \tfrac{3}{16} - f_{23} - f_{32} = 0$$

or, finally,

$$\frac{3 - 4\lambda h}{1 - \lambda h} f_{33} - f_{23} - f_{32} = \tfrac{3}{16}$$

Along an insulated boundary λ is, of course, equal to zero and, in particular, the last equation becomes simply

$$3f_{33} - f_{23} - f_{32} = \tfrac{3}{16}$$

In any event, the system of equations thus obtained can be solved just as in the preceding case.

To obtain the proper finite-difference relation at a lattice point adjacent to an irregular boundary, as suggested in Fig. 9.23, it is convenient to use divided difference approximations for the derivatives. Thus suppose that in Fig. 9.23 the boundary points A and B are at the distances $\theta_A h$ and $\theta_B h$ from E, where θ_A and θ_B are each less than 1. Then using second divided differences in the x and y directions as approximations to $\partial^2 f/\partial x^2$ and $\partial^2 f/\partial y^2$, we have [see Exercise 26, Sec. 5.1]

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_E = 2\left[\frac{f_D}{h(h + \theta_B h)} + \frac{f_E}{-h(\theta_B h)} + \frac{f_B}{\theta_B h(\theta_B h + h)}\right]$$

and

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_E = 2\left[\frac{f_C}{h(h + \theta_A h)} + \frac{f_E}{-h(\theta_A h)} + \frac{f_A}{\theta_A h(\theta_A h + h)}\right]$$

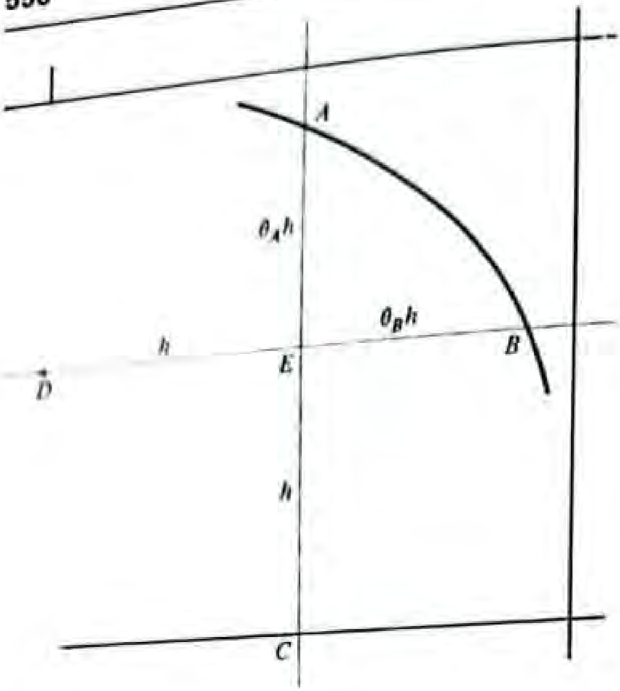


Figure 9.23
Geometry of a lattice point adjacent to an irregular boundary.

or

$$\text{(10)}\quad \nabla^2 f\,|_E = \frac{2}{h^2}\left[\frac{f_A}{\theta_A(1+\theta_A)} + \frac{f_B}{\theta_B(1+\theta_B)} + \frac{f_C}{1+\theta_A} + \frac{F_D}{1+\theta_B} - \frac{\theta_A+\theta_B}{\theta_A\theta_B} f_E\right] = 0$$

This is the equation which is to be used at those points of the lattice whose immediate neighbors in the lattice fall outside the boundary of the region.

There is another way in which the finite-difference approximation to the Laplacian can be used to determine the value of the solution at the points of the lattice. It is a simple iterative method which proceeds as follows:† We first recall that the finite-difference approximation to the Laplacian [Eq. (3)] expresses the value of the solution at any mesh point as the average of the values at the four adjacent points. Thus, after an initial estimate for the value of the solution at each mesh point has been made, they can be corrected and improved by systematically moving through the lattice and replacing each value according to Eq. (3). In doing this, each value as soon as it is corrected should be used in all subsequent calculations. Of course, in regions with irregular boundaries, Eq. (10) must be used to correct the values of the solution at those mesh points whose neighbors lie outside the boundary.

