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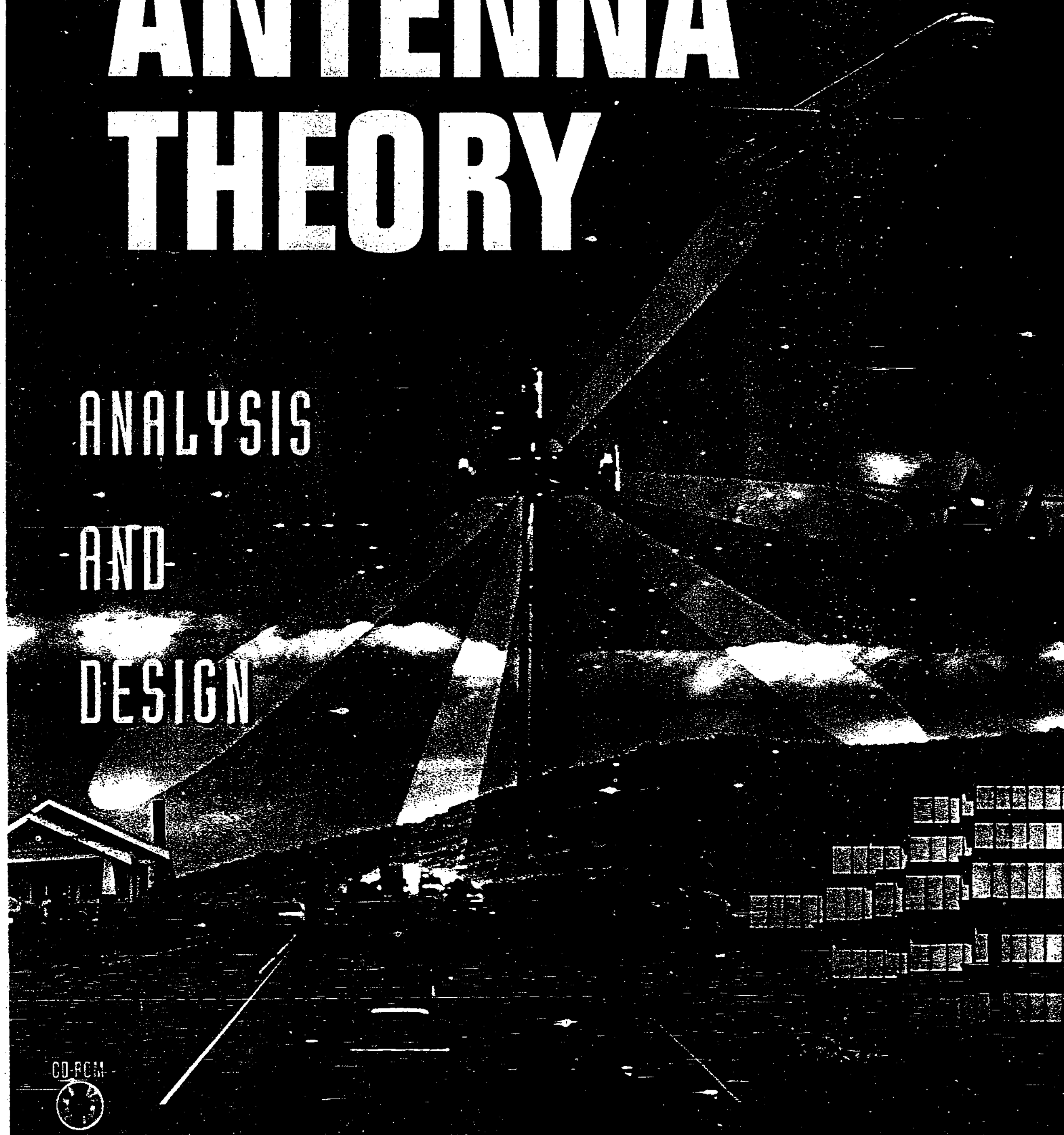
ANTENNA THEORY

ANALYSIS AND DESIGN



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CONSTANTINE A. BALANIS

SOLUTIONS MANUAL TO ACCOMPANY

ANTENNA THEORY

ANALYSIS AND DESIGN

THIRD EDITION

CONSTANTINE A. BALANIS
Arizona State University

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Preface

This Solutions Manual consists of solutions for all the problems found in ***Antenna Theory: Analysis and Design*** (3rd edition, 2005) at the end of Chapters 2–16. There are 596 problems, most of them with multiple parts. The degree of difficulty and length varies. While certain solutions need special functions, found in tabular and graphical forms in the appendices, others require the use of the computer program. These computer programs are contained in a CD, which is included with the book. All of the computer programs, especially those at the end of Chapters 6, 11, 13 and 14 have been developed to design, respectively, uniform and nonuniform arrays, log-periodic arrays, horns and microstrip patch antennas. In some cases, the computer programs also perform analysis on the designs. The programs at the end of Chapters 2, 4, 5, 7, 8, 10 and 16 are primarily developed for analysis. The problems have been designed to test the student's grasp of this text's material and to apply the concepts to the analysis and design of many practical radiators. In this third edition, more emphasis has been placed on design. To accomplish this, equations, procedures, examples, graphs, end-of-the-chapter problems, and computer programs have been developed.

This manual has been prepared to assist the instructor in making homework and test assignments, and to provide one set of solutions for all of the problems. There are undoubtedly errors which have been overlooked. In addition, the solutions contained in this manual are not necessarily the simplest and/or the best. The author would, therefore, appreciate having errors brought to his attention and solicits alternate solutions to the problems.

This Solutions Manual for the third edition has been prepared from the manuals of the first and second editions and many other new problems provided by the author.

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CHAPTER 2

Extract

2-1. (a) $d\Omega = \sin\theta \, d\theta \, d\phi$

$$\Omega_A = \int_{45^\circ}^{60^\circ}\int_{30^\circ}^{60^\circ} d\Omega = \int_{\pi/4}^{\pi/3}\int_{\pi/6}^{\pi/3} \sin\theta \, d\theta \, d\phi$$

$$= (\phi)\Big|_{\pi/4}^{\pi/3} (-\cos\theta)\Big|_{\pi/6}^{\pi/3}$$

$$= \left(\frac{\pi}{3} - \frac{\pi}{4}\right)(-0.5 + 0.866)$$

$$\Omega_A = \left(\frac{\pi}{12}\right)(0.366) = 0.09582 \text{ sterads}$$

$$\Omega_A = \begin{cases} 0.09582 \text{ sterads} \\ 0.09582\left(\frac{180}{\pi}\right)\left(\frac{180}{\pi}\right) = 314.5585 \text{ (degrees)}^2 \end{cases}$$

Approximate

$$\Omega_A \simeq \left(\frac{\pi}{3} - \frac{\pi}{4}\right)\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

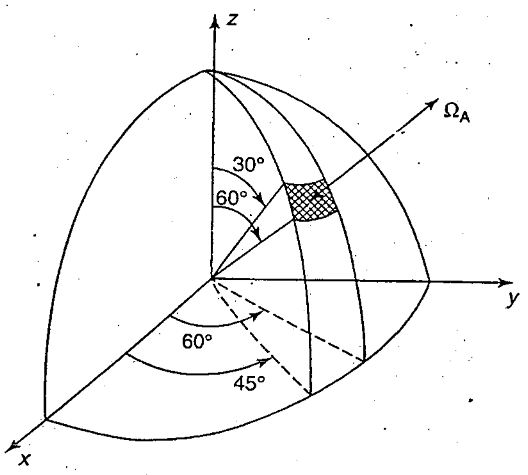
$$\simeq \left(\frac{\pi}{12}\right)\left(\frac{\pi}{6}\right) = \frac{\pi^2}{72}$$

$$\Omega_A \simeq 0.13708 \text{ sterads}$$

$$\Omega_A \simeq (60 - 45)(60 - 30)$$

$$\simeq 450 \text{ (degrees)}^2 \text{ or error of}$$

$$\left(\frac{450 - 314.5585}{314.5585}\right) \times 100 = 43.06\%$$



(b) $$D_0 = \frac{4\pi}{\Omega_A(\text{sterads})} = \frac{4\pi}{0.09582} = 131.1456 \text{ (dimensionless)}$$
$$= 10\log_{10}(131.1456) = 21.1775 \text{ dB}$$

or

$$D_0 = \frac{4\pi\left(\frac{180}{\pi}\right)\left(\frac{180}{\pi}\right)}{\Omega_A \text{ (degrees)}^2} = 131.1456 \text{ (dimensionless)} = 21.1775 \text{ dB}$$

$$D_0 = \begin{cases} 131.1456 \text{ (dimensionless)} \\ 21.1775 \text{ (dB)} \end{cases}$$

2-2. $\underline{\mathcal{W}} = \underline{\mathcal{E}} \times \underline{\mathcal{H}} = Re[\underline{E}e^{j\omega t}] \times Re[\underline{H}e^{j\omega t}]$

Using the identity $Re[\underline{A}e^{j\omega t}] = \frac{1}{2}[\underline{E}e^{j\omega t} + \underline{E}^* e^{-j\omega t}]$

The instant Poynting vector can be written as

$$\begin{aligned}
\underline{\mathcal{W}} &= \left\{\tfrac{1}{2}[\underline{E}e^{j\omega t} + \underline{E}^* e^{-j\omega t}]\right\} \times \left\{\tfrac{1}{2}[\underline{H}e^{j\omega t} + \underline{H}^* e^{-j\omega t}]\right\} \\
&= \tfrac{1}{2}\left\{\tfrac{1}{2}[\underline{E} \times \underline{H}^* + \underline{E}^* \times \underline{H}] + \tfrac{1}{2}[\underline{E} \times \underline{H}e^{j2\omega t} + \underline{E}^* \times \underline{H}^* e^{-j\omega t}]\right\} \\
&= \tfrac{1}{2}\left\{\tfrac{1}{2}[\underline{E} \times \underline{H}^* + (\underline{E} \times \underline{H}^*)^*] + \tfrac{1}{2}[\underline{E} \times \underline{H}e^{j2\omega t} + (\underline{E} \times \underline{H}e^{j\omega t})^*]\right\}
\end{aligned}$$

Using the above identity again, but this time in reverse order, we can write that

$$\underline{\mathcal{W}} = \tfrac{1}{2}[\text{Re}(E \times H^*)] + \tfrac{1}{2}[\text{Re}(\underline{E} \times \underline{H}e^{j2\omega t})]$$

2-3. (a) $\underline{W}_{\text{rad}} = \frac{1}{2}[\underline{E} \times \underline{H}^*] = \frac{E^2}{2\eta}\hat{a}_r = \frac{5^2}{2(120\pi)}\hat{a}_r = 0.03315\hat{a}_r$ watts/m^2

(b)
$$\begin{aligned}
P_{\text{rad}} &= \oint_s W_{\text{rad}}\, ds = \int_0^{2\pi}\int_0^{\pi} (0.03315)(r^2 \sin\theta\, d\theta\, d\phi) \\
&= \int_0^{2\pi}\int_0^{\pi} (0.03315)(100)^2 \cdot \sin\theta\, d\theta\, d\phi \\
&= 2\pi(0.03315)(100)^2 \int_0^{\pi} \sin\theta\, d\theta = 2\pi(0.03315)(100)^2 \cdot (2) \\
&= 4165.75 \text{ watts}
\end{aligned}$$

2-4. a. $\mathcal{U}(\theta) = \cos\theta$

$$\mathcal{U}(\theta_h) = 0.5 = \cos\theta_h \Rightarrow \theta_h = \cos^{-1}(0.5) = 60^\circ$$
$$\Rightarrow \Theta_h = 2(60^\circ) = 120^\circ = \frac{2\pi}{3} \text{ rads.}$$
$$\mathcal{U}(\theta_n) = 0 = \cos\theta_n \Rightarrow \theta_n = \cos^{-1}(0) = 90^\circ$$
$$\Rightarrow \Theta_n = 2(90^\circ) = 180^\circ = \pi \text{ rads.}$$

b. $\mathcal{U}(\theta) = \cos^2\theta$

$$\mathcal{U}(\theta_h) = 0.5 = \cos^2\theta_h \Rightarrow \theta_h = \cos^{-1}(0.5)^{1/2} = 45^\circ$$
$$\Rightarrow \Theta_h = 2(45) = 90^\circ = \pi/2 \text{ rads}$$
$$\mathcal{U}(\theta_n) = 0 = \cos^2\theta_n \Rightarrow \theta_n = \cos^{-1}(0) = 90^\circ$$
$$\Rightarrow \Theta_n = 2(90^\circ) = 180^\circ = \pi \text{ rads}$$

c. $\mathcal{U}(\theta) = \cos(2\theta)$

$$\mathcal{U}(\theta_h) = 0.5 = \cos(2\theta_h) \Rightarrow \theta_h = \tfrac{1}{2}\cos^{-1}(0.5) = 30^\circ$$
$$\Rightarrow \Theta_h = 2(30^\circ) = 60^\circ = \pi/3 \text{ rads}$$
$$\mathcal{U}(\theta_n) = 0 = \cos(2\theta_n) \Rightarrow \theta_n = \tfrac{1}{2}\cos^{-1}(0) = 45^\circ$$
$$\Rightarrow \Theta_n = 2(45^\circ) = 90^\circ = \pi/2 \text{ rads}$$

d. $\mathcal{U}(\theta) = \cos^2(2\theta)$

$$\mathcal{U}(\theta_h) = 0.5 = \cos^2(2\theta_h) \Rightarrow \theta_h = \tfrac{1}{2}\cos^{-1}(0.5)^{1/2} = 22.5^\circ$$
$$\Rightarrow \Theta_h = 2(22.5^\circ) = 45^\circ = \frac{\pi}{4} \text{ rads}$$
$$\mathcal{U}(\theta_n) = 0 = \cos^2(2\theta_n) \Rightarrow \theta_n = \tfrac{1}{2}\cos^{-1}(0) = 45^\circ$$
$$\Rightarrow \Theta_n = 2(45^\circ) = 90^\circ = \pi/2 \text{ rads}$$

e. $\mathcal{U}(\theta) = \cos(3\theta)$

$$\mathcal{U}(\theta_h) = \cos(3\theta_h) = 0.5 \Rightarrow \theta_h = \frac{1}{3}\cos^{-1}(0.5) = 20^\circ$$
$$\Rightarrow \Theta_h = 2(20^\circ) = 40^\circ = 0.698 \text{ rads}$$
$$\mathcal{U}(\theta_n) = \cos(3\theta_n) = 0 \Rightarrow \theta_n = \frac{1}{3}\cos^{-1}(0) = 30^\circ$$
$$\Rightarrow \Theta_n = 2(30^\circ) = 60^\circ = \pi/3 \text{ rads}$$

f. $\mathcal{U}(\theta) = \cos^2(3\theta)$

$$\mathcal{U}(\theta_h) = 0.5 = \cos^2(3\theta_h) \Rightarrow \theta_h = \frac{1}{3}\cos^{-1}(0.5)^{1/2} = 15^\circ$$
$$\Rightarrow \Theta_h = 2(15^\circ) = 30^\circ = \pi/6 \text{ rads}$$
$$\mathcal{U}(\theta_n) = 0 = \cos^2(3\theta_n) \Rightarrow \theta_n = \frac{1}{3}\cos^{-1}(0) = 30^\circ$$
$$\Rightarrow \Theta_n = 2(30^\circ) = 60^\circ = \pi/3 \text{ rads}$$

2-5. Using the results of Problem 2-4 and a nonlinear solver to find the half power beamwidth of the radiation intensity represented by the transcentendal functions, we have that:

(a) $\mathcal{U}(\theta) = \cos\theta\cos(2\theta) \Rightarrow \begin{cases} \text{HPBW} = 55.584^\circ \\ \text{FNBW} = 90^\circ \end{cases}$

(b) $\mathcal{U}(\theta) = \cos^2\theta\cos^2(2\theta) \Rightarrow \begin{cases} \text{HPBW} = 40.985^\circ \\ \text{FNBW} = 90^\circ \end{cases}$

(c) $\mathcal{U} = \cos\theta\cos(3\theta) \Rightarrow \begin{cases} \text{HPBW} = 38.668 \\ \text{FNBW} = 60^\circ \end{cases}$

(d) $\mathcal{U} = \cos^2\theta\cos^2(3\theta) \Rightarrow \begin{cases} \text{HPBW} = 28.745^\circ \\ \text{FNBW} = 60^\circ \end{cases}$

(e) $\mathcal{U} = \cos(2\theta)\cos(3\theta) \Rightarrow \begin{cases} \text{HPBW} = 34.942^\circ \\ \text{FNBW} = 60^\circ \end{cases}$

(f) $\mathcal{U} = \cos^2(2\theta)\cos^2(3\theta) \Rightarrow \begin{cases} \text{HPBW} = 25.583^\circ \\ \text{FNBW} = 60^\circ \end{cases}$

2-6. (a) $D_0 = \dfrac{4\pi U_{\max}}{P_{\text{rad}}} = \dfrac{4\pi(200\times10^{-3})}{0.9(125.66\times10^{-3})} = 22.22 = 13.47 \text{ dB}$

$G_0 = \epsilon_{cd}\cdot D_0 = 0.9(22.22) = 20 = 13.01 \text{ dB}$

(b) $D_0 = \dfrac{4\pi U_{\max}}{P_{\text{rad}}} = \dfrac{4\pi(200\times10^{-3})}{(125.66\times10^{-3})} = 20 = 13.01 \text{ dB}$

$G_0 = \epsilon_{cd}\cdot D_0 = 0.9\cdot(20) = 18 = 12.55 \text{ dB}$

2-7. $\boxed{\mathcal{U} = B_0\cos^2\theta}$

(a)
$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi/2} \mathcal{U}\sin\theta\, d\theta = B_0 2\pi \int_0^{\pi/2} \cos^2\theta\sin\theta\, d\theta$$
$$= 2\pi B_0 \int_0^{\pi/2} \cos^2\theta\, d\,(-\cos\theta)$$
$$P_{\text{rad}} = -2\pi B_0 \frac{\cos^3\theta}{3}\bigg|_0^{\pi/2} = -2\pi B_0\left[\frac{-1}{3}\right] = \frac{2\pi}{3}B_0 = 10 \Rightarrow B_0 = \frac{15}{\pi}$$
$$\mathcal{U} = \frac{15}{\pi}\cos^2\theta \Rightarrow W_{\text{rad}}\bigg|_{\max} = \frac{U}{r^2}\bigg|_{\max} = \frac{15}{\pi}\frac{\cos^2\theta}{r^2}\bigg|_{\max}$$
$$= \frac{15}{\pi(10^3)^2} = 4.7746\times10^{-6} \text{ watts/m}^2 @\ \theta = 0^\circ$$
$$W_{\text{rad}}\bigg|_{\max} = 4.7746\times10^{-6} \text{ watts/m}^2 @\ \theta = 0^\circ$$

(b)
$$\Omega_A \text{ (exact)} = \int_0^{2\pi}\int_0^{\pi} U_n\cos^2\theta\sin\theta\, d\theta\, d\phi$$
$$= \frac{2\pi}{3} \text{ steradians} = 2.0944 \text{ sterads}$$
$$\mathcal{U} = 0.5 = \cos^2\theta_h \Rightarrow \theta_h = \cos^{-1}(0.5)^{1/2} = 45^\circ$$
$$\Rightarrow \Theta_h = 2(45^\circ) = 90^\circ = \pi/2 \text{ rads}$$
$$\Omega_A\left(\begin{matrix}\text{Kraus'}\\ \text{approx}\end{matrix}\right) = \Theta_h^2 = (\pi/2)^2 = \frac{\pi^2}{4} = 2.4674 \text{ sterads}$$

(c)

$$D_0 \text{ (exact)} = \frac{4\pi}{\Omega_A \text{ (exact)}} = \frac{4\pi}{2\pi/3} = 6 = 7.782 \text{ dB}$$

$$D_0 \text{ (approx/Kraus')} = \frac{4\pi}{\Omega_A \text{ (approx)}} = \frac{4\pi}{\pi^2/4} = \frac{16}{\pi} = 5.093 = 7.0697$$

(d) G_0 Assuming lossless antenna ($P_{\text{in}} = P_{\text{rad}}$)

$$G_0 \text{ (exact)} = D_0 \text{ (exact)} = 6 = 7.782 \text{ dB}$$

$$G_0 \text{ (approx)} = D_0 \text{ (approx)} = 5.093 = 7.0697 \text{ dB}$$

$\boxed{\mathcal{U} = B_0 \cos^3\theta}$

(a)

$$P_{\text{rad}} = -2\pi B_0 \left(-\frac{1}{4}\right) = \frac{\pi}{2} B_0 = 10 \Rightarrow B_0 = 20/\pi$$

$$W_{\text{rad}}\Big|_{\max} = \frac{20}{\pi}\frac{1}{r^2} = \frac{20}{\pi} \times 10^{-6} = 6.366 \times 10^{-6} \text{ watts/m}^2$$

(b) Ω_A (exact) $= (\pi/2) = 1.5708$ sterads

$$\mathcal{U} = 0.5 = \cos^3\theta_h \Rightarrow \theta_h \cos^{-1}(0.5)^{1/3} = 37.467^\circ$$

$$\Rightarrow \Theta_h = 2(37.467^\circ) = 74.934^\circ = 1.30785 \text{ rads } \Omega_A \text{ (approx)}$$

$$= (1.30785)^2 = 1.71 \text{ sterads}$$

(c) D_0 (exact) $= 4\pi/\pi/2 = 8 = 9.031$ dB

$$D_0 \text{ (approx)} = \frac{4\pi}{1.71} = 7.347 = 8.66 \text{ dB}$$

(d) Assuming lossless antenna $\Rightarrow$ Gain = Directivity (see part c)

2-8. $\mathcal{U}(\theta,\phi) = \cos^n(\theta) \qquad 0 \leqslant \theta \leqslant \pi/2, 0 \leqslant \phi \leqslant 2\pi$

(a) $\mathcal{U}_n(\theta_n,\phi) = 0.5 = \cos^n(5^\circ) = [\cos(5^\circ)]^n = (0.99619)^n$

$$0.5 = (0.99619)^n$$

$$\log_{10}(0.5) = \log[(0.99619)^n] = n\log_{10}(0.99619) = n(-0.00166)$$

$$-0.30103 = -0.00166n$$

$$\boxed{n = 181.34}$$

(b) $\mathcal{U}(\theta,\phi) = \cos^{181.34}(\theta); \quad \mathcal{U}_{\max} = 1, \theta = 0°$