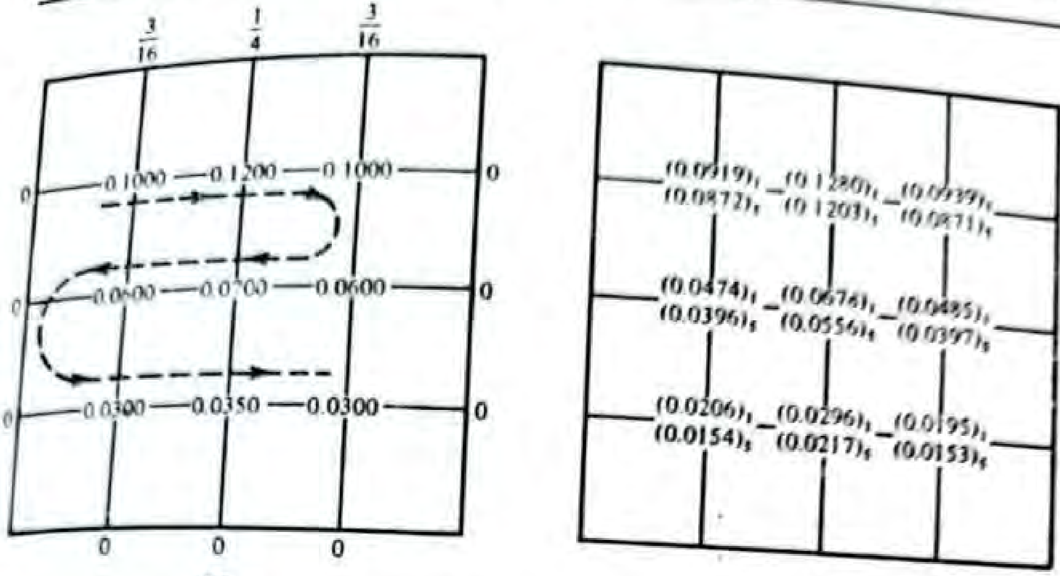
As an illustration of this method, let us reconsider the problem we have just worked. Beginning with the estimates shown in Fig. 9.24*a*, we have for the first refinement of f_{13} the value

$$\frac{0.1875 + 0.0000 + 0.1200 + 0.0600}{4} = 0.0919$$

Continuing through the lattice as indicated‡ using the corrected values as soon as

† This procedure is usually known as the **Liebmann method**, after the German scientist who first proposed it in a paper in 1918.

‡ Any other path would serve just as well.



(a) (b)

Figure 9.24
Data from an iterative solution of Laplace's equation.

they become available (but taking no advantage of the symmetry of the problem), we obtain the values shown in Fig. 9.24*b*. Values bearing the subscript 1 were obtained by a single iteration; values bearing the subscript 5 were obtained after five iterations.

CASE 2. PARABOLIC EQUATIONS (THE ONE-DIMENSIONAL HEAT EQUATION) For the one-dimensional heat equation

$$\frac{\partial^2 f}{\partial x^2} = a^2 \frac{\partial f}{\partial t}$$

the region of the xt plane over which a solution is sought is always infinite, because of the infinite increase of time. Thus a typical lattice would be that shown in Fig. 9.20*c*. As a finite-difference approximation to the heat equation we have, using Eqs. (1*b*) and (2*a*),

$$\frac{1}{h^2}(f_{i+1,j} - 2f_{i,j} + f_{i-1,j}) = \frac{a^2}{k}(f_{i,j+1} - f_{i,j})$$

or, setting $m = k/a^2h^2$ and solving for $f_{i,j+1}$,

(11)
$$f_{i,j+1} = mf_{i+1,j} + (1 - 2m)f_{i,j} + mf_{i-1,j}$$

Clearly, it would be convenient to choose h and k so that the value of m was $\frac{1}{2}$.

The values of f on the boundary are of course provided by the data of the problem. Thus the given initial condition $f(x,0)$ provides the values of $f_{00}, f_{10}, f_{20}, \ldots$. Similarly, end conditions of the form

$$f(0,t) = g_1(t) \qquad f(l,t) = g_2(t)$$

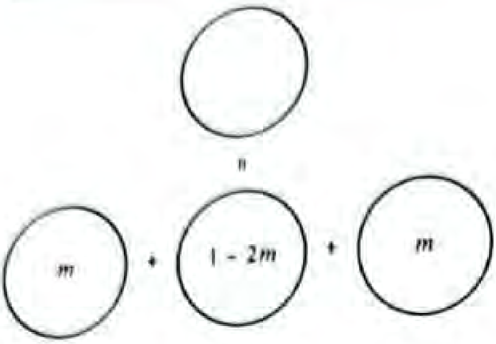


Figure 9.25
The iteration formula for the one-dimensional heat equation.

where g_1 and g_2 are usually, though not necessarily, constant, furnish the values of $f_{01}, f_{02}, f_{03}, \ldots$ and $f_{l1}, f_{l2}, f_{l3}, \ldots$. Insulated end conditions can, of course, be handled as we outlined above in our discussion of Laplace's equation.