$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi/2} \mathcal{U}(\theta,\phi)\sin\theta\, d\theta\, d\phi = 2\pi \int_0^{\pi/2} \cos^{181.34}(\theta)\sin\theta\, d\theta$$
$$= 2\pi\left[-\frac{\cos^{182.34}(\theta)}{182.34}\right] = \left[-0 + \frac{1}{182.34}\right]2\pi = \frac{2\pi}{182.34} = 0.03446$$
$$D_0 = \frac{4\pi\mathcal{U}_{\max}}{P_{\text{rad}}} = \frac{4\pi(1)}{2\pi}(182.34) = 2(182.34) = 364.68$$

$$\boxed{D_0 = 364.68 = 25.62 \text{ dB}}$$

(c) *Kraus' Approximation* (2–27):

$$D_0 \simeq \frac{41{,}253}{\Theta_{1d}\Theta_{2d}} = \frac{41{,}253}{(10)(10)} = 412.53 = 26.15 \text{ dB}$$

$$\boxed{D_0 \simeq 412.53 = 26.15 \text{ dB}}$$

(d) *Tai & Pereira* (2-30b):

$$D_0 \simeq \frac{72{,}815}{\Theta_{1d}^2 + \Theta_{2d}^2} = \frac{72{,}815}{2(10)^2} = \frac{72{,}815}{200} = 364.075 = 25.61 \text{ dB}$$

$$\boxed{D_0 \simeq 364.075 = 25.61 \text{ dB}}$$

2-9.
$$\mathcal{U}(\theta,\phi) = \begin{cases} 1 & 0° \leqslant \theta \leqslant 20° \\ 0.342\csc(\theta) & 20° \leqslant \theta \leqslant 60° \\ 0 & 60° \leqslant \theta \leqslant 180° \end{cases} \quad 0° \leqslant \phi \leqslant 360°$$

$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} u(\theta,\phi)\sin\theta\, d\theta\, d\phi = 2\pi\left[\int_0^{20°}\sin\theta\, d\theta \right.$$
$$\left. + \int_{20°}^{60°} 0.342\csc(\theta)\times\sin\theta\, d\theta\right]$$
$$= 2\pi\left\{-\cos\theta\Big|_0^{\pi/9} + 0.342\cdot\theta\Big|_{\pi/9}^{\pi/3}\right\}$$
$$= 2\pi\left\{\left[-\cos\left(\frac{\pi}{9}\right)+1\right] + 0.342\left(\frac{\pi}{3}-\frac{\pi}{9}\right)\right\}$$
$$= 2\pi\left\{[-0.93969+1] + 0.342\pi\left(\frac{2}{9}\right)\right\}$$
$$= 2\pi\{0.06031 + 0.23876\} = 1.87912$$
$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{4\pi(1)}{1.87912} = 6.68737 = 8.25255 \text{ dB}$$

2-10. (a) $D_0 \simeq \frac{41{,}253}{\Theta_{1d}\Theta_{2d}} = \frac{41{,}253}{30(35)} = 39.29 = 15.94$ dB

$$A_{em} = \frac{\lambda^2}{4\pi} D_0$$

(b) $D_0 \simeq \frac{72{,}815}{\Theta_{1d}^2 + \Theta_{2d}^2} = \frac{72{,}815}{(30)^2 + (35)^2} = 34.27 = 15.35$ dB

$$A_{em} = \frac{\lambda^2}{4\pi} D_0$$

2-11. $D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}}$

(a) $U = \sin\theta \sin\phi$ for $0 \leqslant \theta \leqslant \pi, 0 \leqslant \phi \leqslant \pi$

$U|_{\max} = 1$ and it occurs when $\theta = \phi = \pi/2$.

$$P_{\text{rad}} = \int_0^\pi \int_0^\pi U \sin\theta \, d\theta \, d\phi = \int_0^\pi \sin\phi \, d\phi \int_0^\pi \sin^2\theta \, d\theta = 2\left(\frac{\pi}{2}\right) = \pi.$$

Thus $D_0 = \frac{4\pi(1)}{\pi} = 4 = 6.02$ dB

The half-power beamwidths are equal to

$$\text{HPBW (az.)} = 2[90^\circ - \sin^{-1}(1/2)] = 2(90^\circ - 30^\circ) = 120^\circ$$
$$\text{HPBW (el.)} = 2[90^\circ - \sin^{-1}(1/2)] = 2(90^\circ - 30^\circ) = 120^\circ$$

In a similar manner, it can be shown that for

(b) $U = \sin\theta \sin^2\phi \Rightarrow D_0 = 5.09 = 7.07$ dB

HPBW (el.) = 120°, HPBW (az.) = 90°

(c) $U = \sin\theta \sin^3\phi \Rightarrow D_0 = 6 = 7.78$ dB

HPBW (el.) = 120°, HPBW (az.) = 74.93°

(d) $U = \sin^2\theta \sin\phi \Rightarrow D_0 = 12\pi/8 = 4.71 = 6.73$ dB

HPBW (el.) = 90°, HPBW (az.) = 120°

(e) $U = \sin^2\theta \sin^2\phi \Rightarrow D_0 = 6 = 7.78$ dB, HPBW (az.) = HPBW (el.) = 90°

(f) $U = \sin^2\theta \sin^3\phi \Rightarrow D_0 = 9\pi/4 = 7.07 = 8.49$ dB

HPBW (el.) = 90°, HPBW (az.) = 74.93°

2-12. Using the half-power beamwidths found in the previous problem (Problem 2-11), the directivity for each intensity using Kraus' and Tai & Pereira's formulas is given by

$U = \sin\theta \cdot \sin\phi$;

(a) $D_0 \simeq \dfrac{41253}{\Theta_{1d}\Theta_{2d}} = \dfrac{41253}{120(120)} = 2.86 = 4.57$ dB

(b) $D_0 \simeq \dfrac{72{,}815}{\Theta_{1d}^2 + \Theta_{2d}^2} = \dfrac{72{,}815}{(120)^2 + (120)^2} = 2.53 = 4.03$ dB

$U = \sin\theta \cdot \sin^2\phi$:

(a) $D_0 \simeq 3.82 = 5.82$ dB

(b) $D_0 \simeq 3.24 = 5.10$ dB

$U = \sin\theta \cdot \sin^3\phi$;

(a) $D_0 \simeq 4.59 = 6.62$ dB

(b) $D_0 \simeq 3.64 = 5.61$ dB

$U = \sin^2\theta \cdot \sin\phi$;

(a) $D_0 \simeq 3.82 = 5.82$ dB

(b) $D_0 \simeq 3.24 = 5.10$ dB

$U = \sin^2\theta \cdot \sin^2\phi$;

(a) $D_0 \simeq 5.09 = 7.07$ dB

(b) $D_0 \simeq 4.49 = 6.53$ dB

$U = \sin^2\theta \cdot \sin^3\phi$:

(a) $D_0 \simeq 6.12 = 7.87$ dB

(b) $D_0 \simeq 5.31 = 7.25$ dB

2-13. (a) $D_0 = \dfrac{4\pi}{\Theta_{1r}\Theta_{2r}} = \dfrac{4\pi}{(1.5064)^2} = 5.5377 = 7.433$ dB

(b) $D_0 = \dfrac{32\ln(2)}{\Theta_{1r}^2 + \Theta_{2r}^2} = \dfrac{32\ln(2)}{(1.5064)^2 + (1.5064)^2} = 4.88725 = 6.8906$ dB

2-14. (a) $D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{U_{\max}}{U_0}$

$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} U \sin\theta \, d\theta \, d\phi = 2\pi \int_0^{\pi} U \sin\theta \, d\theta = 2\pi \left\{ \int_0^{30^\circ} \sin\theta \, d\theta \right.$$

$$\left. + \int_{30^\circ}^{60^\circ} (0.5) \sin\theta \, d\theta + \int_{60^\circ}^{90^\circ} (0.1) \sin\theta \, d\theta \right\}$$

$$= 2\pi \left\{ (-\cos\theta)\Big|_0^{30^\circ} + \left(-\frac{\cos\theta}{2}\right)\Big|_{30^\circ}^{60^\circ} + (-0.1\cos\theta)\Big|_{60^\circ}^{90^\circ} \right\}$$

$$= 2\pi \left\{ (-0.866 + 1) + \left(\frac{-0.5 + 0.866}{2}\right) + \left(\frac{-0 + 0.5}{10}\right) \right\}$$

$$P_{\text{rad}} = 2\pi\{-0.866 + 1 - 0.25 + 0.433 + 0.05\} = 2\pi(0.367)$$

$$= 0.734 \cdot \pi = 2.3059$$

$$D_0 = \frac{1(4\pi)}{2.3059} = 5.4496 = 7.3636 \text{ dB}$$

(b) D_0 (dipole) $= 1.5 = 1.761$ dB

D_0 (above dipole) $= (7.3636 - 1.761)$ dB $= 5.6026$ dB

D_0 (above dipole) $= \frac{5.45}{1.5} = 3.633 = 5.603$ dB

2-15. (a) $P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi) \sin\theta \, d\theta \, d\phi = \int_0^{2\pi} \sin^2\phi \, d\phi \cdot \int_0^{\pi/2} \cos^4\theta \sin\theta \, d\theta$

$$= (\pi)\left(\frac{1}{5}\right) = \frac{\pi}{5}.$$

$U_{\max} = U(\theta = 0^\circ, \phi = \pi/2) = 1.$

$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{4\pi}{(\pi/5)} = 20 = 13.0 \text{ dB}$$

(b) Elevation Plane: θ varies, ϕ fixed

$\rightarrow$ Choose $\phi = \pi/2$.

$U(\theta, \phi = \pi/2) = \cos^4\theta, \quad 0 \leqslant \theta \leqslant \pi/2.$

$\cos^4\left[\frac{\text{HPBW(el.)}}{2}\right] = \frac{1}{2}$

HPBW (el.) $= 2 \cdot \cos^{-1}\{\sqrt{0.5}\}^{1/2} = 65.5^\circ.$

2-16. (a) $P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi) \sin\theta \, d\theta \, d\phi = 2\pi$

$$\cdot \left\{ \int_0^{30^\circ} \sin\theta \, d\theta + \int_{30^\circ}^{90^\circ} \frac{\cos\theta \cdot \sin\theta}{0.866} \, d\theta \right\}$$

$$= 2\pi\left\{\int_0^{\pi/6} \sin\theta\, d\theta + \int_{\pi/6}^{\pi/2} \frac{1}{0.866}\cos\theta\cdot\sin\theta\, d\theta\right\}$$

$$= 2\pi\left\{-\cos\theta|_0^{\pi/6} + \frac{1}{0.866}\left(-\frac{\cos^2\theta}{2}\right)\Big|_{\pi/6}^{\pi/2}\right\} = 2\pi[-0.866 + 1 + 0.433]$$

$$= 3.5626$$

$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{4\pi(1)}{3.5626} = 3.5273 = 5.4745 \text{ dB}$$

(b) $U = \dfrac{\cos(\theta)}{0.866} = 0.5 \Rightarrow \cos\theta = 0.5(0.866) = 0.433, \theta = \cos^{-1}(0.433) = 64.34°$

$$\Theta_{1r} = 2(64.34) = 128.68° = 2.246 \text{ rad} = \Theta_{2r}$$

$$D_0 \simeq \frac{4\pi}{\Theta_{1r}\Theta_{2r}} = \frac{4\pi}{(2.246)^2} = 2.4912 = 3.9641 \text{ dB}$$

2-17. a. 35 dB

b. $20\log_{10}\left|\dfrac{E_{\max}}{E_s}\right| = 35, \log_{10}\left|\dfrac{E_{\max}}{E_s}\right| = \dfrac{35}{20} = 1.75$

$$\left|\frac{E_{\max}}{E_s}\right| = 10^{1.75} = 56.234$$

2-18. a. $U = \sin\theta, U_{\max} = 1, \quad P_{\text{rad}} = \displaystyle\int_0^{2\pi}\int_0^{\pi} U \sin\theta\, d\theta\, d\phi$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^2\theta\, d\theta\, d\phi = \pi^2$$

$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{4\pi}{\pi^2} = \frac{4}{\pi} = 1.2732$$

b. HPBW $= 120°, 2\pi/3$

The directivity based on (2–33a) is equal to.

$$D_0 = \frac{101}{120° - 0.0027(120°)^2} = 1.2451$$

while that based on (2–33b) is equal to,

$$D_0 = -172.4 + 191\sqrt{0.818 + \frac{1}{120°}} = 1.2245$$

c. Computer Program $D_0 = 1.2732$

2-19. a. $U = \sin^3\theta, U_{\max} = 1, P_{\text{rad}} = \displaystyle\int_0^{2\pi}\int_0^{\pi} \sin^4\theta\, d\theta\, d\phi = \frac{3}{4}\pi^2,$

$$D_0 = \frac{4\pi}{\frac{3}{4}\pi^2} = \frac{16}{3\pi} = 1.6976$$

b. HPBW = 74.93°

From (2–33a), $D_0 = \dfrac{101}{(74.93°) - 0.0027(74.93°)^2} = 1.68971$

From (2–33b), $D_0 = -172.4 + 191\sqrt{0.818 + \dfrac{1}{74.93°}} = 1.75029$

c. Computer program $D_0 = 1.69766$

The value of $D_0 (= 1.6976)$ is similar to that of (4–91) or 1.643

2-20. a. $U = J_1{}^2(\text{ka}\sin\theta)$,

$a = \lambda/10, \text{ka}\sin\theta = \dfrac{\pi}{5}\sin\theta.\quad \text{HPBW} = 93.10°$

From (2–33a) $D_0 = 101/[(93.10) - 0.0027(93.10)^2] = 1.449120$

From (2–33b) $D_0 = -172.4 + 191\sqrt{0.818 + \dfrac{1}{93.10}} = 1.477271$

$$a = \lambda/20, \text{ka}\sin\theta = \frac{\pi}{10}\sin\theta, \quad \text{HPBW} = 91.10°.$$

From (2-33a), $D_0 = 1.47033$, From (2-33b), $D_0 = 1.502$

b. $a = \dfrac{\lambda}{10}.\quad P_{\text{rad}} = \displaystyle\int_0^{2\pi}\int_0^{\pi} J_1^2(\text{ka}\sin\theta)\cdot\sin\theta\, d\theta\, d\phi = 0.7638045$

$U_{\max} = 0.0893,\ D_0 = \dfrac{4\pi(0.0893)}{0.7638045} = 1.469193$

$a = \dfrac{\lambda}{20},\quad P_{\text{rad}} = \displaystyle\int_0^{2\pi}\int_0^{\pi} J_1{}^2(\pi/10\cdot\sin\theta)\cdot\sin\theta\, d\theta\, d\phi = 0.202604$

$U_{\max} = 0.0240714,\quad D_0 = \dfrac{4\pi(0.0240714)}{0.202604} = 1.49257.$

If the radius of loop is smaller than $\lambda/20$, the directivity approaches to 1.5.

2-21. Using the numerical techniques, the directivity for each intensity of (Prob. 2-11) with 10° uniform divisions is equal to $U = \sin\theta\cdot\sin\phi$;

(a) Midpoint; $D_0 = \dfrac{4\pi U_{\max}}{P_{\text{rad}}}$

$$U_{\max} = 1.\ P_{\text{rad}} = \frac{\pi}{18}\left(\frac{\pi}{18}\right)\sum_{j=1}^{18}\sin\phi_j\sum_{i=1}^{18}\sin^2\theta_i$$

$$\theta_i = \frac{\pi}{36} + (i-1)\frac{\pi}{18}, \quad i = 1, 2, 3, \ldots, 18$$

$$\phi_j = \frac{\pi}{36} + (j-1)\frac{\pi}{18}, \quad j = 1, 2, 3, \ldots, 18$$

$$P_{\text{rad}} = \left(\frac{\pi}{18}\right)^2(11.38656)(8.9924) = 3.119$$

$$D_0 = \frac{4\pi(1)}{3.119} = 4.03 = 6.05\text{ dB}$$

(b) Trailing edge of each division

$$\text{Trailing edge; } \theta_i = i(\pi/18),\ i = 1,2,3,\ldots,18$$
$$\phi_j = j(\pi/18),\ j = 1,2,3,\ldots,18$$
$$P_{\text{rad}} = \left(\frac{\pi}{18}\right)^2 (11.25640)(8.96985) = 3.076$$
$$D_0 = \frac{4\pi(1)}{3.119} = 4.09 = 6.11 \text{ dB}$$

In a similar manner

$U = \sin\theta \cdot \sin^2\phi$;

(a) $P_{\text{rad}} = 2.463 \Rightarrow D_0 = 5.10 = 7.07$ dB
(b) $P_{\text{rad}} = 2.451 \Rightarrow D_0 = 5.13 = 7.10$ dB

$U = \sin\theta \cdot \sin^3\phi$;

(a) $P_{\text{rad}} = 2.092 \Rightarrow D_0 = 6.01 = 7.79$ dB
(b) $P_{\text{rad}} = 2.086 \Rightarrow D_0 = 6.02 = 7.80$ dB

$U = \sin^2\theta \cdot \sin\phi$;

(a) $P_{\text{rad}} = 2.469 \Rightarrow D_0 = 4.74 = 6.76$ dB
(b) $P_{\text{rad}} = 2.618 \Rightarrow D_0 = 4.80 = 6.81$ dB

$U = \sin^2\theta \cdot \sin^2\phi$;

(a) $P_{\text{rad}} = 2.092 \Rightarrow D_0 = 6.01 = 7.79$ dB
(b) $P_{\text{rad}} = 2.086 \Rightarrow D_0 = 6.02 = 7.80$ dB

$U = \sin^2\theta \cdot \sin^3\phi$;

(a) $P_{\text{rad}} = 1.777 \Rightarrow D_0 = 7.07 = 8.49$ dB
(b) $P_{\text{rad}} = 1.775 \Rightarrow D_0 = 7.08 = 8.50$ dB

2-22. Using the computer program Directivity of Chapter 2, the directivities for each radiation intensity of Problem 2-11 are equal to

a. $U = \sin\theta \sin\phi$; $P_{\text{rad}} = 3.1318$

$$U_{\max} = 1; \quad D_0 = \frac{4\pi \cdot U_{\max}}{3.1318} = 4.0125 \Rightarrow 6.034 \text{ dB}$$

b. $U = \sin\theta \cdot \sin^2\phi$; $P_{\text{rad}} = 2.4590$

$$U_{\max} = 1; \quad D_0 = \frac{4\pi \cdot 1}{2.4590} = 5.110358 \Rightarrow 7.0845 \text{ dB}$$

c. $U = \sin\theta \cdot \sin^3\phi$; $P_{\text{rad}} = 2.0870$

$$U_{\max} = 1; \quad D_0 = \frac{4\pi \cdot 1}{2.0870} = 6.02124 \Rightarrow 7.80 \text{ dB}$$

d. $U = \sin^2\theta \sin\phi$; $P_{\text{rad}} = 2.6579$

$$U_{\max} = 1; \quad D_0 = \frac{4\pi \cdot 1}{2.6579} = 4.72793 \Rightarrow 6.746 \text{ dB}$$

e. $U = \sin^2\theta \cdot \sin^2\phi$; $P_{\text{rad}} = 2.0870$

$$D_0 = \frac{4\pi \cdot 1}{2.0870} = 6.02126 \Rightarrow 7.7968 \text{ dB}$$

f. $U = \sin^2\theta \cdot \sin^3\phi$; $P_{\text{rad}} = 1.7714$

$$D_0 = \frac{4\pi \cdot 1}{1.7714} = 7.09403 \Rightarrow 8.5089 \text{ dB}$$

2-23. (a) $E|_{\max} = \cos\left[\frac{\pi}{4}(\cos\theta - 1)\right]|_{\max} = 1$ at $\theta = 0°$.

$$0.707E_{\max} = 0.707 \cdot (1) = \cos\left[\frac{\pi}{4}(\cos\theta_1 - 1)\right]$$

$$\frac{\pi}{4}(\cos\theta_1 - 1) = \pm\frac{\pi}{4} \Rightarrow \theta_1 = \begin{cases} \cos^{-1}(2) = \text{does not exist} \\ \cos^{-1}(0) = 90° = \frac{\pi}{2} \text{ rad.} \end{cases}$$

$$\Theta_{1r} = \Theta_{2r} = 2\left(\frac{\pi}{2}\right) = \pi$$

$$D_0 \simeq \frac{4\pi}{\Theta_{1r}\Theta_{2r}} = \frac{4\pi}{\pi^2} = \frac{4}{\pi} = 1.273 = 1.049 \text{ dB}$$

(b) Using the computer program of Chapter 2

$$D_0 = 2.00789 = 3.027 \text{ dB}$$

Since the pattern is not very narrow, the answer obtained using Kraus' approximate formula is not as accurate.

2-24. a. $E|_{\max} = \cos\left(\frac{\pi}{4}(\cos\theta + 1)\right)|_{\max} = 1$ at $\theta = \pi$.

$0.707 \cdot = \cos\left(\frac{\pi}{4}(\cos\theta_1 + 1)\right)$

$$\frac{\pi}{4}(\cos\theta_1 + 1) = \pm\frac{\pi}{4} \Rightarrow \theta_1 = \begin{cases} \cos^{-1}(-2) \rightarrow \text{does not exist.} \\ \cos^{-1}(0) \rightarrow 90° \rightarrow \frac{\pi}{2} \text{ rad} \end{cases}$$

$$\Theta_{1r} = \Theta_{2r} = 2\left(\frac{\pi}{2}\right) = \pi.$$

$$D_0 \simeq \frac{4\pi}{\pi^2} = \frac{4}{\pi} = 1.273 = 1.049 \text{ dB}$$

b. Computer Program

$D_0 = 2.0078$ $= 3.027$ dB

2-25. a. $P_{rad} = \int_0^{2\pi}\int_0^{\pi/2} U_0 \sin(\pi \sin\theta)\cdot \sin\theta \, d\theta \, d\phi = 2\pi \cdot U_0 \cdot \frac{\pi}{2} J_1(\pi) = U_0 \pi^2 J_1(\pi)$

$$D_0 = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi U_0}{U_0 \pi^2 J_1(\pi)} = \frac{4}{\pi}\cdot\frac{1}{J_1(\pi)} = 4.4735$$

$$\leftarrow \frac{\pi}{2} J_1(\pi) = 0.44707273561622$$

b. Computer Program

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi/2} U_0 \sin(\pi\sin\theta)\sin\theta \, d\theta \, d\phi = 2\pi\cdot(0.44707273561618)$$

$$D_0 = 4.4735$$

2-26. (a) Using the computer program of Chapter 2.

$$D_0 = 14.0707 = 11.48 \text{ dB}$$

(b) $U|_{max} = \left[\frac{\sin(\pi\sin\theta)}{\pi\sin\theta}\right]^2_{max} = 1 \quad \text{when } \theta = 0°.$

$$U = \tfrac{1}{2}U_{max} = \tfrac{1}{2}(1) = \left[\frac{\sin(\pi\sin\theta_1)}{\pi\sin\theta_1}\right]^2$$

Iteratively we obtain $\theta_1 = 26.3°$. Therefore

$$\Theta_{1d} = \Theta_{2d} = 2(26.3°) = 52.6°.$$

and $D_0 \simeq \frac{41{,}253}{(52.6)^2} = 14.91 = 11.73$ dB using the Kraus' formula

(c) For Tai and Pereira's formula

$$D_0 = \frac{72{,}815}{2\cdot\Theta_{1d}^2} = \frac{72{,}815}{2(52.6)^2} = 13.16 = 11.19 \text{ dB}$$

2-27. $U = \frac{1}{2\eta}|E|^2 = \frac{1}{2\eta}\sin\theta\cos^2\phi \Rightarrow U_{max} = \frac{1}{2\eta}$

(a) $P_{rad} = 2\cdot\int_0^{\pi/2}\int_0^{\pi}\frac{1}{2\eta}\sin^2\theta\cos^2\phi \, d\theta \, d\phi = \frac{1}{\eta}\left(\frac{\pi}{4}\right)\left(\frac{\pi}{2}\right) = \frac{\pi^2}{8\eta}$

$$D_0 = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi\left(\frac{1}{2\eta}\right)}{\frac{\pi^2}{8\eta}} = \frac{16}{\pi} = 5.09 = 7.07 \text{ dB}$$

(b) $U_{\max} = \dfrac{1}{2\eta}$ at $\theta = \pi/2, \phi = 0$

In the elevation plane through the maximum $\phi = 0$ and $U = \dfrac{1}{2\eta}\sin\theta$.
The 3-dB point occurs when

$$U = 0.5\ U_{\max} = 0.5\left(\frac{1}{2\eta}\right) = \frac{1}{2\eta}\sin\theta_1 \Rightarrow \theta_1 = \sin^{-1}(0.5) = 30^\circ$$

Therefore $\Theta_{1d} = 2(90 - 30) = 120^\circ$
In the azimuth plane through the maximum $\theta = \pi/2$ and $U = \dfrac{1}{2\eta}\cos^2\phi$.
The 3-dB point occurs when $U = 0.5\ U_{\max} = 0.5\left(\dfrac{1}{2\eta}\right) = \dfrac{1}{2\eta}\cos^2\theta_1$
$\Rightarrow \phi_1 = \cos^{-1}(0.707) = 45^\circ$, $\Theta_{2d} = 2(90^\circ - 45^\circ) = 90^\circ$.
Therefore using Kraus' formula $D_0 \simeq \dfrac{41{,}253}{120 \cdot (90)} = 3.82 = 5.82$ dB

(c) Using Tai and Pereira's formula

$$D_0 \simeq \frac{72{,}815}{\Theta_{1d}^2 + \Theta_{2d}^2} = \frac{72{,}815}{(120)^2 + (90)^2} = 3.24 = 5.10 \text{ dB}$$

(d) Using the computer program of Chapter 2.

$$D_0 = 5.16425 = 7.13 \text{ dB}$$

2-28. $\mathcal{U} = \left[\dfrac{J_1(\text{ka}\sin\theta)}{\sin\theta}\right]^2 = (\text{ka})^2\left[\dfrac{J_1(\text{ka}\sin\theta)}{\text{ka}\sin\theta}\right]^2 = \mathcal{U}_0\left[\dfrac{J_1(\text{ka}\sin\theta)}{\text{ka}\sin\theta}\right]^2$

(a) $\mathcal{U}_{\max} = \mathcal{U}_0\left(\frac{1}{2}\right)^2 = \dfrac{\mathcal{U}_0}{4}$ and it occurs when $\text{ka}\sin\theta = 0 \Rightarrow \theta = 0^\circ$.
The 3-dB point is obtained using

$$\mathcal{U} = \tfrac{1}{2}\,\mathcal{U}_{\max} = \frac{\mathcal{U}_0}{8} = \mathcal{U}_0\left[\frac{J_1(\text{ka}\sin\theta)}{\text{ka}\sin\theta}\right]^2 \Rightarrow \frac{J_1(\text{ka}\sin\theta)}{\text{ka}\sin\theta} = 0.3535$$

with the aid of the $J_1(x)/x$ tables of Appendix V.

$x = \text{ka}\sin\theta_1 = 1.61 \Rightarrow \theta_1 = \sin^{-1}(1.61/2\pi) = 14.847^\circ$
$\Rightarrow \Theta_{1r} = 29.694^\circ$

(b) Since $\Theta_{1r} = \Theta_{2r} = 29.694^\circ$, the directivity is equal to

$$D_0 \simeq \frac{41{,}253}{(29.694)^2} = 46.79 = 16.70 \text{ dB}$$

2-29. $G_0 = 16 \text{ dB} \Rightarrow 16 = 10\log_{10} G_0(\text{dimensionless}) \Rightarrow G_0(\text{dim}) = 10^{1.6} = 39.81$

$r = 100 \text{ meters} = 10{,}000 \text{ cm} = 10^4 \text{ cm}$

$P_{\text{rad}} = e_{cd}P_{\text{in}} = (1)P_{\text{in}} = 8 \text{ watts}$

$f = 1{,}900 \text{ MHz} \Rightarrow \lambda = 30 \times 10^9/1.9 \times 10^9 = 15.789 \text{ cm}$

(a) $W_0 = \dfrac{P_{\text{rad}}}{4\pi r^2} = \dfrac{8}{4\pi(10^4)^2} = \dfrac{8}{4\pi \times 10^8}$

$= \dfrac{2}{\pi} \times 10^{-8} = 0.6366 \times 10^{-8} \text{ watts/cm}^2$

$W_0 = 0.6366 \times 10^{-8} = 6.366 \times 10^{-9} \text{ watts/cm}^2$

$W_{\text{max}} = W_0 G_0(\text{dim}) = 6.366 \times 10^{-9}(39.81) = 253.438 \times 10^{-9}$

$$\boxed{W_{\text{max}} = 253.438 \times 10^{-9} \text{ watts/cm}^2}$$

(b) $D_0(\lambda/4 \text{ monopole}) = 1.643$

$A_{em} = \dfrac{\lambda^2}{4\pi}D_0 = \dfrac{\lambda^2}{4\pi}(1.643) = \dfrac{1.643(15.789)^2}{4\pi} = 32.5938 \text{ cm}^2$

$A_{em} = 32.5938 \text{ cm}^2$

$P(\text{received}) = W_{\text{max}}\, A_{em} = (253.438 \times 10^{-9})(32.5938)$

$$\boxed{P(\text{received}) = 8.2606 \times 10^{-6} \text{ watts}}$$

2-30. (a) Linear because $\Delta\phi = 0$.

(b) Linear because $\Delta\phi = 0$.

(c) Circular because

1. $E_x = E_y$
2. $\Delta\phi = \pi/2$.

CCW because E_y leads E_x. $\text{AR} = 1, \tau = 90°$

(d) Circular because

1. $E_x = E_y$
2. $\Delta\phi = -\pi/2$

CW because E_y lags E_x. $\text{AR} = 1, \tau = 90°$

(e) Elliptical because $\Delta\phi$ is not multiples of $\pi/2$. CCW because E_y leads E_x.
AR = OA/OB

Letting $E_x = E_y = E_0$

$$\left.\begin{array}{l} \mathrm{OA} = E_0[0.5(1+1+\sqrt{2})]^{1/2} = 1.30656E_0 \\ \mathrm{OB} = E_0[0.5(1+1-\sqrt{2})]^{1/2} = 0.541196E_0 \end{array}\right\} \Rightarrow \mathrm{AR} = \frac{1.30656}{0.541196} = 2.414$$

$$\tau = 90^\circ - \tfrac{1}{2}\tan^{-1}\left[\frac{2(1)\cos(45^\circ)}{1-1}\right] = 90^\circ - \tfrac{1}{2}\tan^{-1}\left(\frac{1.414}{0}\right)$$

$$= 90^\circ - \tfrac{1}{2}(90^\circ) = 45^\circ$$

(f) Elliptical because $\Delta\phi$ is not multiples of $\pi/2$ CW because E_y lags E_x.

$$\text{From above}\quad \left.\begin{array}{l} \mathrm{OA} = 1.30656E_0 \\ \mathrm{OB} = 0.541196E_0 \end{array}\right\} \Rightarrow \mathrm{AR} = \frac{1.30656}{0.541196} = 2.414$$

From above $\tau = 90^\circ - \frac{1}{2}(90^\circ) = 45^\circ$

(g) Elliptical because

1. $E_x \neq E_y$
2. $\Delta\phi$ is not zero or multiples of π.

CCW because E_y leads E_x.

$$\left.\begin{array}{l} \mathrm{OA} = E_y\left\{\tfrac{1}{2}[0.25+1+0.75]\right\}^{1/2} = E_y \\ \mathrm{OB} = E_y\left\{\tfrac{1}{2}[0.25+1-0.75]\right\}^{1/2} = 0.5E_y \end{array}\right\} \Rightarrow \mathrm{AR} = \frac{1}{0.5} = 2.$$

$$\tau = 90^\circ - \tfrac{1}{2}\tan^{-1}\left(\frac{0}{-0.75}\right) = 90^\circ - \tfrac{1}{2}(180^\circ) = 0^\circ$$

(h) Elliptical because

1. $E_x \neq E_y$
2. $\Delta\phi$ is not zero or multiples of π.

CW because E_y lags E_x.

$$\text{From above}\quad \left.\begin{array}{l} \mathrm{OA} = E_y \\ \mathrm{OB} = 0.5E_y \end{array}\right\} \Rightarrow \mathrm{AR} = \frac{1}{0.5} = 2$$

$$\tau = 90^\circ - \tfrac{1}{2}(180^\circ) = 0^\circ.$$

2-31. $\mathcal{E}_x(z,t) = \mathrm{Re}[E_x e^{j(\omega t + kz + \phi_x)}] = E_x\cos(\omega t + kz + \phi_x)$

$\mathcal{E}_y(z,t) = \mathrm{Re}[E_y e^{j(\omega t + kz + \phi_y)}] = E_y\cos(\omega t + kz + \phi_y)$

where E_x and E_y are real positive constants.

Choosing $z = 0$ and letting $\Delta\phi = \phi_y - \phi_x = \phi_y - 0 = \phi$

$$\begin{array}{l} \mathcal{E}_x(t) = E_x\cos(\omega t) \\ \mathcal{E}_y(t) = E_y\cos(\omega t + \phi) \end{array} \qquad (1)$$

and

$$\mathcal{E}(t) = \sqrt{\mathcal{E}_x{}^2 + \mathcal{E}_y{}^2} = \sqrt{E_x{}^2 \cos^2(\omega t) + E_y{}^2 \cos^2(\omega t + \phi)} \tag{2}$$

The maximum and minimum values of (2) are the major and minor axes of the polarization ellipse. Squaring (2) and using the half-angle identity, equation (2) can be written as

$$\mathcal{E}^2(t) = \tfrac{1}{2}\{E_x{}^2 + E_y{}^2 + E_x{}^2\cos(2\omega t) + E_y{}^2\cos^2[2(\omega t + \phi)]\} \tag{3}$$

Since E_x and E_y are constants, the maximum and minimum values of (3) occur when $f(t) = E_x{}^2\cos(2\omega t) + E_y{}^2\cos[2(\omega t + \phi)]$ is maximum or minimum. These are found by differentiating (4) and setting it equal to zero. Thus

$$\frac{df}{d(2\omega t)} = -E_x{}^2\sin(2\omega t) - E_y{}^2\sin[2(\omega t + \phi)] = 0 \tag{4}$$

or

$$\begin{aligned} E_x^2\sin(2\omega t) &= -E_y^2\sin[2(\omega t + \phi)] \\ &= -E_y^2\{\sin 2\omega t\cos 2\phi + \cos 2\omega t\sin 2\phi\} \end{aligned} \tag{5}$$

Dividing (5) by $\cos(2\omega t)$ yields

$$E_x^2\tan(2\omega t) = -E_y^2\tan[2\omega t]\cos(2\phi) + \sin(2\phi)]$$

or

$$\tan(2\omega t) = \frac{-E_y^2\sin(2\phi)}{E_x^2 + E_y^2\cos(2\phi)}$$

from which we obtain that

$$\cos(2\omega t) = \frac{E_x^2 + E_y^2\cos(2\phi)}{\pm\rho} \tag{6}$$

$$\cos(2\omega t + 2\phi) = \frac{E_y^2 + E_x^2\cos(2\phi)}{\pm\rho} \tag{7}$$

where

$$\rho = \sqrt{E_x^4 + E_y^4 + 2E_x^2E_y^2\cos(2\phi)} \tag{8}$$

Substituting (6)–(8) into (3) yields

$$\mathcal{E}^2 = \tfrac{1}{2}\left[E_x^2 + E_y^2 \pm \frac{1}{\rho}(\rho^2)\right]$$

whose maximum value is

$$\mathcal{E}_{\text{max}} = \text{OA} = \left\{\tfrac{1}{2}[E_x^2 + E_y^2 + (E_x^4 + E_y^4 + 2E_x^2E_y^2\cos 2\phi)^{1/2}]\right\}^{1/2}$$

$$\mathcal{E}_{\text{min}} = \text{OB} = \left\{\tfrac{1}{2}[E_x^2 + E_y^2 - (E_x^4 + E_y^4 + 2E_x^2E_y^2\cos 2\phi)^{1/2}]\right\}^{1/2}$$

The tilt angle τ can be obtained by expanding (1) and writing the two as

$$\frac{\mathcal{E}_x^2}{E_x^2} - \frac{2\mathcal{E}_x\mathcal{E}_y\cos\phi}{E_x\,E_y} + \frac{\mathcal{E}_y^2}{E_y^2} = \sin^2\phi \tag{9}$$

which is the equation of a tilted ellipse. Choosing a coordinate system whose principal axes coincide with the major and minor axes of the tilted ellipse, we can write that

$$\begin{aligned} \mathcal{E}_x &= \mathcal{E}'_x\sin(z) - \mathcal{E}'_y\cos(z) \\ \mathcal{E}_y &= \mathcal{E}'_x\cos(z) + \mathcal{E}'_y\sin(z) \end{aligned} \tag{10}$$

where $\mathcal{E}'_x$ and $\mathcal{E}'_y$ are the new field values along the new principal axes x', y', z'. Substituting (10) into (9) yields

$$\frac{2\mathcal{E}'_x\mathcal{E}'_y\cos(z)\sin(z)}{E_x^2} - \frac{2\mathcal{E}'_x\mathcal{E}'_y\cos(z)\sin(z)}{E_y^2} - \frac{2\mathcal{E}'_x\mathcal{E}'_y\cos\phi}{E_x\,E_y}(\sin^2 z - \cos^2 z) = 0$$

which when solved for the tilt angle τ reduces to

$$\tan\left[2\left(\frac{\pi}{2} - \tau\right)\right] = \frac{2E_x\,E_y\cos\phi}{E_x^2 - E_y^2}$$

or

$$\tau = \frac{\pi}{2} - \tfrac{1}{2}\tan^{-1}\left(\frac{2E_x\,E_y\cos\phi}{E_x^2 - E_y^2}\right)$$

For more details on the tilt angle derivation, see J.D. Kraus, Antennas, McGraw - Hill, 1950, pp. 464–476.

2-32. (a)

$$\begin{aligned} \hat{\rho}_w &= \hat{a}_x\cos\phi_1 + \hat{a}_y\sin\phi_1 \\ \hat{\rho}_a &= \hat{a}_x\cos\phi_2 + \hat{a}_y\sin\phi_2 \\ \text{PLF} &= |\hat{\rho}_w\cdot\hat{\rho}_a|^2 = |(\hat{a}_x\cos\phi_1 + \hat{a}_y\sin\phi_1)\cdot(\hat{a}_x\cos\phi_2 + \hat{a}_y\sin\phi_2)|^2 \\ &= |\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2|^2 = |\cos(\phi_1 - \phi_2)|^2 \end{aligned}$$

(b)

$$\begin{aligned} \hat{\rho}_w &= \hat{a}_x\sin\theta_1\cos\phi_1 + \hat{a}_y\sin\theta_1\sin\phi_1 + \hat{a}_z\cos\theta_1 \\ \hat{\rho}_a &= \hat{a}_x\sin\theta_2\cos\phi_2 + \hat{a}_y\sin\theta_2\sin\phi_2 + \hat{a}_z\cos\theta_2 \\ \text{PLF} &= |\hat{\rho}_w\cdot\hat{\rho}_a|^2 = |\sin\theta_1\cos\phi_1\sin\theta_2\cos\phi_2 + \sin\theta_1\sin\phi_1\sin\theta_2\cdot\sin\phi_2 \\ &\qquad + \cos\theta_1\cdot\cos\theta_2|^2 \\ \text{PLF} &= |\sin\theta_1\cdot\sin\theta_2(\cos\phi_1\cdot\cos\phi_2 + \sin\phi_1\sin\phi_2) + \cos\theta_1\cos\theta_2|^2 \\ \text{PLF} &= |\sin\theta_1\sin\theta_2\cos(\phi_1 - \phi_2) + \cos\theta_1\cos\theta_2|^2 \end{aligned}$$

2-33. Assuming electric field is x-polarized

(a) $\underline{E}_w = \hat{a}_x E_1 e^{-jkz} \Rightarrow \hat{\rho}_w = \hat{a}_x$

$$\underline{E}_a = (\hat{a}_\theta - j\hat{a}_\phi)E_0 f(r, \theta, \phi) \Rightarrow \hat{\rho}_a = \left(\frac{\hat{a}_\theta - j\hat{a}_\phi}{\sqrt{2}}\right)$$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \tfrac{1}{2}|\hat{a}_x \cdot \hat{a}_\theta - j\hat{a}_x \cdot \hat{a}_\phi|^2$$

since $\hat{a}_\theta = \hat{a}_x \cos\theta\cos\phi + \hat{a}_y \cos\theta\sin\phi - \hat{a}_z \sin\theta$

$$\hat{a}_\phi = -\hat{a}_x \sin\phi + \hat{a}_y \cos\phi$$

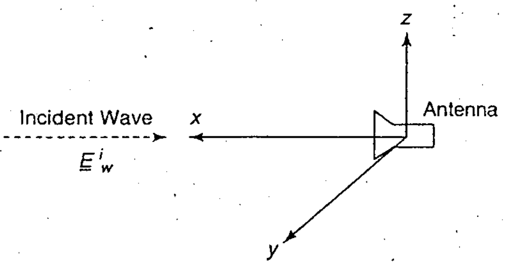
$$\text{PLF} = \tfrac{1}{2}(\cos^2\theta\cos^2\phi + \sin^2\phi)$$

(b) when $\underline{E}_a = (\hat{a}_\theta + j\hat{a}_\theta)E_0 f(r, \theta, \phi)$, PLF is also

$$\text{PLF} = \tfrac{1}{2}(\cos^2\theta\cos^2\phi + \sin^2\phi)$$

A more general, but also more complex, expression can be derived when the incident electric field is of the form $\underline{E}_w = (a\hat{a}_x + b\hat{a}_y)e^{-jkz}$ where a, b are real constants. It can be shown (using the same procedure) that

$$\text{PLF} = \frac{1}{\sqrt{2(a^2+b^2)}}[(a\cos\theta\cos\phi + b\sin\theta\sin\phi)^2 + (a\sin\phi - b\cos\phi)^2]^{1/2}$$



2-34. (a) $\underline{E}_w = E_0(j\hat{a}_y + 3\hat{a}_z)e^{+jkx}$

1. **Elliptical polarization; $AR = \frac{3}{1} = 3$; Left Hand (CCW)**
   a. 2 components orthogonal to direction of propagation
   b. Not of same magnitude
   c. 90° phase difference between them
   d. y component is leading the z component *or* z component is lagging the y component

(b) $\underline{E}_a = E_a(\hat{a}_y + 2\hat{a}_z)e^{-jkx}$

1. **Linear polarization; $AR = \infty$; No rotation**
   a. 2 components orthogonal to direction of propagation.
   b. Not of the same magnitude
   c. 0° phase difference between them,

(c) PLF $= |\hat{\rho}_w \cdot \hat{\rho}_a|^2$

$$\underline{E}_w = E_0(j\hat{a}_y + 3\hat{a}_z)e^{+jkx} = E_0 \underbrace{\left(\frac{j\hat{a}_y + 3\hat{a}_z}{\sqrt{10}}\right)}_{\hat{\rho}_w} \sqrt{10}e^{+jkx}$$

$$\hat{\rho}_w = \left(\frac{j\hat{a}_y + 3\hat{a}_z}{\sqrt{10}}\right)$$

$$\underline{E}_a = E_a(\hat{a}_y + 2\hat{a}_z)e^{-jkx} = E_0 \underbrace{\left(\frac{\hat{a}_y + 2\hat{a}_z}{\sqrt{5}}\right)}_{\hat{\rho}_a} \sqrt{5}e^{-jkx}$$

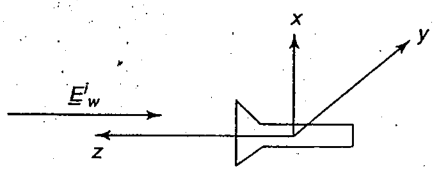
$$\hat{\rho}_a = \left(\frac{\hat{a}_y + 2\hat{a}_z}{\sqrt{5}}\right)$$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\frac{(j\hat{a}_y + 3\hat{a}_z)}{\sqrt{10}} \cdot \frac{(\hat{a}_y + 2\hat{a}_z)}{\sqrt{5}}\right|^2 = \frac{|j+6|^2}{50} = \frac{37}{50}$$

$$\text{PLF} = \frac{37}{50} = \boxed{0.740 = -1.31 \text{ dB}}$$

2-35. $\underline{E}_w^i = (\hat{a}_x + j\hat{a}_y)E_0e^{+jkz}$

$$\underline{E}_a = (\hat{a}_x + 2\hat{a}_y)E_1\frac{e^{-jkr}}{r}\Big|_{\theta=0^\circ\ z-\text{axis}} = (\hat{a}_x + 2\hat{a}_y)E_1\frac{e^{-jkz}}{z}$$