Once the values of f at the lattice points on the boundary have been determined from the conditions of the problem, the determination of the solution over the rest of the lattice proceeds in a straightforward way, using the extrapolation pattern provided by Eq. (11) and shown in Fig. 9.25. First, the values of $f_{11}, f_{21}, \ldots, f_{l-1,1}$ are calculated from the known values of $f_{00}, f_{10}, f_{20}, \ldots, f_{l0}$. Then using these new values and the boundary values f_{01} and f_{l1}, the solution is "marched" forward by calculating the values of f at the lattice points in the third row, and so on as far as desired.

CASE 3. HYPERBOLIC EQUATIONS (THE ONE-DIMENSIONAL WAVE EQUATION) The solution procedure for the one-dimensional wave equation

$$a^2 \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial t^2}$$

is very much like that for the one-dimensional heat equation. Because of the increase of time, the region of the xt plane over which a solution is sought is always infinite. Hence a solution must be found by "marching" forward a step at a time from known initial values. The finite-difference approximation which provides the necessary extrapolation formula comes immediately from Eqs. (2a) and (2b), and is

$$a^2\left[\frac{1}{h^2}(f_{i+1,j} - 2f_{i,j} + f_{i-1,j})\right] = \frac{1}{k^2}(f_{i,j+1} - 2f_{i,j} + f_{i,j-1})$$

or, setting $m^2 = a^2k^2/h^2$ and solving for $f_{i,j+1}$

(12) $$f_{i,j+1} = 2(1 - m^2)f_{i,j} + m^2(f_{i+1,j} + f_{i-1,j}) - f_{i,j-1}$$

If the dimensions of the lattice, h and k, are chosen so that $m = 1$, Eq. (12) assumes the especially convenient form

(13) $$f_{i,j+1} = f_{i+1,j} + f_{i-1,j} - f_{i,j-1}$$

Equations (12) and (13) are shown schematically in Fig. 9.26.

In a particular problem the wave equation would be accompanied by end conditions of the form

$$f(0, t) = g_1(t) \qquad \text{and} \qquad f(l,t) = g_2(t)$$

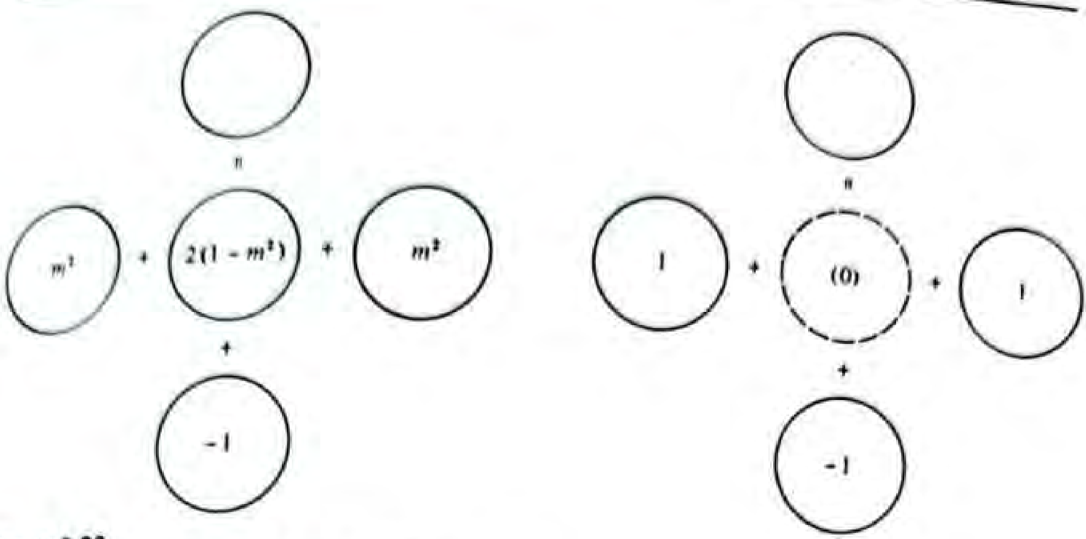


Figure 9.26
Iterative formulas for the one-dimensional wave equation.

where usually, though not necessarily, g_1 and g_2 would be identically zero, and by initial conditions of the form

$$f(x,0) = \phi(x) \quad \text{and} \quad \left.\frac{\partial f}{\partial t}\right|_{x,0} = \theta(x)$$

The two end conditions provide the values of f at the mesh points on the left- and right-hand boundaries of the grid. The first of the two initial conditions provides the values of f on the lowest row of mesh points. The second initial condition provides the values of f at the mesh points in the second row of the lattice, since the approximation

$$\left.\frac{\partial f}{\partial t}\right|_{x_i,0} = \frac{f_{i1} - f_{i0}}{k} = \theta(x_i)$$

becomes

$$f_{i1} = f_{i0} + k\theta(x_i)$$

With these values known, either formula (12) or (13) allows us to calculate the values of f at the lattice points in the third row, and thus the solution can be "marched" forward as far as desired.