(a) $\underline{E}_w^i = \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)\sqrt{2}E_0e^{+jkz}$

Circular: 2 components, same amplitude, 90° phase difference

(b) Clockwise (y component is leading the x component)

(c) $\underline{E}_a = \left(\frac{\hat{a}_x + 2\hat{a}_y}{\sqrt{5}}\right)\sqrt{5}E_1\frac{e^{-jkz}}{z}$

Linear: 2 components, 0° phase difference

(d) No rotation

(e) $\hat{\rho}_w = \left(\dfrac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right), \quad \hat{\rho}_a = \left(\dfrac{\hat{a}_x + 2\hat{a}_y}{\sqrt{5}}\right)$

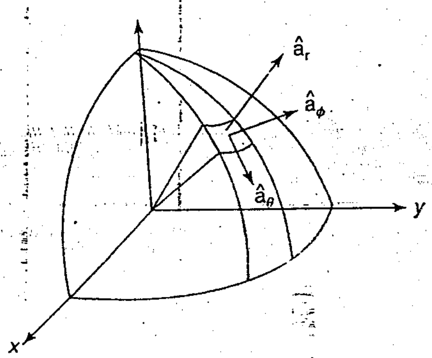
$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left[\left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right) \cdot \left(\frac{\hat{a}_x + 2\hat{a}_y}{\sqrt{5}}\right)\right]^2 = \frac{|1+j2|^2}{10} = \frac{5}{10}$$

$$\text{PLF} = \frac{5}{10} = 0.5 = 10\log_{10}(0.5) = -3 \text{ dB}$$

2-36. (a) $\underline{E}_a = E_0(j\hat{a}_\theta + 2\hat{a}_\phi) f_0(\theta_0, \phi_0)\dfrac{e^{-jkr}}{r} = E_0 \underbrace{\left(\dfrac{j\hat{a}_\theta + 2\hat{a}_\phi}{\sqrt{5}}\right)}_{\hat{\rho}_a} \sqrt{5} f_0(\theta_0, \phi_0)\dfrac{e^{-jkr}}{r}$

$$\hat{\rho}_a = \left(\frac{j\hat{a}_\theta + 2\hat{a}_\phi}{\sqrt{5}}\right)$$

Elliptical, CW



(b) $\underline{E}_w = E_1(2\hat{a}_\theta + j\hat{a}_\phi) f_1(\theta_0, \phi_0)\dfrac{e^{+jkr}}{r}$

$$= E_1 \underbrace{\left(\frac{2\hat{a}_\theta + j\hat{a}_\phi}{\sqrt{5}}\right)}_{\hat{\rho}_w} \sqrt{5} f_1(\theta_0, \phi_0)\frac{e^{+jkr}}{r}$$

$$\hat{\rho}_w = \left(\frac{2\hat{a}_\theta + j\hat{a}_\phi}{\sqrt{5}}\right)$$

Elliptical, CW

(c) $\text{PLF} = |\hat{\rho}_a \cdot \hat{\rho}_w|^2 = \left|\left(\dfrac{j\hat{a}_\theta + 2\hat{a}_\phi}{\sqrt{5}}\right) \cdot \left(\dfrac{2\hat{a}_\theta + j\hat{a}_\phi}{\sqrt{5}}\right)\right|^2 = \left|\dfrac{2j + j2}{\sqrt{25}}\right|^2 = \left|\dfrac{4j}{\sqrt{25}}\right|^2$

$$\boxed{\text{PLF} = \frac{16}{25} = 0.64 = -1.938 \text{ dB}}$$

2-37. (a) $\underline{E}_w = E_0(\hat{a}_x \pm j\hat{a}_y)e^{-jkz} \Rightarrow \hat{\rho}_w = \frac{1}{\sqrt{2}}(\hat{a}_x \pm j\hat{a}_y)$

$\underline{E}_a \simeq E_1(\hat{a}_\theta - j\hat{a}_\phi)f(r,\theta,\phi) \Rightarrow \hat{\rho}_a = \frac{1}{\sqrt{2}}(\hat{a}_\theta - j\hat{a}_\phi)$

$\text{PLF} = \frac{1}{2}|(\hat{a}_x \pm j\hat{a}_y)\cdot(\hat{a}_\theta - j\hat{a}_\phi)|^2 = \frac{1}{2}|(\hat{a}_x\cdot\hat{a}_\theta \pm \hat{a}_y\cdot\hat{a}_\phi) - j(\hat{a}_x\hat{a}_\phi \mp \hat{a}_y\hat{a}_\theta)|^2$

Converting the spherical unit vectors to rectangular, as it was done in Problem 2.32, leads to

$$\text{PLF} = \frac{1}{2}(\cos\theta \pm 1)^2$$

(b) When

$\underline{E}_w = E_0(\hat{a}_x \pm j\hat{a}_y)e^{-jkz}$

$\underline{E}_a \simeq E_1(\hat{a}_\theta + j\hat{a}_\phi)f(r,\theta,\phi)$ the PLF is equal to

$\text{PLF} = \frac{1}{2}(\cos\theta \mp 1)^2$

2-38. $\underline{E}_w = (\hat{a}_\theta\cos\phi - \hat{a}_\phi\sin\phi\cos\theta)f(r,\theta,\phi)$ or

$$\underline{E}_w = \left[\frac{\hat{a}_\theta\cos\phi - \hat{a}_\phi\sin\phi\cos\theta}{\sqrt{\cos^2\phi + \sin^2\phi\cos^2\theta}}\right]\sqrt{\cos^2\phi + \sin^2\phi\cos^2\theta}\cdot f(r,\theta,\phi)$$

Thus $\hat{\rho}_w = \dfrac{\hat{a}_\theta\cos\phi - \hat{a}_\phi\sin\phi\cos\theta}{\sqrt{\cos^2\phi + \sin^2\phi\cos^2\theta}}$

and

$$\text{PLF} = |\hat{\rho}_w\cdot\hat{\rho}_a|^2 = \left|\left(\frac{\hat{a}_\theta\cos\phi - \hat{a}_\phi\sin\phi\cos\theta}{\sqrt{\cos^2\phi + \sin^2\phi\cos^2\theta}}\right)\cdot\hat{a}_x\right|^2$$

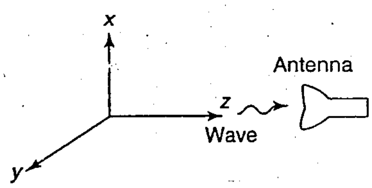
Transforming the rectangular unit vector to spherical using

$$\hat{a}_x = \hat{a}_r\sin\theta\cos\phi + \hat{a}_\theta\cos\theta\cos\phi - \hat{a}_\phi\sin\phi$$

the PLF reduces to $\text{PLF} = \dfrac{\cos^2\theta}{\cos^2\phi + \sin^2\phi\cos^2\theta}$

The same answer is obtained by transforming the spherical unit vectors to rectangular, as was done in Prob. 2-32.

2-39. $\underline{E}_a \simeq (2\hat{a}_x \pm j\hat{a}_y)f(r,\theta,\phi) = \left(\dfrac{2\hat{a}_x \pm j\hat{a}_y}{\sqrt{5}}\right)\sqrt{5}f(r,\theta,\phi)$



(a) $\hat{\rho}_w = \left(\dfrac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}\right) \Rightarrow$ Wave is Right Hand (RH)

$$\hat{\rho}_a = \left(\frac{2\hat{a}_x \pm j\hat{a}_y}{\sqrt{5}}\right)$$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2$$

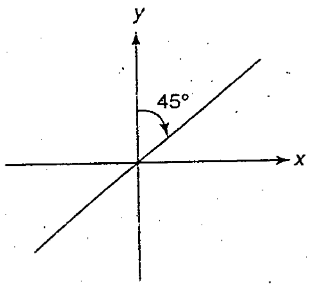
$$= \begin{cases} \dfrac{9}{10} = -0.4576 \text{ dB using the} + \text{sign} & \text{(Antenna is LH in receiving mode and RH in transmitting)} \\ \dfrac{1}{10} = -10 \text{ dB using the} - \text{sign} & \text{(Antenna is RH in receiving mode and LH in transmitting)} \end{cases}$$

(b) $\hat{\rho}_w = \left(\dfrac{\hat{a}x + j\hat{a}y}{\sqrt{2}}\right) \Rightarrow$ Wave is Left Hand (LH)

$$\hat{\rho}_a = \left(\frac{2\hat{a}_x \pm j\hat{a}_y}{\sqrt{5}}\right)$$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2$$

$$= \begin{cases} \dfrac{1}{10} = -10 \text{ dB using the} + \text{sign} & \text{(Antenna is LH in receiving mode and RH in transmitting)} \\ \dfrac{9}{10} = -0.4545 \text{ dB using the} - \text{sign} & \text{(Antenna is RH in receiving mode and LH in transmitting)} \end{cases}$$



2-40. for $\hat{\rho}_w$

$$\hat{\rho}_w = \frac{\hat{a}_x + \hat{a}_y}{\sqrt{2}}; \text{PLF} = \left|\frac{\hat{a}_x + \hat{a}_y}{\sqrt{2}} \cdot \frac{4\hat{a}_x + j\hat{a}_y}{\sqrt{17}}\right|^2$$

$$\text{PLF} = \frac{1}{34}|(\hat{a}_x \cdot 4\hat{a}_x) + (\hat{a}_y \cdot j\hat{a}_y)|^2 = \frac{1}{34}|4 + j|^2$$

$$= 0.5$$

2-41. (a) RHCP; $\hat{\rho}_a = \dfrac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}$

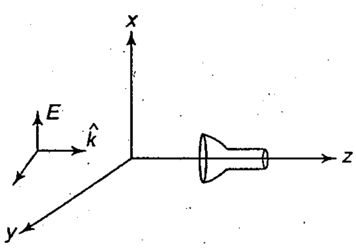
$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\frac{2\hat{a}_x + j\hat{a}_y}{\sqrt{5}} \cdot \frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}\right|^2 = 0.9 = -0.46 \text{ (dB)}$$

(b) LHCP; $\hat{\rho}_a = \dfrac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\frac{2\hat{a}_x + j\hat{a}_y}{\sqrt{5}} \cdot \frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right|^2 = 0.1 = -10.0 \text{ (dB)}$$

2-42. $\underline{E}^i = (\hat{a}_x - j\hat{a}_y)E_0 e^{-jkz} = \left(\dfrac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}\right)\sqrt{2}E_0 e^{-jkz}$

$$\hat{\rho}_w = \frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}} \qquad \text{CW}$$



(a) $\underline{E}^a = (\hat{a}_x + j\hat{a}_y)E_1 e^{+jkz}$

$$= \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)\sqrt{2}E_1 e^{+jkz}$$

$$\hat{\rho}_a = \frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}} \qquad \text{CW}$$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\left(\frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}\right) \cdot \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)\right|^2 = \left(\frac{1 - j^2}{2}\right)^2 = 1$$

PLF = 1 = 0 dB

(b) $\underline{E}^a = \left(\dfrac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}\right)\sqrt{2}E_1 e^{+jkz}$

$$\hat{\rho}_a = \frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}$$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\left(\frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}\right) \cdot \left(\frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}\right)\right|^2 = \left|\frac{1 + j^2}{2}\right|^2 = 0$$

PLF = 0 = $-\infty$ dB

2-43. $\underline{E}^i = \hat{a}_x E_0 e^{-jkz}, \hat{\rho}_w = \hat{a}_x$

$$\underline{E}^a = (\hat{a}_x + j\hat{a}_y)E_1 e^{+jkz} = \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)\sqrt{2}E_1 e^{+jkz}$$

$$\hat{\rho}_a = \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)$$

(a) $A_{em} = \frac{\lambda^2}{4\pi} e_o D_0 |\hat{\rho}_a \cdot \hat{\rho}_w|^2 = \frac{\lambda^2}{4\pi} G_0 |\hat{\rho}_a \cdot \hat{\rho}_w|^2$

$(\leftarrow e_o D_0 = G_0)$

At 10 GHz $\Rightarrow \lambda = \frac{c}{f} = \frac{3 \times 10^8}{10 \times 10^9} = \frac{3 \times 10^8}{10^{10}} = 3 \times 10^{-2}$

$$G_0 = 10 = 10 \log_{10} G_0(\text{dim}) \Rightarrow G_0(\text{dim}) = 10^1 = 10$$

$$A_{em} = \frac{\lambda^2}{4\pi} G_0 |\hat{\rho}_a \cdot \hat{\rho}_w|^2 = \frac{(3 \times 10^{-2})^2}{4\pi}(10)\left|\hat{a}_x \cdot \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)\right|^2$$

$$= \frac{9 \times 10^{-4}}{4\pi}(10)\left(\tfrac{1}{2}\right) = \frac{9 \times 10^{-3}}{4\pi}\left(\tfrac{1}{2}\right) = (0.7162 \times 10^{-3})\left(\tfrac{1}{2}\right)$$

$$A_{em} = 0.3581 \times 10^{-3} \text{ m}^2$$

(b) $P_T = A_{em} W^i = (0.3581 \times 10^{-3})(10 \times 10^{-3}) = 3.581 \times 10^{-6}$ watts

$P_T = 3.581 \times 10^{-6}$ watts

2-44. $\underline{E}_a = (2\hat{a}_x \pm j\hat{a}_y)Ee^{-jkz}$

$$\hat{\rho}_a = \frac{2\hat{a}_x \pm j\hat{a}_y}{\sqrt{5}}$$

(a) $\underline{E}_w = \hat{a}_x E_w \Rightarrow \hat{\rho}_w = \hat{a}_x$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\frac{2}{\sqrt{5}}\right|^2 = \frac{4}{5} = 0.8 = -0.9691 \text{ dB}$$

(b) $\underline{E}_w = \hat{a}_y E_w \Rightarrow \hat{\rho}_w = \hat{a}_y$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\frac{1}{\sqrt{5}}\right|^2 = \frac{1}{5} = 0.2 = -6.9897 \text{ dB}$$

2-45. (a) $E_y = E'_y + E''_y = 3\cos\omega t + 2\cos\omega t = 5\cos\omega t$

$$E_x = E'_x + E''_x = 7\cos\left(\omega t + \frac{\pi}{2}\right) + 3\cos\left(\omega t - \frac{\pi}{2}\right)$$

$$= -7\sin\omega t + 3\sin\omega t = -4\sin\omega t$$

$$\text{AR} = \frac{5}{4} = 1.25$$

(b) At $\omega t = 0$, $\vec{E} = 5\hat{a}_y$

At $\omega t = \pi/2 \Rightarrow \vec{E} = -4\hat{a}_x \Rightarrow$ Rotation in CCW

2-46. (a) PLF $= \frac{1}{2}$ independent of $\psi \rightarrow$ must have CP

$\therefore$ AR $= 1$.

(b) Polarization will be elliptical with major axies aligned with x-axis.

guess: AR $= 2$

verify: $\hat{\rho}_w = (2\hat{a}_x + ja_y)/\sqrt{5}$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = |\frac{2\cos\psi + j\sin\psi}{\sqrt{5}}|^2 = \frac{4\cos^2\psi + \sin^2\psi}{5}$$

$\psi = 0$: PLF $= 0.8$

$\psi = 90°$: PLF $= 0.2$

(c) PLF $= 1$ at $\psi = 45°$ and $225°$

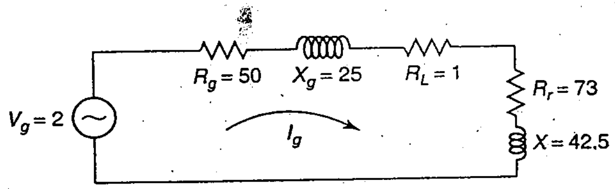
PLF $= 0$ at $\psi = 135°$ and $315°$

Polarization must be linear with that angle of 45°

$\therefore$ AR $= \infty$

2-47. $$I_g = \frac{2}{(50+1+73)+j(25+42.5)} = \frac{2}{124+j67.5}$$

$$= (12.442 - j6.7724) \times 10^{-3} = 14.166 \times 10^{-3}\angle - 28.56°$$



(a) $P_s = \frac{1}{2}\text{Re}(V_g \cdot I_g^*) = Re(12.442 + j6.7724) \times 10^{-3} = 12.442 \times 10^{-3}$ W

(b) $P_r = \frac{1}{2}|I_g|^2 R_r = 7.325 \times 10^{-3}$ W

(c) $P_L = \frac{1}{2}|I_g|^2 R_L = 0.1003 \times 10^{-3}$ W

The remaining supplied power is dissipated as heat in the internal resistor of the generator, or

$$P_g = \frac{1}{2}|I_g|^2 R_g = 5.0169 \times 10^{-3} \text{ W}$$

Thus

$$P_r + P_L + P_g = (7.325 + 0.1003 + 5.0169) \times 10^{-3} = 12.4422 \times 10^{-3} = P_s$$

2-48. The impedance transfer equation of

$$Z_{in} = Z_c \left[\frac{Z_L + jZ_c \tan(kl)}{Z_c + jZ_L \tan(kl)}\right]$$

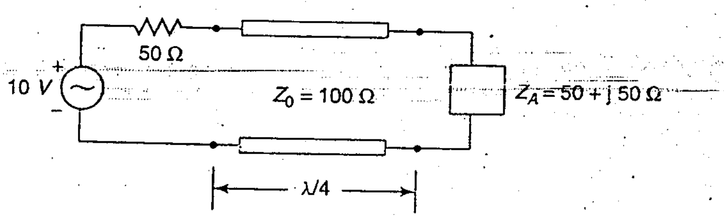
reduces for $l = \lambda/2$ to $Z_{in} = Z_L$.

Therefore the equivalent load impedance at the terminals of the generator is the same as that for Problem 2-47.

Thus the supplied, radiated, and dissipated powers are the same as those of Problem 2-47.

2-49. (a) $Z_{in} = \dfrac{(100)^2}{50 + j50} = \dfrac{10000}{5000} = (50 - j50) = 100 - j100\ \Omega$

$I_g = \dfrac{10}{150 - j100} = \dfrac{10}{180.3\angle -33.7^\circ} = 0.05546\angle 33.7^\circ A$



(b) $P_s = \frac{1}{2}\text{Re}\{V_g I_g^*\} = \frac{1}{2} \times 10 \times 0.05546 \times \cos(33.7^\circ)$

$= 0.231$ W

(c) $P_A = \frac{1}{2}|I_g|^2 \text{Re}\{Z_{in}\} = \frac{1}{2} \times (0.05546)^2 \times 100 = 0.1538$ W

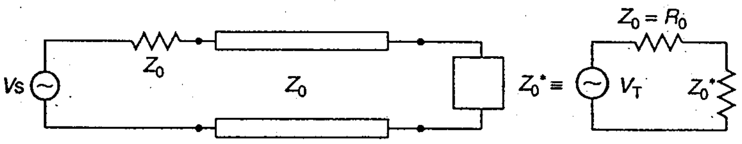
$P_{rad} = e_{cd} P_A = 0.96 \times 0.1538 = 0.148$ W

2-50.

$$\text{Gain} = \frac{P_{rad}}{P_{accepted}} \text{Directivity}$$

$$\text{Realized Gain} = \frac{P_{rad}}{P_{available}} \text{Directivity}$$

$$\frac{\text{Gain}}{\text{Realized Gain}} = \frac{P_{available}}{P_{accepted}}$$



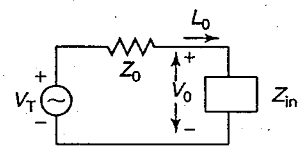


Fig. 1.

$$P_{\text{available}} = \frac{1}{2}\frac{\left(\frac{V_s}{\sqrt{2}}\right)^2}{Z_0} = \frac{V_s^2}{4Z_0}$$

$$V(x) = A(e^{-jkx} + \Gamma(0)e^{jkx})$$

$$I(x) = \frac{A}{Z_0}(e^{-jkx} - \Gamma(0)e^{jkx})$$

$$V(0) = A(1+\Gamma(0))$$

$$I(0) = \frac{A}{Z_0}(1-\Gamma(0))$$

From Fig. 1;

$$-V_s + I(0)Z_0 + V(0) = 0$$

$$-V_s + \frac{A}{Z_0}(1-\Gamma(0))Z_0 + A(1+\Gamma(0)) = 0$$

$$-V_s + A - A\Gamma(0) + A + A\Gamma(0) = 0$$

$$2A = V_s \rightarrow A = \frac{V_s}{2}$$

$$P_{\text{accepted}} = \text{Re}[V(0)I^*(0)]$$

$$V(0) = \frac{V_s}{2}(1+\Gamma(0))$$

$$I(0) = \frac{V_s}{2Z_0}(1-\Gamma(0))$$

$$\Gamma(0) = \frac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0}$$

$$\Rightarrow V(0) = \frac{V_s}{2}\left(1 + \frac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0}\right)$$

$$= \frac{V_s}{2}\left(1 + \frac{R_{\text{in}} + jX_{\text{in}} - Z_0}{R_{\text{in}} + jX_{\text{in}} + Z_0}\right)$$

$$= \frac{V_s}{2}\left(\frac{R_{\text{in}} + jX_{\text{in}} + Z_0 + R_{\text{in}} + jX_{\text{in}} - Z_0}{R_{\text{in}} + jX_{\text{in}} + Z_0}\right)$$

$$V(0) = \frac{V_s(R_{\text{in}} + jX_{\text{in}})}{R_{\text{in}} + jX_{\text{in}} + Z_0}$$

$$I(0) = \frac{V_s}{2Z_0}\left(1 - \frac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0}\right) = \frac{V_s}{2Z_0}\left(\frac{Z_{\text{in}} + Z_0 - Z_{\text{in}} + Z_0}{Z_{\text{in}} + Z_0}\right)$$

$$I(0) = \frac{V_s}{Z_{\text{in}} + Z_0} = \frac{V_s}{R_{\text{in}} + jX_{\text{in}} + Z_0}$$

$$\text{Re}[V(0)I(0)^*] = \text{Re}\left[\frac{V_sR_{\text{in}} + jV_sX_{\text{in}}}{R_{\text{in}} + Z_0 + jX_{\text{in}}} \times \frac{V_s}{R_{\text{in}} + Z_0 - jX_{\text{in}}}\right]$$

$$P_{\text{accepted}} = \text{Re}\left(\frac{V_s^2(R_{\text{in}} + jX_{\text{in}})}{(R_{\text{in}} + Z_0)^2 + X_{\text{in}}^2}\right) = \frac{V_s^2 R_{\text{in}}}{(R_{\text{in}} + Z_0)^2 + X_{\text{in}}^2}$$

$$\frac{\text{Gain}}{\text{Realized Gain}} = \frac{\dfrac{V_s^2}{4Z_0}}{\dfrac{V_s^2 R_{\text{in}}}{(R_{\text{in}} + Z_0)^2 + X_{\text{in}}^2}} = \frac{(R_{\text{in}} + Z_0)^2 + X_{\text{in}}^2}{4Z_0 R_{\text{in}}}$$

2-51. (a) $R_L = R_{hf}(\underline{2\text{-}90b}) = \dfrac{l}{C}\sqrt{\dfrac{\omega\mu_o}{2\sigma}}$

$$= \frac{\lambda/60}{2\pi(\lambda/200)} \cdot \sqrt{\frac{2\pi \times 10^9(4\pi \times 10^{-7})}{2(5.7 \times 10^7)}}$$

$$= 0.4415 \times 10^{-2} = 0.004415 \text{ (ohms)}$$

(b) $R_r(\underline{4-19}) = 80\pi^2\left(\dfrac{l}{\lambda}\right)^2 = 80\pi^2\left(\dfrac{1}{60}\right)^2 = 0.21932$

$\Rightarrow R_{\text{in}} = R_r = 0.21932$ ohms (because of assumed constant current)

(c) $e_{cd}(\underline{2-90}) = \dfrac{R_r}{R_L + R_r} = \dfrac{0.21932}{0.21932 + 0.004415} = 0.98027$

$e_{cd} = 98.027\%$

(d) $$Z_L = (R_L + R_{in}) + jX_{in} = (0.21932 + 0.004415) + jX_{in} = 0.2237 + jX_{in}$$

$$X_{in} \simeq -120\frac{\ln(l/2a) - 1}{\tan\left(\frac{kl}{2}\right)} = -120\frac{\left[\ln\left(\frac{\lambda/60}{\lambda/100}\right) - 1\right]}{\tan\left(\frac{2\pi}{2\lambda}\frac{\lambda}{60}\right)}$$

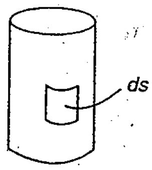
$$= -120 \cdot \left[\frac{0.51003 - 1}{0.05241}\right] = +1{,}120.03$$

$$|\Gamma| = \frac{Z_L - Z_c}{Z_L + Z_c} = \frac{(0.2237 + j1{,}120.03) - 50}{(0.2237 + j1{,}120.03) + 50} = 0.9999$$

$$\text{VSWR} = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.9999}{1 - 0.9999} = 9{,}999 \simeq \infty.$$

2-52. Radiation Efficiency of a dipole

$I_z(z) = I_0 \cos\left[\frac{\pi}{l} z'\right], -l/2 \leqslant z' \leqslant l/2$



$H_\phi(r = a)|_{\text{at the surface}} = \frac{I_0}{2\pi a} \cos\left[\frac{\pi}{l} z\right]$

$ds = a\, d\phi\, dz \Rightarrow$ differential patch of area.

$dw \Rightarrow$ power loss into this patch.

$dw = \frac{1}{2}|H_\phi|^2 R_s a\, d\phi\, dz$

(time avs) ($\leftarrow R_s$ = skin resistance)

$$dW = \left(\frac{I_0}{2\pi a}\right)^2 \cdot \frac{R_s}{2} \cos^2\left[\frac{\pi}{l} Z\right] a\, d\phi\, dz$$

$$W(\text{total loss}) = \int_{-l/2}^{l/2} \int_{\phi=0}^{2\pi} \frac{{I_0}^2 R_s}{8\pi^2 \cdot a^2} \cos^2\left[\frac{\pi}{l} Z\right] a\, d\phi\, dz$$

$$W = \frac{{I_0}^2}{8\pi^2 a^2} \cdot 2\pi a \cdot R_s \int_{-l/2}^{l/2} \cos^2\left[\frac{\pi}{l} z\right] dz = \frac{{I_0}^2}{4\pi} \frac{l \cdot R_s}{a} \cdot \frac{1}{2}$$
$$= \frac{1}{2} {I_0}^2 R_L$$
$$R_L = \frac{1}{2} \cdot \frac{l R_s}{2\pi a}$$

2-53. $E = \begin{cases} 1 & 0 < \theta \leqslant 45^\circ \\ 0 & 45^\circ < \theta \leqslant 90^\circ \\ \frac{1}{2} & 90^\circ < \theta \leqslant 180^\circ \end{cases}$

(a) $U = \dfrac{r^2 E^2}{2\eta} = \dfrac{r^2 |E|^2}{\eta}, \qquad U_{\max} = \dfrac{r^2}{\eta} = \dfrac{1}{120\pi}$

$$P_{\text{rad}} = \frac{r^2}{\eta}\int_0^{2\pi} d\phi \left[\int_0^{45^\circ} \sin\theta\, d\theta + \int_{90^\circ}^{180^\circ} \frac{1}{4}\sin\theta\, d\theta\right]$$

$$= \frac{r^2}{\eta}[2\pi]\left[-\cos\theta\Big|_0^{45^\circ} + \frac{1}{4}(-\cos\theta)\Big|_{90^\circ}^{180^\circ}\right]$$

$$= \frac{2r^2\pi}{\eta}\left[-\cos 45^\circ + \cos 0^\circ - \frac{1}{4}\cos 180^\circ + \frac{1}{4}\cos 90^\circ\right]$$

$$P_{\text{rad}} = 0.54289\frac{2\pi r^2}{\eta}$$

$$D = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{4\pi \frac{r^2}{\eta}}{0.54289(2\pi)r^2/\eta} = 3.684$$

(b) When the field is equal to 10 v/m, for $\theta = 0^\circ$.

$$\Rightarrow E = \begin{cases} 10 \text{ v/m} & 0 < \theta \leqslant 45^\circ \\ 0 & 45^\circ < \theta \leqslant 90^\circ \\ \frac{1}{2} \times 10 \text{ v/m} & 90^\circ < \theta \leqslant 180^\circ \end{cases}$$

$$P_{\text{rad}} = \frac{r^2}{\eta}\left[\int_0^{2\pi}\left\{\int_0^{45^\circ} |E|^2 \sin\theta\, d\theta + \int_{90^\circ}^{180^\circ} |E|^2 \sin\theta\, d\theta\right\} d\phi\right]$$

$$P_{\text{rad}} = r^2(0.54289)\left(\frac{2\pi}{\eta}\right)|10|^2 = 36{,}193$$

$$P_{\text{rad}} = \tfrac{1}{2}|I|^2 R_r = |I_{\text{rms}}|^2 \cdot R_r$$

$$\Rightarrow R_r = \frac{36{,}193}{|I_{\text{rms}}|^2} = \frac{36{,}193}{25} = 1{,}447.72$$

2-54.
```
Input parameters:
------------------
The lower bound of theta in degrees = 0
The upper bound of theta in degrees = 90
The lower bound of phi in degrees = 0
The upper bound of phi in degrees = 360
```

```
Output parameters:
------------------
Radiated power (watts) = 0.1566
Partial Directivity (theta) (dimensionless) = 80.2511
Partial Directivity (theta) (dB) = 19.0445
Partial Directivity (phi) (dimensionless) = 80.2511
Partial Directivity (phi) (dB) = 19.0445
Directivity (dimensionless) = 80.2511
Directivity (dB) = 19.0445
```

Using Table 12.1

$$a = 3\lambda, b = 2\lambda$$
$$D_0 = 4\pi\left(\frac{ab}{\lambda^2}\right) = 4\pi(6) = 24\pi$$
$$D_0 = 75.398 = 18.774 \text{ dB}$$

Since the maximum $|E_\theta| = |E_\phi| = |\underline{E}|$ then the maximum directivity

$$D_0 = D_\theta = D_\phi$$

2-55.
```
Input parameters:
-----------------
The lower bound of theta in degrees = 0
The upper bound of theta in degrees = 90
The lower bound of phi in degrees = 0
The upper bound of phi in degrees = 360

Output parameters:
------------------
Radiated power (watts) = 0.0330
Partial Directivity (theta) (dimensionless) = 62.4635
Partial Directivity (theta) (dB) = 17.9563
Partial Directivity (phi) (dimensionless) = 62.4635
Partial Directivity (phi) (dB) = 17.9563
Directivity (dimensionless) = 62.4635
Directivity (dB) = 17.9563
```

Using Table 12.1

$$a = 3\lambda, b = 2\lambda$$
$$D_0 = 0.81\left(4\pi\frac{ab}{\lambda^2}\right) = 0.81(24\pi)$$
$$= 61.072 = 17.858 \text{ dB}$$

Since the maximum $|E_\theta| = |E_\phi| = |\underline{E}|$, then the maximum directivity

$$D_0 = D_\theta = D_\phi$$

2-56. Input parameters:

```
-----------------
The lower bound of theta in degrees = 0
The upper bound of theta in degrees = 90
The lower bound of phi in degrees = 0
The upper bound of phi in degrees = 360

Output parameters:
------------------
Radiated power (watts) = 0.4863
Partial Directivity (theta) (dimensionless) = 4.2443
Partial Directivity (theta) (dB) = 6.2780
Partial Directivity (phi) (dimensionless) = 4.2443
Partial Directivity (phi) (dB) = 6.2780
Directivity (dimensionless) = 4.2443
Directivity (dB) = 6.2780
```

Using Table 12.1

$$f = 10 \text{ GHz} \Rightarrow \lambda = 3 \text{ cm} \Rightarrow a = \frac{2.286}{3}\lambda = 0.762\lambda$$

$$b = \frac{1.016}{3}\lambda \doteq 0.3387\lambda$$

$$D_0 = 0.81\left(4\pi\frac{ab}{\lambda^2}\right) = 0.81(4\pi)(0.762)(0.3387)$$

$$= 2.627 = 4.194 \text{ dB}$$

Since the maximum $|E_\theta| = |E_\phi| = |\underline{E}|$, then the maximum directivity

$$D_0 = D_\theta = D_\phi$$

2-57. Input parameters:

```
-----------------
The lower bound of theta in degrees = 0
The upper bound of theta in degrees = 90
The lower bound of phi in degrees = 0
The upper bound of phi in degrees = 360
```

```
Output parameters:
------------------

Radiated power (watts) = 0.0338
Partial Directivity (theta) (dimensionless) = 92.9470
Partial Directivity (theta) (dB) = 19.6824
Partial Directivity (phi) (dimensionless) = 92.9470
Partial Directivity (phi) (dB) = 19.6824
Directivity (dimensionless) = 92.9470
Directivity (dB) = 19.6824
```

Using Table 12.2

$$a = 1.5\lambda$$

$$D_0 = \frac{4\pi}{\lambda^2}(\pi a^2) = \left(\frac{2\pi a}{\lambda}\right)^2 = 9\pi^2$$

$$D_0 = 88.826 = 19.485 \text{ dB}$$

Since the maximum $|E_\theta| = |E_\phi| = |\underline{E}|$, then the maximum directivity

$$D_0 = D_\theta = D_\phi$$

2-58. Input parameters:

```
-----------------

The lower bound of theta in degrees = 0
The upper bound of theta in degrees = 90
The lower bound of phi in degrees = 0
The upper bound of phi in degrees = 360

Output parameters:
------------------

Radiated power (watts) = 0.0418
Partial Directivity (theta) (dimensionless) = 75.1735
Partial Directivity (theta) (dB) = 18.7606
Partial Directivity (phi) (dimensionless) = 75.1735
Partial Directivity (phi) (dB) = 18.7606
Directivity (dimensionless) = 75.1735
Directivity (dB) = 18.7606
```

Using Table 12.2

$$a = 1.5\lambda$$

$$D_0 = 0.836\left(\frac{2\pi a}{\lambda}\right)^2 = 0.836(9\pi^2)$$

$$D_0 = 74.2589 = 18.71 \text{ dB}$$

Since the maximum $|E_\theta| = |E_\phi| = |\underline{E}|$, then the maximum directivity

$$D_0 = D_\theta = D_\phi$$

2-59. Input parameters:

```
------------------
The lower bound of theta in degrees = 0
The upper bound of theta in degrees = 90
The lower bound of phi in degrees = 0
The upper bound of phi in degrees = 360

Output parameters:
------------------
Radiated power (watts) = 0.4952
Partial Directivity (theta) (dimensionless) = 6.3439
Partial Directivity (theta) (dB) = 8.0236
Partial Directivity (phi) (dimensionless) = 6.3439
Partial Directivity (phi) (dB) = 8.0236
Directivity (dimensionless) = 6.3439
Directivity (dB) = 8.0236
```

Using Table 12.2

$$f = 10 \text{ GHZ} \Rightarrow \lambda = 3 \text{ cm} \Rightarrow a = \frac{1.143}{3}\lambda = 0.381\lambda$$

$$D_0 = 0.836\left(\frac{2\pi a}{\lambda}\right)^2 = 0.836[2\pi(0.381)]^2$$

$$D_0 = 4.791 = 6.804 \text{ dB}$$

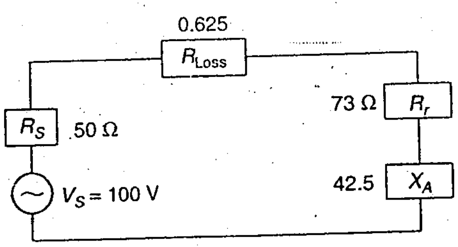
Since the maximum $|E_\theta| = |E_\phi| = |\underline{E}|$, then the maximum directivity

$$D_0 = D_\theta = D_\phi$$

2-60. $f = 150$ MHz, $\lambda = 2$m

$\Rightarrow$ 1 m dipole is $\frac{\lambda}{2}$ in electrical length

$\Rightarrow R_r = 73\ \Omega, Z_{in} = 73 + j42.5\ \Omega$



a. $I_{\text{ant}} = \dfrac{V_s}{50+73+0.625+j42.5} = 0.765\angle -18.97^\circ A$

b. $P_{\text{dissip}} = P_{\text{Loss}} = \frac{1}{2}|I_{\text{ant}}|^2 \cdot R_{\text{Loss}} = 189$ mW

c. $P_{\text{rad}} = \frac{1}{2}|I_{\text{ant}}|^2 \cdot R_r = 21.36\ W$

d. $E_{cd} = \dfrac{R_r}{R_r + R_{\text{Loss}}} = \dfrac{73}{73+0.625} = 99\%$

2-61. $\underline{E} = \hat{a}_\theta E_\theta \simeq \hat{a}_\theta j\eta \dfrac{kI_0 l}{4\pi r} e^{-jkr}\sin\theta = -j\eta\dfrac{kI_0e^{-jkr}}{4\pi r}[\underbrace{-\hat{a}_\theta l\sin\theta}_{\underline{l}_e}]$

a. $\underline{l}_e = -\hat{a}_\theta l \sin\theta$

b. $|\underline{l}_e|_{\max} = |-\hat{a}_\theta l\sin\theta|_{\max} = l \quad @\ \theta = 90^\circ$

c. $|\underline{l}_e|_{\max}/l = 1$

2-62.

$$\underline{E} = \hat{a}_\theta E_\theta = \hat{a}_\theta j\eta\frac{I_0e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]$$

$$= j\eta\frac{kI_0e^{-jkr}}{4\pi r}\left[-\hat{a}_\theta\frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{k\sin\theta}\right]$$

$$= j\eta\frac{kI_0e^{-jkr}}{4\pi r}\left[-\underbrace{\hat{a}_\theta\frac{\lambda}{\pi}\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}}_{\underline{l}_e}\right]$$

$$\underline{l}_e = -\hat{a}_\theta\frac{\lambda}{\pi}\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} = -\hat{a}_\theta 0.3183\lambda\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

$$|\underline{l}_e|_{\max} = \left|-\hat{a}_\theta 0.3183\lambda\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right|_{\max} = 0.3183\lambda \quad @\ \theta = 90^\circ$$

$$\frac{|\underline{l}_e|_{\max}}{l} = \frac{0.3183\lambda}{\lambda/2} = 0.6366 = 63.66\% \quad @\ \theta = 90^\circ$$

2-63.

$$\underline{l}_e = -\hat{a}_\theta l\sin\theta,\ l = \lambda/50,\ f = 10\text{ GHz} \Rightarrow \lambda = 3\text{ cm}$$

$$W = \frac{1}{2\eta}|\underline{E}|^2 = 10^{-3}\text{ W/cm} \Rightarrow |\underline{E}| = \sqrt{2\eta W}$$

$$= \sqrt{2(377)(10^{-3})} = 0.8683\text{ V/cm}$$

$$V_{oc}|_{\max} = |\underline{E}^i||\underline{l}_e|_{\max} = (0.8683)\left(\frac{\lambda}{50}\right) = 52.1\times 10^{-3}\text{ Volts}$$

2-64. Since$|l_e|_{max} = l/2 \Rightarrow |V_{oc}|_{max} = \frac{1}{2}(V_{oc}$ of dipole with uniform current)

$V_{oc}|_{max} = \frac{1}{2}(52.1 \times 10^{-3}) = 26.05 \times 10^{-3}$ Votts (see Problem 2-63)

2-65. $|l_e|_{max} = 0.3183\lambda \Rightarrow |V_{oc}| = |l_e|_{max}|E^i|$. . From Problem 2-63 solution

$|V_{oc}| = 0.8683(0.3183\lambda) = 0.27638\lambda = 0.27638(3) = 0.82914$ Votts

2-66. Using equation (2-94), the effective aperture of an antenna can be written as

$$A_e = \frac{|V_T|^2 \cdot R_T}{2\, W_i |Z_t|^2}, \text{ where } W_i = |E|^2/2\eta$$

Defining the effective length l_e as $V_T = E \cdot l_e$ reduces A_e to

$$A_e = \frac{\eta R_T l_e^2}{|Z_t|^2} \Rightarrow l_e = \sqrt{\frac{A_e |Z_t|^2}{\eta R_T}}$$

For maximum power transfer and lossless antenna ($R_L = 0$)

$$X_A = -X_T, R_r = R_T \Rightarrow |Z_t| = 2R_r = 2R_T$$

Thus $l_e = \sqrt{\frac{4A_{em} \cdot R_T^2}{\eta R_T}} = 2\sqrt{\frac{A_{em} R_T}{\eta}} = 2 \cdot \sqrt{\frac{A_{em} R_r}{\eta}}$

2-67. $A_{em} = 2.147 = \left(\frac{\lambda^2}{4\pi}\right) \cdot e_{cd} \cdot (1 - |\Gamma|^2) \cdot |\hat{\rho}_w \cdot \hat{\rho}_a|^2 \cdot D_0$

$$\Gamma = \frac{75 - 50}{75 + 50} = 0.2; \lambda = \frac{3 \times 10^8}{100 \times 10^6} = 3 \text{ m}$$

$$\therefore D_0 = \frac{2.147}{\frac{3^2}{4\pi}[(1 - (0.2)^2]} = 3.125$$

2-68. $d = 1$ m, $f = 3$ GHz, $\varepsilon_{ap} = 68\% \Rightarrow \lambda = \frac{3 \times 10^8}{3 \times 10^9} = 0.1$ m

(a) $A_p = \pi r^2 = \pi\left(\frac{d}{z}\right)^2 = \frac{\pi d^2}{4} = \frac{\pi (1)^2}{4} = \boxed{\frac{\pi}{4} = 0.785 \text{ m}^2}$

(b) $\varepsilon_{ap} = \frac{A_{em}}{A_p} \Rightarrow A_{em} = \varepsilon_{ap} A_p$

$A_{em} = \varepsilon_{ap} A_p = 0.68(0.785) = \boxed{0.534 \text{ m}^2}$

(c) $A_{em} = \frac{\lambda^2}{4\pi} D_0 \Rightarrow D_0 = \frac{4\pi}{\lambda^2} A_{em}$

$D_0 = \frac{4\pi}{\lambda^2} A_{em} = \frac{4\pi}{(0.1)^2}(0.534) = \frac{4\pi}{0.01}(0.534) = 671.044$

$D_0 = \boxed{671.044 = 28.268 \text{ dB}}$

(d) $P_L = A_{em} W_L = 0.534(10 \times 10^{-6})$

$P_L = \boxed{5.34 \times 10^{-6} \text{ watts}}$

2-69. $W_i = 10^{-3}$ W/m^2

$$A_{em} = \frac{\lambda^2}{4\pi} \cdot D_0, D_0 = 20 \text{ dB} = 10 \log_{10} x \Rightarrow x = 100$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{10 \times 10^9} = 0.03 \text{ m} = 3 \times 10^{-2} \text{ m}$$

$$A_{em} = \frac{(3 \times 10^{-2})^2}{4\pi} \cdot 100 = \frac{9 \times 10^{-4}}{4\pi} \cdot (100) = 0.716 \times 10^{-2} = 7.16 \times 10^{-3}$$

$$P_{rec} = 10^{-3} \cdot \left(\frac{9 \times 10^{-2}}{4\pi}\right) = \frac{9 \times 10^{-5}}{4\pi} = 0.716 \times 10^{-5} = 7.16 \times 10^{-6} \text{ watts}$$

$P_{rec} = 7.16 \times 10^{-6}$ watts.