There are, of course, many questions concerning refinements in the solution procedures we have outlined, as well as their accuracy, which we have not discussed. These, however, we must leave to more advanced texts.†

EXERCISES

1 Work the example considered under Case 1, using the Liebmann method but taking advantage of the symmetry of the problem.

2 Work the example considered under Case 1, taking as initial estimates the values obtained by solving Eqs. (4) through (9). Explain.

† See, for example, E. Isaacson and H. B. Keller, "Analysis of Numerical Methods," Wiley, New York, 1966; K. S. Kunz, "Numerical Analysis," McGraw-Hill, New York, 1957; S. H. Crandall, "Engineering Analysis," McGraw-Hill, New York, 1956.

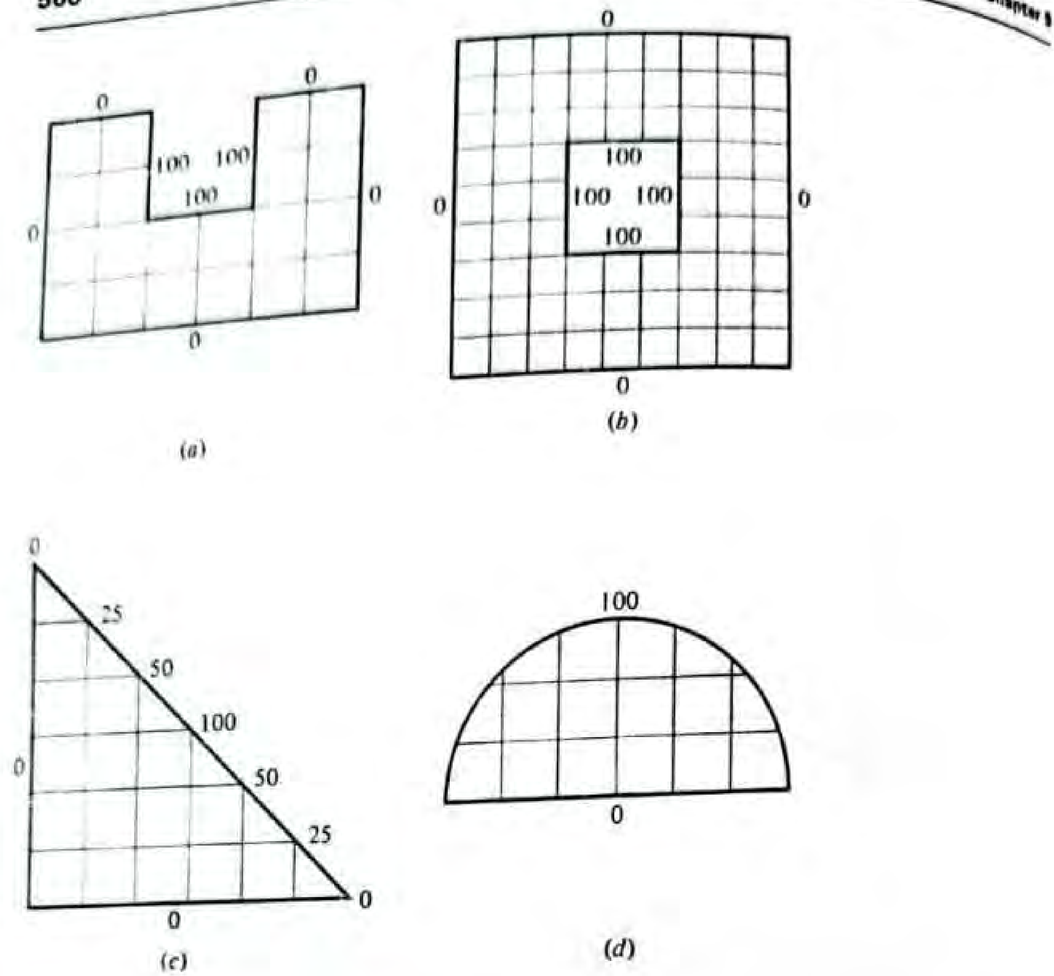


Figure 9.27

3 Work the example considered under Case 1 if the upper edge is perfectly insulated, the lower edge is maintained at the temperature 100 and the two vertical edges are maintained at the temperature 0.