2-70. $A_p = 10 \text{ cm}^2, f = 10 \text{ GHz} \Rightarrow \lambda = 30 \times 10^9/10 \times 10^9 = 3 \text{ cm}, W^i = 10 \times 10^{-3} \text{ W/cm}^2$

(a) $A_{em} = \frac{\lambda^2}{4\pi} D_0 = \frac{\lambda^2}{4\pi} G_0 = A_p = 10$

$\Rightarrow G_0 = \frac{4\pi(10)}{\lambda^2} = \frac{4\pi(10)}{(3)^2} = 13.96 = 11.45 \text{ dB}$

(b) $P_r = A_{em} W^i(\text{PLF}) = \frac{1}{2}(10)(10 \times 10^{-3}) = 100 \times 10^{-3}/2 = 0.05$ Watts

$P_r = 0.05$ Watts

$$\text{PLF} = \left| \hat{a}_x \cdot \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)\right|^2 = \tfrac{1}{2}$$

2-71. $\underline{W}_{\text{rad}} = \underline{W}_{\text{ave}} \simeq C_0 \frac{1}{r^2} \cos^4(\theta)\hat{a}_r \quad (0 \leqslant \theta \leqslant \pi/2, 0 \leqslant \phi \leqslant 2\pi)$

a. $$P_{\text{rad}} = \int_0^{2\pi} \int_0^{\pi/2} \underline{W}_{\text{rad}} \cdot d\underline{s} = \int_0^{2\pi} \int_0^{\pi/2} \hat{a}_r W_{\text{rad}} \cdot \hat{a}_r r^2 \sin\theta \, d\theta \, d\phi$$

$$= C_0 \int_0^{2\pi} \int_0^{\pi/2} \cos^4\theta \sin\theta \, d\theta \, d\phi = 2\pi C_0 \int_0^{\pi/2} \cos^4\theta \sin\theta \, d\theta$$

$$= 2\pi C_0 \left(-\frac{\cos^5\theta}{5}\right)_0^{\pi/2}$$

$$P_{\text{rad}} = 2\pi C_0 \left(0 + \frac{1}{5}\right) = \frac{2\pi}{5} C_0 = 1.2566 C_0$$

b. $$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} \Rightarrow U_{\max} = r^2 \left. W_{\text{rad}}\right|_{\max} = C_0 \left.\cos^4\theta\right|_{\max} = C_0$$

$$D_0 = \frac{4\pi C_0}{2\pi C_0/5} = 10 = 10 \log_{10}(10) = 10 \text{ dB}$$

c. $D_0 = 10$ toward $\theta = 0°$

d. $A_{em} = \dfrac{\lambda^2}{4\pi} D_0 \quad \lambda = \dfrac{c}{f} = \dfrac{3 \times 10^8 \text{ m/sec}}{1 \times 10^9} = 0.3 \text{ m}$

$$A_{em} = \frac{(0.3)^2}{4\pi}(10) = \frac{0.09}{4\pi}(10) = \frac{0.225}{\pi} = 0.0716 \text{ m}^2$$

e. $P_L = A_{em} W^i = 0.0716 \times (10 \times 10^{-3}) = 0.716 \times 10^{-3}$ Watts

2-72. $A_{em} = \dfrac{\lambda^2}{4\pi} e_t D_0 = \dfrac{\lambda^2}{4\pi} G_0$

a. $G_0 = 14.8 \text{ dB} \Rightarrow G_0(\text{power ratio}) = 10^{1.48} = 30.2$

$f = 8.2 \text{ GHZ} \Rightarrow \lambda = 3.6585 \text{ cm}$

$$A_{em} = \frac{(3.6585)^2}{4\pi}(30.2) = 32.167 \text{ cm}^2$$

The physical aperture is equal to $A_p = 5.5(7.4) = 40.7 \text{ cm}^2$

b. $G_0 = 16.5 \text{ dB} \Rightarrow G_0(\text{power ratio}) = 10^{1.65} = 44.668$

$f = 10.3 \text{ GHZ} \Rightarrow \lambda = 2.912 \text{ cm}$

$$A_{em} = \frac{(2.912)^2}{4\pi}(44.668) = 30.142 \text{ cm}^2$$

c. $G_0 = 18.0 \text{ dB} \Rightarrow G_0(\text{power ratio}) = 10^{1.8} = 63.096$

$f = 12.4 \text{ GHZ} \Rightarrow \lambda = 2.419 \text{ cm}$

$$A_{em} = \frac{(2.419)^2}{4\pi}(63.096) = 29.389 \text{ cm}^2$$

2-73. $A_{em} = \dfrac{\lambda^2}{4\pi} D_0$

From Problem 2-54:

Computer Program **Directivity**: $D_0 = 80.2511 \Rightarrow A_{em} = \dfrac{\lambda^2}{4\pi}(80.2511) = 6.386\lambda^2$

Table 12.1: $D_0 = 75.398 \Rightarrow A_{em} = \dfrac{\lambda^2}{4\pi}(75.398) = 6.0\lambda^2$

2-74. $A_{em} = \dfrac{\lambda^2}{4\pi} D_0$

From Problem 2-55:

Computer Program **Directivity**: $D_0 = 62.4635 \Rightarrow A_{em} = \dfrac{\lambda^2}{4\pi}(62.4635) = 4.972\lambda^2$

Table 12.1: $D_0 = 61.072 \Rightarrow A_{em} = \dfrac{\lambda^2}{4\pi}(61.072) = 4.86\lambda^2$

2-75. $A_{em} = \frac{\lambda^2}{4\pi} D_0$

From Problem 2-56:

Computer Program **Directivity**: $D_0 = 4.2443 \Rightarrow A_{em} = \frac{\lambda^2}{4\pi}(4.2443) = 0.3378\lambda^2$

Table 12.1: $D_0 = 2.627 \Rightarrow A_{em} = \frac{\lambda^2}{4\pi}(2.627) = 0.20905\lambda^2$

2-76. $A_{em} = \frac{\lambda^2}{4\pi} D_0$

From Problem 2-57:

Computer Program **Directivity**: $D_0 = 92.947 \Rightarrow A_{em} = \frac{\lambda^2}{4\pi}(92.947) = 7.396\lambda^2$

Table 12.2: $D_0 = 88.826 \Rightarrow A_{em} = \frac{\lambda^2}{4\pi}(88.826) = 7.068\lambda^2$

2-77. $A_{em} = \frac{\lambda^2}{4\pi} D_0$

From Problem 2-58:

Computer Program **Directivity**: $D_0 = 75.1735 \Rightarrow A_{em} = \frac{\lambda^2}{4\pi}(75.1735) = 5.982\lambda^2$

Table 12.2: $D_0 = 74.2589 \Rightarrow A_{em} = \frac{\lambda^2}{4\pi}(74.2589) = 5.909\lambda^2$

2-78. $A_{em} = \frac{\lambda^2}{4\pi} D_0$

From Problem 2-59:

Computer Program **Directivity**: $D_0 = 8.0236 \Rightarrow A_{em} = \frac{\lambda^2}{4\pi}(8.0236) = 0.638\lambda^2$

Table 12.2: $D_0 = 4.791 \Rightarrow A_{em} = \frac{\lambda^2}{4\pi}(4.791) = 0.3813\lambda^2$

2-79. Gain = 30 dB, $f = 2$ GHZ, $P_{\text{rad}} = 5$ W

Receiving antenna VSWR = 2, efficiency = 95%

$$\underline{E}_R = (2\hat{a}_x + j\hat{a}_y)F_R(\theta, \phi), \text{ Use Friis transmission formula (2-118)}$$

$$P_r = P_t e_{cdt} e_{cdr}(1 - |\Gamma_t|^2)(1 - |\Gamma_r|^2)\left(\frac{\lambda}{4\pi R}\right)^2 D_t(\theta_t, \phi_t) D_r(\theta_r, \phi_r) \cdot \text{PLF}$$

$$P_r = 10^{-14} \text{ W}, e_{cdt} = 1 \text{ (we assume that)}, e_{cdr} = 0.95, 1 - |\Gamma_t|^2 = 1$$

$$\text{Since VSWR} = 2 \Rightarrow |\Gamma_r| = \left|\frac{\text{VSWR} - 1}{\text{VSWR+1}}\right| = \frac{2-1}{2+1} = \frac{1}{3}, (1 - |\Gamma_r|^2) = 8/9$$

$$\lambda = \frac{3 \times 10^8}{2 \times 10^9} = 0.15 \text{ m}, R = 4000 \times 10^3 \text{ m},$$

$$\text{Hence } \left(\frac{\lambda}{4\pi R}\right)^2 = \left(\frac{0.15}{4\pi 4000 \times 10^3}\right)^2 = 8.9 \times 10^{-18}$$

$$D_t = 30 \text{ dB} = 10^3, \text{PLF} \Rightarrow \begin{cases} \rho_t = \frac{1}{\sqrt{2}}(\hat{a}_x - j\hat{a}_y) \Rightarrow |\hat{\rho}_t \cdot \hat{\rho}_r|^2 = 0.1 \\ \rho_r = \frac{1}{\sqrt{5}}(2\hat{a}_x + j\hat{a}_y) \end{cases}$$

$$\Rightarrow 10^{-14} = 5(1)(0.95)(1)\left(\frac{8}{9}\right)(8.9 \times 10^{-18})(10^3)D_r(0.1)$$

$$D_r = 2.661$$

$$\text{Hence } A_{em} = \frac{\lambda^2}{4\pi} 2.661 = 0.00476 \text{ m}^2$$

2-80. $U(\theta, \phi) = \begin{cases} \cos^4(\theta), & 0° \leqslant \theta \leqslant 90° \\ 0, & 90° \leqslant \theta \leqslant 180° \end{cases} \quad 0° \leqslant \phi \leqslant 360°$

$$A_{em} = \frac{\lambda^2}{4\pi} D_0$$

$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}}$$

$$P_{\text{rad}} = \int_0^{2\pi} \int_0^{\pi} U(\theta, \phi) \sin\theta \, d\theta \, d\phi = 2\pi \int_0^{\pi/2} \cos^4(\theta) \sin\theta \, d\theta = 2\pi \left[-\frac{\cos^5\theta}{5}\right]_0^{\pi/2}$$

$$P_{\text{rad}} = 2\pi\left(-0 + \frac{1}{5}\right) = \frac{2\pi}{5},$$

$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{4\pi(1)}{2\pi/5} = 10$$

$$A_{em} = \frac{\lambda^2}{4\pi} D_0 = \frac{\lambda^2}{4\pi} \cdot 10 = \frac{10\lambda^2}{4\pi}, \lambda = \frac{3 \times 10^8}{10^{10}} = 3 \times 10^{-2} = 0.03 \text{ m}$$

$$A_{em} = \frac{10(0.03)^2}{4\pi} = \frac{10 \cdot (3 \times 10^{-2})^2}{4\pi} = \frac{10 \cdot (9 \times 10^{-4})}{4\pi} = 7.16197 \times 10^{-4}$$

$$A_{em} = 7.16197 \times 10^{-4}$$

2-81. 1 status mile = 1609.3 meters, 22,300(status miles) = 3.588739×10^7 m

a. $P_i = \dfrac{P_{\text{rad}}}{4\pi R^2} = \dfrac{8 \times 10^{-14}}{4\pi \times (3.58874)} = 4.943 \times 10^{-16}$ watts/m^2.

b. $A_{em} = \dfrac{\lambda^2}{4\pi} D_0$, $(\leftarrow D_0 = 60 \text{ dB} = 10^6)$

$(\leftarrow \lambda = 0.15 \text{ m})$

$$A_{em} = \frac{(0.15)^2}{4\pi} \cdot 10^6 = 1790.493 \text{ m}^2$$

$$P_{\text{received}} = A_{em} \cdot P_i = (1790.493) \cdot (4.943 \times 10^{-16})$$
$$= 8.85 \times 10^{-13} \text{ watts.}$$

2-82. $A_{em} = 0.7162 \text{ m}^2$

$$A_{em} = \left(\frac{\lambda}{4\pi}\right)^2 \cdot e_{cd}(1 - |\Gamma|^2)|\hat{\rho}_w \cdot \hat{\rho}_a|^2 \cdot D_0$$

$$D_0 = \frac{A_{em}}{\left(\frac{\lambda}{4\pi}\right)^2 (1 - 1\Gamma|^2)}, \Gamma = \frac{75 - 50}{75 + 50} = 0.2, \lambda = \frac{3 \times 10^8}{100 \times 10^6} = 3 \text{ m}$$

$$D_0 = \frac{0.7162}{\frac{3^2}{4\pi}(1 - |0.2|^2)}$$

$$D_0 = 1.0417$$

2-83. $P_r = W_i A_{em} = W_i e_{cd}(1 - |\Gamma|^2)\left(\frac{\lambda^2}{4\pi}\right) D_0 |\hat{\rho}_w \cdot \hat{\rho}_a|^2$

$$W_i = 5 \text{ W/m}^2, e_{cd} = 1(\text{lossless}), \quad \Gamma = \frac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0} = \frac{73 - 50}{73 + 50} = 0.187$$

$$\lambda = \frac{3 \times 10^8}{10 \times 10^6} = 30 \text{ m}, D_0 = 2.156 \text{ dB} = 1.643, \text{PLF} = 1$$

$$P_r = (5)(1)(1 - (0.187)^2)\left(\frac{30^2}{4\pi}\right)(1.643)(1) = 567.78 \text{ watts}$$

$P_r = 567.78$ watts.

2-84. $\frac{P_r}{P_t} = \left(\frac{\lambda}{4\pi R}\right)^2 G_{0r} G_{0t}, G_{0r} = G_{0t} = 16.3 \Rightarrow G_0(\text{power ratio}) = 42.66$

$f = 10 \text{ GHz} \Rightarrow \lambda = 0.03$ meters.

$P_t = 200$ m watts $= 0.2$ watts

a. $R = 5$ m: $P_r = \left[\frac{0.03}{4\pi(5)}\right]^2 (42.66)^2(0.2) = 82.9\ \mu$watts

b. $R = 50$ m: $P_r = 0.829\ \mu$watts

c. $R = 500$ m: $P_r = 8.29\ n$watts

The VSWR was not needed because the gain was given.

2-85. $\frac{P_r}{P_t} = |\hat{\rho}_t \cdot \hat{\rho}_r|^2 \left(\frac{\lambda}{4\pi R}\right)^2 G_{0t} G_{0r}$

$G_{0t} = 20 \text{ dB} \Rightarrow G_{0t}(\text{power ratio}) = 10^2 = 100$

$G_{0r} = 15 \text{ dB} \Rightarrow G_{0r}(\text{power ratio}) = 10^{1.5} = 31.623$

$f = 1 \text{ GHz} \Rightarrow \lambda = 0.3 \text{ meters}$

$R = 1 \times 10^3 \text{meters}$

a. For $|\hat{\rho}_t \cdot \hat{\rho}_r|^2 = 1$

$$P_r = \left(\frac{0.3}{4\pi \times 10^3}\right)^2 (100)(31.623)\ (150 \times 10^{-3}) = 270.344\ \mu\text{watts}$$

b. When transmitting antennas is circularly polarized and receiving antenna is linearly polarized, the PLF is equal to

$$|\hat{\rho}_t \cdot \hat{\rho}_r|^2 = \left|\left(\frac{\hat{a}_x \pm j\hat{a}_y}{\sqrt{2}}\right) \cdot \hat{a}_x\right|^2 = \frac{1}{2}$$

Thus

$$P_r = \tfrac{1}{2}(270.344 \times 10^{-6}) = 135.172 \times 10^{-6} = 135.172\ \mu\text{watts}$$

2-86. Lossless: $e_{cd} = 1$, polarization matched: $|\hat{\rho}_w \cdot \hat{\rho}_a|^2 = 1$, line matched: $(1 - |\Gamma|^2) = 1$

$D_0 = 20 \text{ dB} = 10^2 = 100 = D_{0r} = D_{0t}$

$P_r = P_t \left(\frac{\lambda}{4\pi R}\right)^2 D_{0t} D_{0r} = 10 \left(\frac{\lambda}{4\pi \cdot 50\lambda}\right)^2 (100)(100) = 0.253 \text{ watts}$

$P_r = 0.253 \text{ watts}$

2-87. Lossless: $e_{cd} = 1, \text{PLF} = 1$. Line matched: $(1 - |\Gamma|^2) = 1$.

$D_0 = 30 \text{ dB} = 10^3 = 1000 = D_{0r} = D_{0t}$

$P_r = P_t \left(\frac{\lambda}{4\pi \cdot 100\lambda}\right)^2 \cdot (1000)^2 = 20 \cdot \left(\frac{1}{4\pi}\right)^2 \cdot 100 = 12.665 \text{ watts}$

2-88. $G_{0r} = 20 \text{ dB} = 100, G_{0t} = 25 \text{ dB} = 316.23. \lambda = \frac{3 \times 10^8}{3 \times 10^9} = 0.1 \text{ m}.$

$= P_t \cdot |\hat{\rho}_t \cdot \hat{\rho}_r|^2 \cdot \left(\frac{\lambda}{4\pi R}\right)^2 \cdot G_{0r} \cdot G_{0t}$

$= 100 \cdot (1) \cdot \left(\frac{0.1}{4\pi \times 500}\right)^2 (100)(316.23)$

$P_r = 8 \times 10^{-4} \text{ watts}$

2-89. $f = 10$ GHZ, $\rightarrow \lambda = \dfrac{3 \times 10^8}{10^{10}} = 0.03$ m

$G_{0t} = G_{0r} = 15$ dB $= 10^{1.5} = 31.62$

$R = 10$ km $= 10^4$ m

$P_r \geq 10$ nw $= 10^{-8}$ w

$|\hat{\rho}_t \cdot \hat{\rho}_r|^2 = -3$ dB $= \frac{1}{2}$

Friis Transmission Equation:

$$\frac{P_r}{P_t} = G_{0t}G_{0r} \cdot \left(\frac{\lambda}{4\pi R}\right)^2 \cdot |\hat{\rho}_t \cdot \hat{\rho}_r|^2$$

$$= (10^{1.5})^2 \cdot \left(\frac{0.03}{4\pi \times 10^4}\right)^2 \cdot \left(\tfrac{1}{2}\right) = 2.85 \times 10^{-11}$$

$$P_t = \frac{P_r}{2.85 \times 10^{-11}}$$

$$P_r \geq 10^{-8} \text{ W} \rightarrow (P_t)_{\text{min}} = 351 \text{ W}$$

2-90. $$\frac{P_r}{P_t} = (\text{PLF})e_t e_r D_{0t} D_{0r} \left(\frac{\lambda}{4\pi R}\right)^2$$

$$= (\text{PLF})(e_{rt} \cdot e_{cdt})(e_{rr} \cdot e_{cdr}) \left(\frac{\lambda}{4\pi R}\right)^2 \cdot D_{0t} \cdot D_{0r}$$

$$\frac{P_r}{P_t} = (1)(e_{rt} \cdot (1))(e_{rr} \cdot (1)) \left(\frac{\lambda}{4\pi R}\right)^2 \cdot D_{0t} \cdot D_{0r}$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{10^8} = 3 \text{ m}, R = 10 \times 10^3 = 10^4.$$

$$\left(\frac{\lambda}{4\pi R}\right)^2 = \left(\frac{3}{4\pi \times 10^4}\right)^2 = \left(\frac{3}{4\pi} \times 10^{-4}\right)^2$$

$$= (0.2387 \times 10^{-4})^2 = 5.699 \times 10^{-2} \times 10^{-8}$$

$$\left(\frac{\lambda}{4\pi R}\right)^2 = 5.699 \times 10^{-10}$$

$$e_{rt} = e_{rr} = (1 - |\Gamma|^2) = \left(1 - \left|\frac{73.3 - 50}{73.3 + 50}\right|^2\right) = \left(1 - \left|\frac{23.3}{123.3}\right|^2\right)$$

$$= (1 - (0.18897)^2) = (1 - 0.0357) = 0.9643$$

$$e_{cdt} = e_{cdr} = 1$$

$$D_{0t} = D_{0r} = 1.643$$

$$\frac{P_r}{P_t} = (0.9643)^2(1.643)^2(5.699 \times 10^{-10})$$
$$= (0.92987)(2.699)\ (5.699 \times 10^{-10})$$
$$= 2.51 \cdot (5.699 \times 10^{-10}) = 14.305 \times 10^{-10}$$
$$P_t = \frac{P_r}{14.305 \times 10^{-10}} = 6.99 \times 10^{-2} \times 10^{10}(1 \times 10^{-6})$$
$$= 6.99 \times 10^2 = 699.$$
$$P_t = 699 \text{ watts}$$

2-91. $\frac{P_r}{P_t} = \left(\frac{\lambda}{4\pi R}\right)^2 \cdot G_{0t} \cdot G_{0r}, \lambda = \frac{3 \times 10^8}{9 \times 10^9} = \frac{3 \times 10^8}{90 \times 10^8} = \frac{1}{30}$

$$R = 10{,}000 \text{ meter} = \frac{10{,}000}{1/30}\lambda = 3 \times 10^5 \lambda$$
$$\frac{P_r}{P_t} = \left[\frac{\lambda}{4\pi(3 \times 10^5 \lambda)}\right]^2 \cdot {G_0}^2 = \frac{10 \times 10^{-6}}{10} = 10^{-6}$$
$${G_0}^2 = 10^{-6}(4\pi \times 3 \times 10^5)^2$$
$$G_0 = 10^{-3}(4\pi \times 3 \times 10^5) = 12\pi \times 10^2 = 1200\pi$$
$$G_0 = 1200\pi = 3{,}769.91 = 10\log_{10}(3{,}769.91) \text{ dB}$$
$$G_0 = 3{,}769.91 = 35.76 \text{ dB}$$

2-92. $R = 16 \times 10^3$ m, $f = 2$ GHz, $G_{0t} = 20$ dB, $P_t = 100$ watts,

$P_r = 5 \times 10^{-9}$ watts $G_{0r} = ?$

$$G_{0t} = 20 \text{ dB} = 10\log_{10}[G_{0t}(\text{dim})] \Rightarrow G_{0t}(\text{dimensionless}) = 10^2 = 100$$
$$G_{0t}(\text{dimensionless}) = 100$$
$$f = 2 \text{ GHz} \Rightarrow \lambda = \frac{3 \times 10^8}{2 \times 10^9} = 0.15 \text{ meters}$$

Friis Transmission Equation (2-119):

$$\frac{P_r}{P_t} = G_{0t}G_{0r}\left(\frac{\lambda}{4\pi R}\right)^2 \text{PLF} \Rightarrow G_{0r} = \frac{P_r}{P_t}\left(\frac{1}{G_{0t}}\right)\left(\frac{4\pi R}{\lambda}\right)^2\left(\frac{1}{\text{PLF}}\right)$$
$$G_{0r} = \frac{5 \times 10^{-9}}{100}\left(\frac{1}{100}\right)\left[\frac{4\pi(16 \times 10^3)}{0.15}\right]^2\left(\frac{2}{1}\right)$$
$$= \frac{10 \times 10^{-9} \times 10^6}{10^4}\left[\frac{4\pi(16)}{0.15}\right]^2 = 10^{-6}(1{,}340.413)^2$$
$$G_{0r} = 1{,}796{,}706.65 \times 10^{-6} = 1.7967 = 2.545 \text{ dB}$$
$$\boxed{G_{0r} = 1.7967 = 2.545 \text{ dB}}$$

2-93. $\sigma = \pi a^2 = 25\pi\lambda^2$

Got = Gor = 16.3 dB $\Rightarrow$ Got (power ratio) $= 10^{1.63} = 42.66$

$f = 10$ GHz $\Rightarrow \lambda = 0.03$ m

$$\frac{P_r}{P_t} = \sigma \frac{\text{Got} \cdot \text{Gor}}{4\pi} \left(\frac{\lambda}{4\pi R_1 \cdot R_2} \right)^2$$

a. $R_1 = R_2 = 200\lambda = 6$ meters;

$$P_r = 25 \cdot \pi\lambda^2 \frac{(42.66)^2}{4\pi} \cdot \left[\frac{\lambda}{4\pi(200\lambda)^2} \right]^2 \cdot (0.2) = 9.00\ n\text{watts}$$

b. $R_1 = R_2 = 500\lambda = 15$ meters;

$P_r = 0.23\ n$watts

2-94. $P_r = P_t\sigma \cdot \dfrac{\text{Got} \cdot \text{Gor}}{4\pi} \cdot \left[\dfrac{\lambda}{4\pi R_1 \cdot R_2} \right]^2, \lambda = \dfrac{3 \times 10^8}{5 \times 10^9} = 0.06$ m

$$P_r = 10^5 \cdot (3) \cdot \frac{150^2}{4\pi} \cdot \left[\frac{0.06}{4\pi(10^6)} \right]^2$$

$P_r = 1.22 \times 10^{-8}$ watts

2-95. $\dfrac{P_r}{P_t} = \sigma \dfrac{\text{Gor} \cdot \text{Got}}{4\pi} \cdot \left[\dfrac{\lambda}{4\pi R_1 R_2} \right]^2 \Rightarrow \sigma = \dfrac{P_r \cdot 4\pi}{P_t \cdot \text{Gor} \cdot \text{Got}} \left[\dfrac{4\pi R_1 \cdot R_2}{\lambda} \right]^2$

$$\lambda = \frac{3 \times 10^8}{3 \times 10^8} = 1 \text{ m}$$

$$\therefore \sigma = \frac{0.1425 \times 10^{-3}(4\pi)}{1000(75)(75)} \left[\frac{4\pi(500)(500)}{1} \right]^2 = 3142 \text{ m}^2$$

2-96. $\sigma = \dfrac{P_r \cdot 4\pi}{P_t \cdot \text{Gor} \cdot \text{Got}} \left[\dfrac{4\pi R_1 R_2}{\lambda} \right]^2$

$$\lambda = \frac{3 \times 10^8}{1 \times 10^8} = 3 \text{ m}$$

$$\sigma = \frac{0.01 \cdot (4\pi)}{1000(75)(75)} \left[\frac{4\pi(700)(700)}{3} \right]^2 = 94{,}114.5 \text{ m}^2$$

$\sigma = 94{,}114.5$ m^2

2-97. $\sigma = 0.85\lambda^2$

$$\frac{P_r}{P_t} = \sigma \cdot \frac{\text{Got} \cdot \text{Gor}}{4\pi} \left(\frac{\lambda}{4\pi R_1 \cdot R_2}\right)^2 |\hat{P}_w \cdot \hat{P}_r|^2$$

$\sigma = 0.85\lambda^2$, Got = Gor = 15 dB $\Rightarrow$ Got = Gor = 31.6228 (dimensionless)

$R_1 = R_2 = 100$ meter $\Rightarrow R_1 = R_2 = 1{,}000\lambda$

$$f = 3 \text{ GHz} \Rightarrow \lambda = \frac{3 \times 10^8}{3 \times 10^9} = 0.1 \text{ meters}$$

$|\hat{\rho}_w \cdot \hat{\rho}_r|^2 = 1$ dB $\Rightarrow |\hat{\rho}_w \cdot \hat{\rho}_r|^2 = 0.7943$

$$\frac{P_r}{P_t} = 0.85\lambda^2 \cdot \frac{(31.6228)^2}{4\pi} \cdot \left(\frac{\lambda}{4\pi \times 10^6 \lambda^2}\right)^2 \cdot (0.7943)$$

$$= \frac{0.85(31.6228)^2(0.7943)}{(4\pi)^3(10^{12})} = 0.3402 \times 10^{-12}$$

$P_r = 0.3402 \times 10^{-12}(10^2) = 0.3402 \times 10^{-10} = 34.02 \times 10^{-12}$ watts

$P_r = 34.02$ $p$watts

2-98. $T_a = T_A e^{-2\alpha l} + T_0(1 - e^{-2\alpha l})$

$T_A = 5°\text{K}$

$$T_0 = 72°\text{F} = \frac{5}{9}(72 - 32) + 273 = 295.2°\text{K}$$

$-4 \text{ dB} = 20 \log_{10} e^{-\alpha} = -\alpha(20) \log_{10} e = -\alpha(20)(0.434)$

$$\alpha = \frac{4}{8.68} = 0.460 \text{ Nepers/100 ft} = 0.0046 \text{ Nepers/ft.}$$

a. $l = 2$ feet;

$$T_a = 5e^{-2(0.0046)2} + 295.2[1 - e^{-2(0.0046)2}] = 4.91 + 5.38 = 10.29°\text{K}$$

b. $l = 100$ feet;

$$T_a = 5e^{-2(0.0046)100} + 295.2[1 - e^{-2(0.0046)100}] = 179.72°\text{K}$$

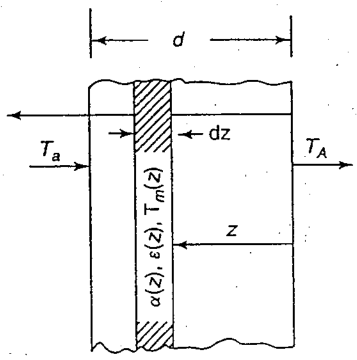
2-99. $T_a = T_A e^{-\int_0^d 2\alpha(z)\,dz} + \int_0^d \epsilon(z) T_m(z) e^{-\int_z^d 2\alpha(z')\,dz'}\,dz$

If $\alpha(z) = \alpha_0$ = Constant

$$T_a = T_A e^{-2\alpha_0 d} + \int_0^d \epsilon(z) T_m(z) e^{-2\alpha_0 (d-z)}\,dz$$

$$T_a = T_A e^{-2\alpha_0 d} + e^{-2\alpha_0 d} + \int_0^d \epsilon(z) T_m(z) e^{+2\alpha_0 z}\,dz$$

If $T_m(z) = T_0 =$ Constant and $\epsilon(z) = \epsilon_0 =$ constant



$$T_a = T_A e^{-2\alpha_0 d} + \epsilon_0 T_0 e^{-2\alpha_0 d} \int_0^d e^{2\alpha_0 z}\, dz$$

$$T_a = T_A e^{-2\alpha_0 d} + \frac{\epsilon_0}{2\alpha_0} T_0 e^{-2\alpha_0 d}(e^{2\alpha_0 d} - 1)$$

For $\epsilon_0 = 2\alpha_0$

$$T_a = T_A e^{-2\alpha_0 d} + T_0 e^{-2\alpha_0 d}(e^{2\alpha_0 d} - 1)$$
$$= T_A e^{-2\alpha_0 d} + T_0(1 - e^{-2\alpha_0 d})$$

CHAPTER 3

3-1.
$$\text{If } \underline{H}_e = j\omega\epsilon\nabla\times\underline{\pi}_e \qquad (1)$$

Maxwell's curl equation $\nabla\times\underline{E}_e = -j\omega\mu\underline{H}_e$ can be written as

$$\nabla\times\underline{E}_e = -j\omega\mu\underline{H}_e = -j\omega\mu(j\omega\epsilon\nabla\times\underline{\pi}_e) = \omega^2\mu\epsilon\nabla\times\underline{\pi}_e$$

or

$$\nabla\times(\underline{E}_e - \omega^2\mu\epsilon\underline{\pi}_e) = \nabla\times(\underline{E}_e - k^2\pi_e) = 0 \qquad \text{where } k^2 = \omega^2\mu\epsilon$$

Letting

$$\underline{E}_e - k^2\underline{\pi}_e = -\nabla\phi_e \Rightarrow \underline{E}_e = -\nabla\phi_e + k^2\underline{\pi}_e \qquad (2)$$

Taking the curl of (1) and using the vector identity of Equation (3–8) leads to

$$\nabla\times\underline{H}_e = j\omega\epsilon\nabla\times\nabla\times\underline{\pi}_e = j\omega\epsilon[\nabla(\nabla\cdot\underline{\pi}_e) - \nabla^2\underline{\pi}_e] \qquad (3)$$

Using Maxwell's equation

$$\nabla\times\underline{H}_e = \underline{J} + j\omega\epsilon\underline{E}_e$$

reduces (3) to

$$\underline{J} + j\omega\epsilon\underline{E}_e = j\omega\epsilon[\nabla(\nabla\cdot\underline{\pi}_e) - \nabla^2\underline{\pi}_e] \qquad (4)$$

Substituting (2) into (4) reduces to

$$\nabla^2\underline{\pi}_e + k^2\underline{\pi}_e = j\frac{\underline{J}}{\omega\epsilon} + [\nabla(\nabla\cdot\underline{\pi}_e) + \nabla\phi_e] \qquad (5)$$

Letting $\phi_e = -\nabla\cdot\underline{\pi}_e$ simplifies 5 to

$$\nabla^2\underline{\pi}_e + k^2\underline{\pi}_e = j\frac{\underline{J}}{\omega\epsilon} \qquad (6)$$

and (2) to

$$\underline{E}_e = \nabla(\nabla\cdot\underline{\pi}_e) + k^2\underline{\pi}_e \qquad (7)$$

Comparing (6) with (3–14) leads to the relation

$$\underline{\pi}_e = -j\frac{1}{\omega\mu\epsilon}\underline{A} \tag{8}$$

3-2. If $\underline{E}_m = -j\omega\mu\nabla \times \underline{\pi}_m$ (1)

Maxwell's curl equation $\nabla \times \underline{H}_m = j\omega\epsilon\underline{E}_m$ can be written as

$$\nabla \times \underline{H}_m = j\omega\epsilon(-j\omega\mu\nabla \times \underline{\pi}_m) = \omega^2\mu\epsilon\nabla \times \underline{\pi}_m$$

or

$$\nabla \times (\underline{H}_m - \omega^2\mu\epsilon\underline{\pi}_m) = \nabla \times (\underline{H}_m - k^2\underline{\pi}_m) = 0 \quad \text{where } k^2 = \omega^2\mu\epsilon$$

Letting

$$\underline{H}_m - k^2\underline{\pi}_m = -\nabla\phi_m \Rightarrow \underline{H}_m = -\nabla\phi_m + k^2\underline{\pi}_m \tag{2}$$

Taking the curl of (1) and using the vector identity of Equation (3–8) leads to

$$\nabla \times \underline{E}_m = -j\omega\mu\nabla \times \nabla \times \underline{\pi}_m = -j\omega\mu[\nabla(\nabla \cdot \underline{\pi}_m) - \nabla^2\underline{\pi}_m] \tag{3}$$

Using Maxwell's equation

$$\nabla \times \underline{E}_m = -\underline{M} - j\omega\mu\underline{H}_m \quad \text{reduces (3) to}$$

$$-\underline{M} - j\omega\mu\underline{H}_m = -j\omega\mu[\nabla(\nabla \cdot \underline{\pi}_m) - \nabla^2\underline{\pi}_m] \tag{4}$$

Substituting (2) into (4) reduces to

$$\nabla^2\underline{\pi}_m + k^2\underline{\pi}_m = j\frac{\underline{M}}{\omega\mu} + [\nabla(\nabla \cdot \underline{\pi}_m) + \nabla\phi_m] \tag{5}$$

Letting $\phi_m = -\nabla \cdot \underline{\pi}_m$ simplifies (5) to

$$\nabla^2\underline{\pi}_m + k^2\underline{\pi}_m = j\frac{\underline{M}}{\omega\mu} \tag{6}$$

and (2) to

$$\underline{H}_m = \nabla(\nabla \cdot \underline{\pi}_m) - k^2\underline{\pi}_m \tag{7}$$

Comparing (6) with (3–25) leads to the relation

$$\underline{\pi}_m = -j\frac{1}{\omega\mu\epsilon}\underline{F}$$

3-3.

$$\underline{A} = \hat{a}_z A_{z1} = \hat{a}_z C_1 \frac{e^{-jkr}}{r}$$

Substituting the above into (3-34) leads to the following terms:

$$\frac{d^2 A_{z1}}{dr^2} = \frac{d}{dr}\left[\frac{d}{dr}\left(C_1 \frac{e^{-jkr}}{r}\right)\right] = C_1 \frac{d}{dr}\left[-jk\frac{e^{-jkr}}{r} - \frac{e^{-jkr}}{r^2}\right]$$

$$= C_1\left[(-jk)^2\frac{e^{-jkr}}{r} + jk\frac{e^{-jkr}}{r^2} - jk\frac{e^{-jkr}}{r^2} + 2\frac{e^{-jkr}}{r^3}\right]$$

$$\frac{2}{r}\frac{dA_{z1}}{dr} = \frac{2}{r}C_1\left(-jk\frac{e^{-jkr}}{r} - \frac{e^{-jkr}}{r^2}\right)$$

$$k^2 A_{z1} = k^2 C_1 \frac{e^{-jkr}}{r}$$

The sum of the above three terms is equal to zero, and it satisfies (3–34) The same conclusion is derived using

$$\underline{A} = \hat{a}_z A_{z2} = \hat{a}_z C_2 \frac{e^{-jkr}}{r}$$

as a solution

3-4. The solution of $\nabla^2 A_z = -\mu J_z$ can be inferred from the solution of Poisson's equation

$$\nabla^2 \phi = -\frac{\rho}{\epsilon} \tag{1}$$

for the potential ϕ. $\rho(x', y', z')$ represents the charge density

We begin with Green's theorem

$$\int_v (\psi \nabla^2 \phi - \phi \nabla^2 \psi)\, dv' = \oint_\Sigma (\psi \nabla \phi - \phi \nabla \psi) \cdot \hat{n}\, da \tag{2}$$

where ψ and ϕ are well behaved functions (nonsingular, continuous, and twice differentiable). For ψ we select a solution of the form

$$\psi = \frac{1}{R} \tag{3}$$

where

$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \tag{3a}$$

By considering the charge at the origin of the coordinate system, it can be shown that (provided $r \neq 0$)

$$\nabla^2 \psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2} = \nabla^2\left(\frac{1}{r}\right) = 0$$

Thus (2) reduces to

$$\int_v \psi \nabla^2 \phi \, dv' = -\frac{1}{\epsilon} \int_v \frac{\rho(x', y', z')}{r} \, dv' \tag{4}$$

To exclude the $r = 0$ singularity of ψ, the observation point x', y', z' is surrounded by a sphere of radius r' and surface Σ'. Therefore the volume V is bounded by the surfaces Σ and Σ', and (2) is broken into two integrals; one over Σ and the other of Σ'. Using (4) reduces (2) to

$$-\frac{1}{\epsilon} \int_v \frac{\rho}{r} \, dv' = \oint_\Sigma (\psi \nabla \phi - \phi \nabla \psi) \cdot \hat{n} \, da + \oint_{\Sigma_0} (\psi \nabla \phi - \phi \nabla \psi) \cdot \hat{n} \, da \tag{5}$$

and

$$\oint_{\Sigma'} (\psi \nabla \phi - \phi \nabla \psi) \cdot \hat{n} \, da = \oint_{\Sigma'} \left[\frac{1}{r'} \nabla \phi - \phi (\nabla \psi)_{r=r'} \right] \cdot \hat{n} \, da$$

$$= -\frac{1}{r'} \oint_{\Sigma'} \frac{\partial \phi}{\partial r} \, da - \frac{1}{r'^2} \oint_{\Sigma'} \phi \, da \tag{5a}$$

Since r' is arbitrary, it can be chosen small enough so that ϕ and $\dfrac{\partial \phi}{\partial r}$ are essentially constant at every point on Σ'. If we make r' progressively smaller, ϕ and its normal derivative approach their limiting values at the center (by hypothesis, both exist and are continuous functions of position). Therefore in the limit as $r' \to 0$, both can be taken outside the integral and we can write that

$$\oint_{\Sigma'} (\psi \nabla \phi - \phi \nabla \psi) \cdot \hat{n} \, da = -4\pi \phi(x, y, z) \tag{6}$$

Since

$$\lim_{r' \to 0} \frac{1}{r'} \oint_{\Sigma'} \frac{\partial \phi}{\partial r} \, da = \lim_{r' \to 0} \frac{1}{r'} \left(\frac{\partial \phi}{\partial r} \right)_{r=r'} \oint_{\Sigma'} da = \lim_{r' \to 0} \frac{1}{r'} \left(\frac{\partial \phi}{\partial r} \right)_{r=r'} (4\pi r'^2) = 0$$

Substituting (6) into (5) reduces it to

$$\phi(x, y, z) = \frac{1}{4\pi\epsilon} \int_v \frac{\rho}{r} \, dv' + \frac{1}{4\pi} \oint_\Sigma \left[\frac{1}{r} \nabla \phi - \phi \nabla \left(\frac{1}{r} \right) \right] \cdot \hat{n} \, da \tag{7}$$

The first term on the right side of (7) accounts for the contributions from the charges within Σ while the second term for those outside Σ. Expansion of Σ to include all charges makes the second term to vanish and to reduce (7) to

$$\phi(x, y, z) = \frac{1}{4\pi\epsilon} \int_v \frac{\rho(x', y', z')}{r} \, dv' \tag{8}$$

By comparing $\nabla^2 A_z = -\mu J_z$ with (1), we can write that

$$A_z(x, y, z) = \frac{\mu}{4\pi} \int_v \frac{J_z(x', y', z')}{r} \, dv' \tag{9}$$

For more details see D.T. Paris and F.K. Hard, *Basic Electromagnetic Theory*, McGraw-Hill, 1969, pp. 128–131.

For the details of the solution of (3-31) see R.E. Collin, *Field Theory of Guided Waves*, McGraw-Hill, 1960, pp. 35–39. It can be shown that

$$A_z = \frac{\mu}{4\pi}\int_v J_z(x', y', z')\frac{e^{-jkr}}{r}\,dv$$

Because of the length of the derivation, it will not be repeated here.

3-5. $\underline{A} \simeq [\hat{a}_r A'_r(\theta,\phi) + \hat{a}_\theta A'_\theta(\theta,\phi) + \hat{a}_\theta A'_\phi(\theta,\phi)]\frac{e^{-jkr}}{r}$

$\underline{E} = -j\omega\underline{A} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla\cdot\underline{A})$

$$\psi = \nabla\cdot\underline{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) - \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\psi = \nabla\cdot\underline{A} = -jk\frac{e^{-jkr}}{r} + \frac{e^{-jkr}}{r^2}[\cdots\cdots] + \frac{e^{-jkr}}{r^3}[\cdots\cdots] + \cdots$$

$$\nabla(\nabla\cdot\underline{A}) = \nabla\psi = \hat{a}_r\frac{\partial\psi}{\partial r} + \hat{a}_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{a}_\phi\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$= \hat{a}_r\left\{\frac{1}{r}[-\omega^2\mu\epsilon e^{-jkr}A'_r(\theta,\phi)] + \frac{1}{r^2}[\cdots] + \frac{1}{r^3}[\cdots] + \cdots\right\}$$

$$+ \hat{a}_\theta\left\{\frac{1}{r}(0) + \frac{1}{r^2}[\cdots] + \frac{1}{r^3}[\cdots] + \cdots\right\}$$

$$+ \hat{a}_\phi\left\{\frac{1}{r}(0) + \frac{1}{r^2}[\cdots] + \frac{1}{r^3}[\cdots] + \cdots\right\}$$

Therefore

$$\underline{E} = -j\omega\underline{A} - j\frac{1}{\omega\mu\epsilon}\nabla(\nabla\cdot\underline{A})$$

$$\underline{E} \simeq -j\omega[\hat{a}_r A'_r + \hat{a}_\theta A'_\theta + \hat{a}_\phi A'_\varphi]\frac{e^{-jkr}}{r}$$

$$-j\frac{1}{\omega\mu\epsilon}\left\{\hat{a}_r\left[\omega^2\mu\epsilon\frac{e^{-jkr}}{r} + \frac{1}{r^2}(\cdots) + \frac{1}{r^3}(\cdots) + \cdots\right]\right.$$

$$+ \hat{a}_\theta\left[\frac{1}{r}(0) + \frac{1}{r^2}(\cdots) + \frac{1}{r^3}(\cdots) + \cdots\right]$$

$$\left. + \hat{a}_\phi\left[\frac{1}{r}(0) + \frac{1}{r^2}(\cdots) + \frac{1}{r^3}(\cdots) + \cdots\right]\right\}$$

or

$$\underline{E} \simeq \frac{1}{r}\{-j\omega e^{-jkr}[\hat{a}_r(0) + \hat{a}_\theta A'_\theta + \hat{a}_\phi A'_\phi]\} + \frac{1}{r^2}[\cdots] + \frac{1}{r^3}[\cdots]\cdots$$

In a similiar manner, it can be shown that

$$\underline{H} = \frac{1}{\mu}\nabla \times \underline{A} = \frac{1}{r}\left\{j\frac{\omega}{\eta}e^{-jkr}[\hat{a}_r(0) + \hat{a}_\theta A'_\phi - \hat{a}_\phi A'_\theta]\right\} + \frac{1}{r^2}[\cdots] + \frac{1}{r^3}[\cdots] + \cdots$$

3-6. Let us assume that within a linear and isotropic medium, but not necessarily homogeneous, there exist two sets of sources $\underline{J}_1$, $\underline{M}_1$ and $\underline{J}_2$, $\underline{M}_2$ which are allowed to radiate simultaneously or individually inside the same medium at the same frequency and produce $\underline{E}_1$, $\underline{H}_1$ and $\underline{E}_2$, $\underline{H}_2$, respectively. For the fields to be valid, they must satisfy Maxwell's equations

$$\nabla \times \underline{E}_1 = -\dot{z}\underline{H}_1 - \underline{M}_1 \quad (1)$$
$$\nabla \times \underline{H}_1 = \dot{y}\underline{E}_1 + \underline{J}_1 \quad (2)$$
$$\nabla \times \underline{E}_2 = -\dot{z}\underline{H}_2 - \underline{M}_2 \quad (3)$$
$$\nabla \times \underline{H}_2 = \dot{y}\underline{E}_2 + \underline{J}_2 \quad (4)$$
$$\text{where } \dot{z} = j\omega(\mu' - j\mu'') \quad (5)$$
$$\dot{y} = \sigma + j\omega(\epsilon' - j\epsilon'') \quad (6)$$