Find the steady-state temperature distribution in each of the following regions, using the indicated lattice.

4 The region shown in Fig. 9.27*a*

5 The region shown in Fig. 9.27*b*

6 The region shown in Fig. 9.27*c*

7 The region shown in Fig. 9.27*d*

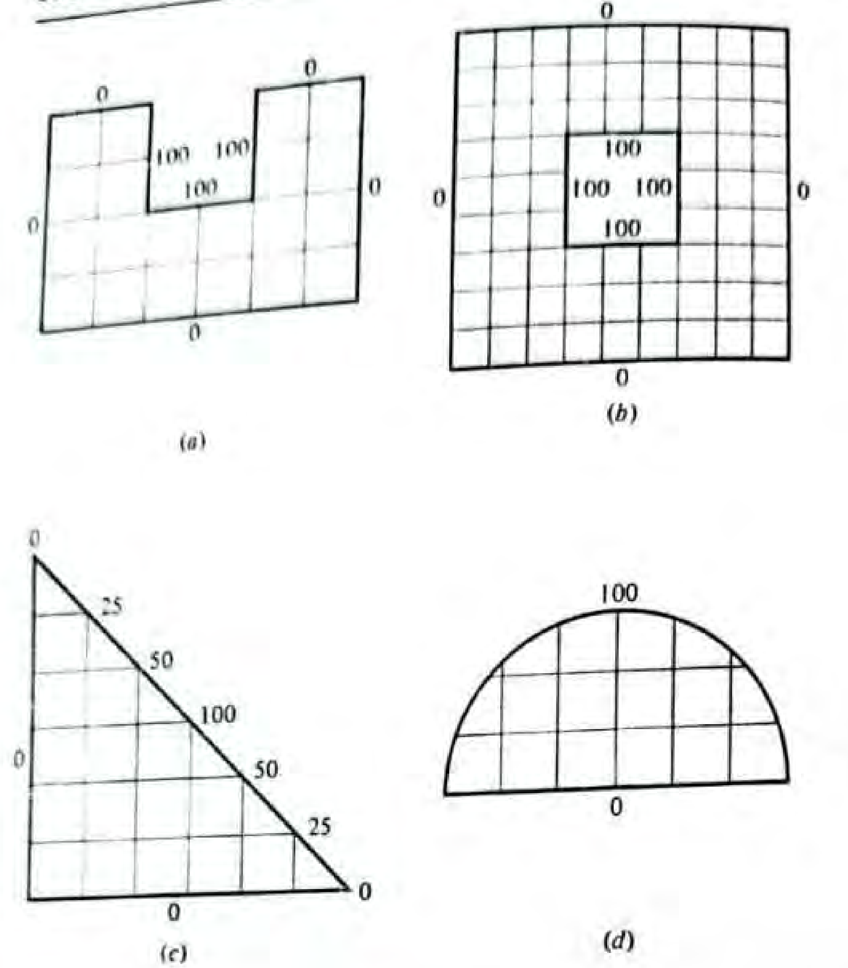
8 Derive a finite difference-approximation for the three-dimensional Laplacian $u_{xx} + u_{yy} + u_{zz}$

9 Using the indicated lattice, find the function which satisfies the nonhomogeneous equation $u_{xx} + u_{yy} = -2$ and vanishes on the boundary of the region shown in Fig. 9.22. (This problem is of importance in the study of the torsion of prismatic bars.)

10 A thin rod is initially at the temperature $u(x,0) = 0$. Find its temperature as a function of x and t if at $t = 0$ the left-hand end of the rod is suddenly raised to the temperature 100 and maintained thereafter at that temperature while the right-hand end is maintained at the temperature 0. Assume a lattice based on a division of the rod into 10 equal segments and (a) $m = \frac{1}{2}$, (b) $m = 1$.

11 A thin rod of unit length is initially at the temperature distribution $u(x,0) = 100(1 - 2x)^2$. If each end of the rod is perfectly insulated, find the temperature in the rod as a function of x and t. Assume a lattice based on a division of the rod into 10 equal segments and $m = \frac{1}{2}$.

12 A thin rod is initially at the temperature $u(x,0) = 0$. At $t = 0$, while the right-hand end is maintained at the temperature 0, the left-hand end is subjected to the periodic temperature condition $u(0,t) = \sin(\pi t/12)$. Assuming a lattice based on a division of the rod into 10 equal segments and $\Delta t = 1$, find the temperature in the rod as a function of x and t. Assume $m = \frac{1}{2}$.



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