If we dot multiply (1) by $\underline{H}_2$ and (4) by $\underline{E}_1$, we can write

$$\underline{H}_2 \cdot \nabla \times \underline{E}_1 = -\dot{z}\underline{H}_2 \cdot \underline{H}_1 - \underline{H}_2 \cdot \underline{M}_1 \quad (7)$$
$$\underline{E}_1 \cdot \nabla \times \underline{H}_2 = \dot{y}\underline{E}_1 \cdot \underline{E}_2 + \underline{E}_1 \cdot \underline{J}_2 \quad (8)$$

Subtracting (7) from (8) reduces to

$$\underline{E}_1 \cdot \nabla \times \underline{H}_2 - \underline{H}_2 \cdot \nabla \times \underline{E}_1 = \dot{y}\underline{E}_1 \cdot \underline{E}_2 + \dot{z}\underline{H}_2 \cdot \underline{H}_1 + \underline{E}_1 \cdot \underline{J}_2 + \underline{H}_2 \cdot \underline{M}_1 \quad (9)$$

which by using the vector identity

$$\nabla \cdot (\underline{A} \times \underline{B}) = \underline{B} \cdot (\nabla \times \underline{A}) - \underline{A} \cdot (\nabla \times \underline{B}) \quad (10)$$

can be written as

$$\nabla \cdot (\underline{H}_2 \times \underline{E}_1) = -\nabla \cdot (\underline{E}_1 \times \underline{H}_2) = \dot{y}\underline{E}_1 \cdot \underline{E}_2 + \dot{z}\underline{H}_2 \cdot \underline{H}_1 + \underline{E}_1 \cdot \underline{J}_2 + \underline{H}_2 \cdot \underline{M}_1 \quad (11)$$

In a similar manner, if we dot multiply (2) by $\underline{E}_2$ and (3) by $\underline{H}_1$, we can write

$$\underline{E}_2 \cdot \nabla \times \underline{H}_1 = \dot{y}\underline{E}_2 \cdot \underline{E}_1 + \underline{E}_2 \cdot \underline{J}_1 \quad (12)$$
$$\underline{H}_1 \cdot \nabla \times \underline{E}_2 = -\dot{z}\underline{H}_1 \cdot \underline{H}_2 - \underline{M}_1 \cdot \underline{M}_2 \quad (13)$$

Substracting (13) from (12) leads to

$$\underline{E}_2 \cdot \nabla \times \underline{H}_1 - \underline{H}_1 \cdot \nabla \times \underline{E}_2 = \dot{y}\underline{E}_2 \cdot \underline{E}_1 + \dot{z}\underline{H}_1 \cdot \underline{H}_2 + \underline{E}_2 \cdot \underline{J}_1 + \underline{H}_1 \cdot \underline{M}_2 \quad (14)$$

which by using (10) can be written as

$$\nabla \cdot (\underline{H}_1 \times \underline{E}_2) = -\nabla \cdot (\underline{E}_2 \times \underline{H}_1) = \dot{y}\underline{E}_2 \cdot \underline{E}_1 + \dot{z}\underline{H}_1 \cdot \underline{H}_2 + \underline{E}_2 \cdot \underline{J}_1 + \underline{H}_1 \cdot \underline{M}_2 \quad (15)$$

Substracting (15) from (11) leads to

$$-\nabla \cdot (\underline{E}_1 \times \underline{H}_2 - \underline{E}_2 \times \underline{H}_1) = \underline{E}_1 \cdot \underline{J}_2 + \underline{H}_2 \cdot \underline{M}_1 - \underline{E}_2 \cdot \underline{J}_1 - \underline{H}_1 \cdot \underline{M}_2 \quad (16)$$

Which is known as the *Lorentz Reciprocity Theorem* in differential form. Taking the volume integral of both sides of (16) and using the divergence theorem on the left side, we can write (16) as

$$-\oint\!\!\!\oint (\underline{E}_1 \times \underline{H}_2 - \underline{E}_2 \times \underline{H}_1) \cdot \mathrm{ds} = \iiint_v (\underline{E}_1 \cdot \underline{J}_2 + \underline{H}_2 \cdot \underline{M}_1 - \underline{E}_2 \cdot \underline{J}_1 - H_1 \cdot M_2)\, dv \quad (17)$$

Which is known as the Lorentz Reciprocity Theorem in integral form.

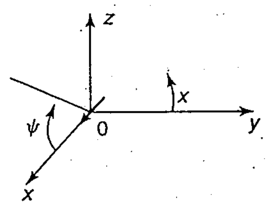
CHAPTER 4

4-1. a. $\sin\psi = \sqrt{1-\cos^2\psi} = \sqrt{1-|\hat{a}_z\cdot\hat{a}_r|^2}$
$= \sqrt{1-(\sin\theta\cdot\cos\phi)^2}$

In far-zone fields

$$E_\psi = j\eta\frac{kI_0\cdot le^{-jkr}}{4\pi r}\cdot\sin\psi = j\eta\frac{k\cdot I_0\cdot le^{-jkr}}{4\pi r}\cdot\sqrt{1-(\sin\theta\cdot\cos\phi)^2}$$

$$H_\chi = j\frac{kI_0le^{-jkr}}{4\pi r}\cdot\sin\psi = \frac{E_\psi}{\eta}$$



b. $U = U_0(1-\sin^2\theta\cos^2\phi)$

$$\therefore\ P_{\text{rad}} = U_0\int_0^{2\pi}\int_0^{\pi}(1-\sin^2\theta\cdot\cos^2\phi)\cdot\sin\theta\, d\theta\, d\phi = U_0\cdot\frac{8\pi}{3}$$

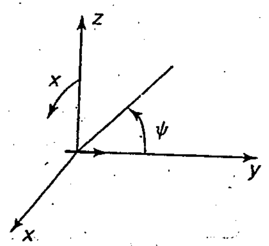
$$D_0 = \frac{4\pi\cdot U_0}{U_0\cdot\frac{8\pi}{3}} = \frac{3}{2} = 1.5$$

4-2. a. $\sin\psi = \sqrt{1-\cos^2\psi} = \sqrt{1-|\hat{a}_y\cdot\hat{a}_r|^2}$
$= \sqrt{1-\sin^2\theta\cdot\sin^2\phi}$

In far-zone, the fields are

$$E_\psi = j\eta \frac{kI_0 l e^{-jkr}}{4\pi r} \cdot \sin\psi = j\eta \frac{kI_0 l e^{-jkr}}{4\pi r} \cdot \sqrt{1 - \sin^2\theta \cdot \sin^2\phi}$$

$$H_\chi \simeq \frac{E_\psi}{\eta} \simeq j \frac{kI_0 e^{-jkr} \cdot l}{4\pi r} \sqrt{1 - \sin^2\theta \sin^2\phi}$$



b. $U = U_0(1 - \sin^2\theta \sin^2\phi)$

$$P_{\text{rad}} = U_0 \int_0^{2\pi} \int_0^{\pi} (1 - \sin^2\theta \cdot \sin^2\phi) \sin\theta \, d\theta \, d\phi$$

$$= U_0 \int_0^{2\pi} \left[\int_0^{\pi} \sin\theta - \sin^3\theta \cdot \sin^2\phi \, d\theta \right] d\phi$$

$$= U_0 \left[\int_0^{2\pi} 2 \, d\phi - \frac{4}{3} \int_0^{2\pi} \sin^2\phi \, d\phi \right] = U_0 \left[4\pi - \frac{4}{3}\pi \right] = \frac{8}{3}\pi \cdot U_0$$

$$D_0 = \frac{4\pi \cdot U_0}{U_0 \cdot \frac{8\pi}{3}} = \frac{3}{2} = 1.5$$

4-3. (a) $\underline{A} = \frac{\mu}{4\pi} \int I_e \frac{e^{-jKR}}{R} \, dl' = \frac{\mu}{4\pi} \int_{-l/2}^{+l/2} \hat{a}_x I_0 \frac{e^{-jkr}}{r} \, dx' = \hat{a}_x \frac{I_0 \mu}{4\pi} \frac{e^{-jkr}}{4\pi r} \int_{-l/2}^{+l/2} dx'$

$$\underline{A} = \hat{a}_x \frac{l\mu I_0 e^{-jkr}}{4\pi r} = \hat{a}_x A_x \Rightarrow A_x = \frac{\mu I_0 l}{4\pi r} e^{-jkr}$$

$$\begin{pmatrix} A_r \\ A_\theta \\ A_\phi \end{pmatrix} = \begin{pmatrix} l \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} A_x \\ 0 \\ 0 \end{pmatrix} \qquad (4\text{-}5)$$

$$A_r = A_x \sin\theta\cos\phi = \frac{\mu I_0 l e^{-jkr}}{4\pi r} \sin\theta\cos\phi$$

$$A_\theta = A_x \cos\theta\cos\phi = \frac{\mu I_0 l e^{-jkr}}{4\pi r} \cos\theta\cos\phi$$

$$A_\phi = -A_x \sin\phi = -\frac{\mu I_0 l e^{-jkr}}{4\pi r} \sin\phi$$

In far-field:

$$\left.\begin{aligned} E_r &\simeq 0 \\ E_\theta &\simeq -j\omega A_\theta \\ E_\phi &\simeq -j\omega A_\phi \end{aligned}\right\} \begin{matrix}(3\text{-}58\text{a}) \\ (3\text{-}58\text{b})\end{matrix} \Rightarrow \begin{aligned} E_r &\simeq 0 \\ E_\theta &= -j\frac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\cos\theta\cos\phi \\ E_\phi &= -j\frac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\sin\phi \end{aligned} \left\} \begin{aligned} H_r &\simeq 0 \\ H_\phi &= \frac{E_\theta}{\eta} \\ H_\theta &= -\frac{E_\phi}{\eta} \end{aligned}\right.$$

(b) $U = \frac{r^2}{2\eta}[|E_\theta|^2 + |E_\phi|^2]$ (2-12a)

$$U = \left(\frac{\omega\mu I_0 l}{4\pi}\right)^2 \frac{1}{2\eta}[\cos^2\theta\cos^2\phi + \sin^2\phi]$$

$$= B_0[\cos^2\theta\cos^2\phi + \sin^2\phi] \quad \left(\underline{\text{see 3-D plot}}\right)$$

$$B_0 = \frac{1}{2\eta}\left(\frac{\omega\mu I_0 l}{4\pi}\right)^2 = \frac{1}{2\eta}\left(\frac{\eta\omega\mu I_0 l}{\eta 4\pi}\right)^2 = \frac{1}{2\eta}\left[\frac{\eta\omega\mu I_0 l}{4\pi\sqrt{\mu/\epsilon}}\right]^2$$

$$= \frac{1}{2\eta}\left[\frac{\eta\omega\sqrt{\mu\epsilon}}{4\pi}I_0 l\right] = \frac{1}{2\eta}\left[\frac{\eta k I_0 l}{4\pi}\right] = \frac{\eta^2}{2\eta}\left(\frac{kI_0 l}{4\pi}\right)^2 = \frac{\eta}{2}\left(\frac{kI_0 l}{4\pi}\right)^2$$

$$B_0 = \frac{\eta}{2}\left(\frac{kI_0 l}{4\pi}\right)^2$$

$$U = B_0(\cos^2\theta\cos^2\phi + \sin^2\phi) \Rightarrow u_{\max} = B_0 \text{ when } \phi = 90°, 270°$$
$$0 \leqslant \theta \leqslant 180°$$

$$P_{\text{rad}} = \int_0^{2\pi}\int_0^\pi U\sin\theta\, d\theta\, d\phi$$

$$= B_0\left\{\int_0^{2\pi}\int_0^\pi \underbrace{\cos^2\theta\cos^2\phi\sin\theta\, d\theta\, d\phi}_{I_1} + \int_0^{2\pi}\int_0^\pi \underbrace{\cos^2\phi\sin\theta\, d\theta\, d\phi}_{I_2}\right\}$$

$$I_1 = \int_0^{2\pi}\cos^2\phi\, d\phi \int_0^\pi \cos^2\theta\sin\theta\, d\theta = \int_0^{2\pi}\cos^2\phi\, d\phi\int_0^\pi \cos^2\theta\, d(-\cos\theta)$$

$$= -\int_0^{2\pi}\left(\frac{1+\cos(2\phi)}{2}\right)d\phi\int_0^\pi(\cos\theta)^2\, d(\cos\theta)$$

$$= -\tfrac{1}{2}\left[\left(\phi + \tfrac{1}{2}\sin 2\phi\right)_0^{2\pi}\right]\left[\frac{\cos^3\theta}{3}\right]_0^\pi$$

$$I_1 = -\tfrac{1}{2}[(2\pi)]\left(-\frac{1}{3} - \frac{1}{3}\right) = \tfrac{1}{2}(2\pi)\left(\frac{2}{3}\right) = \frac{2\pi}{3}$$

$$I_2 = \int_0^{2\pi}\int_0^\pi \cos^2\phi\sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\cos^2\phi\, d\phi\int_0^\pi \sin\theta\, d\theta$$

$$= \int_0^{2\pi}\left(\frac{1+\cos(2\phi)}{2}\right)d\phi\int_0^\pi \sin\theta\, d\theta$$

$$I_2 = \tfrac{1}{2}\left[\pi + \tfrac{1}{2}\sin 2\phi\right]_0^{2\pi}(-\cos\theta)_0^{\pi} = \tfrac{1}{2}(2\pi)[-(-1)+1] = 2\pi$$

$$I_1 + I_2 = \frac{2\pi}{3} + 2\pi = \frac{8\pi}{3}$$

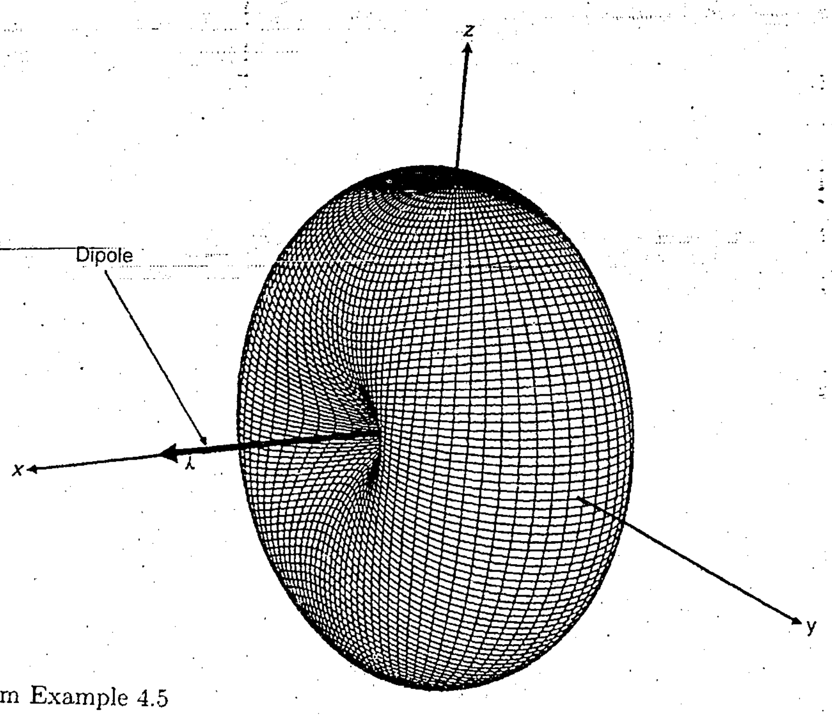
$$P_{\text{rad}} = B_0(I_1 + I_2) = B_0\left(\frac{8\pi}{3}\right)$$

$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{4\pi(B_0)}{\frac{8\pi}{3}(B_0)} = \frac{3}{2} = 1.761 \text{ dB}$$

$$\boxed{D_0 = 1.5 = +1.761 \text{ dB}}$$

c. Computer Program *Directivity*:

$$\boxed{D_0 = 1.4980 = 1.7551 \text{ dB}}$$



4-4. From Example 4.5

$$E_r \simeq 0$$

$$E_\theta \simeq -j\omega A_\theta = -j\frac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\cos\theta\sin\phi$$

$$E_\phi \simeq -j\omega A_\phi = -j\frac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\cos\phi$$

(a) $$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}}$$

$$U(\theta,\phi) = \frac{r^2}{2\eta}[|E_\theta|^2 + |E_\phi|^2] = \frac{1}{2\eta}\left(\frac{\omega\mu I_0 l}{4\pi}\right)^2[\cos^2\theta\sin^2\phi + \cos^2\phi]$$

$$U(\theta,\phi) = \frac{1}{2\eta}\left(\frac{\eta\omega\mu I_0 l}{4\pi\sqrt{\mu/\varepsilon}}\right)^2[\cos^2\theta\sin^2\phi + \cos^2\phi] = B_0(\cos^2\theta\sin^2\phi + \cos^2\phi)$$

$$B_0 = \frac{1}{2\eta}\left(\frac{\eta\omega\sqrt{\mu\varepsilon}I_0 l}{4\pi}\right)^2 = \frac{\eta}{2}\left(\frac{kI_0 l}{4\pi}\right)^2$$

$$U(\theta,\phi) = B_0(\cos^2\theta\sin^2\phi + \cos^2\phi) \Rightarrow U_{\max} = B_0 @ \phi = 0°, 180°$$

(b) $$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi = B_0\int_0^{2\pi}\int_0^{\pi}(\cos^2\theta\sin^2\phi + \cos^2\phi)\sin\theta\, d\theta\, d\phi$$

$$= B_0\left\{\underbrace{\int_0^{2\pi}\int_0^{\pi}\cos^2\theta\sin^2\phi\sin\theta\, d\theta\, d\phi}_{I_1} + \underbrace{\int_0^{2\pi}\int_0^{\pi}\cos^2\phi\sin\theta\, d\theta\, d\phi}_{I_2}\right\}$$

$$I_1 = \int_0^{2\pi}\sin^2\phi\, d\phi\int_0^{\pi}\cos^2\theta\sin\theta\, d\theta$$

$$= \int_0^{2\pi}\left(\frac{1-\cos 2\phi}{2}\right)d\phi\int_0^{\pi}\cos^2\theta\, d(-\cos\theta)$$

$$= -\tfrac{1}{2}\left[\phi - \tfrac{1}{2}\sin 2\phi\right]_0^{2\pi}\left(\frac{\cos^3\theta}{3}\right)_0^{\pi} = -\tfrac{1}{2}(2\pi)\left(\frac{-1-1}{3}\right) = \frac{2\pi}{3}$$

$$I_2 = \int_0^{2\pi}\int_0^{\pi}\cos^2\phi\sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\left(\frac{1+\cos 2\phi}{2}\right)d\phi\int_0^{\pi}\sin\theta\, d\theta$$

$$= \tfrac{1}{2}\left[\phi + \tfrac{1}{2}\sin 2\phi\right]_0^{2\pi}(-\cos\theta)_0^{\pi} = \tfrac{1}{2}(2\pi)(2) = 2\pi$$

$$P_{\text{rad}} = B_0(I_1 + I_2) = B_0\left(\frac{2\pi}{3} + 2\pi\right) = B_0\left(\frac{8\pi}{3}\right)$$

(c) $D_0 = \dfrac{4\pi U_{\max}}{P_{\text{rad}}} = \dfrac{4\pi B_0}{\frac{8\pi}{3}B_0} = \dfrac{3}{2}$ (same as in Problem 4-2 or any other infinitesimal dipole)

(d)
```
Input parameters:
-----------------
The lower bound of theta in degrees = 1
The upper bound of theta in degrees = 180
The lower bound of phi in degrees   = 0
The upper bound of phi in degrees   = 360
```

```
Output parameters:
-------------------
Radiated power (watts)        = 8.4122
Directivity (dimensionless) = 1.4938
Directivity (dB)              = 1.7430
```

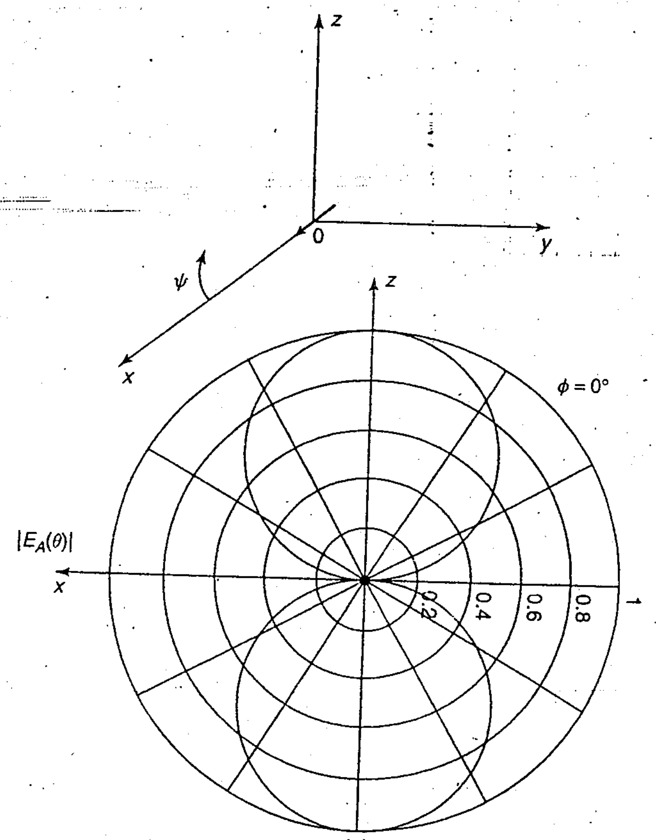
4-5. a. $\phi = 0^\circ$, ($x - z$ plane)

$$E_\psi = j\eta \frac{kI_0 le^{-jkr}}{4\pi r}\sqrt{1-\sin^2\theta}$$

$$\simeq j\eta \frac{kI_0 le^{-jkr}}{4\pi r}\cdot\cos\theta$$

At $\phi = 0^\circ$, $\underline{E}_\psi$ has only $\hat{a}_\theta$ direction.

$\underline{E}_\psi \leftarrow \underline{E}_\theta$ polarization



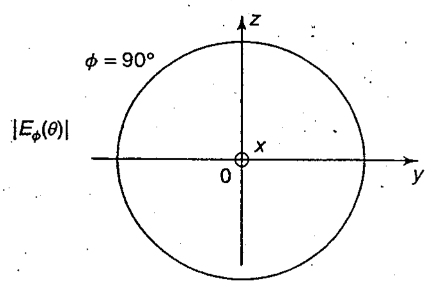
(a)

b. $\phi = 90°$ $(y - z \text{ plane})$

$$E_\psi \simeq j\eta \frac{kI_0 le^{-jkr}}{4\pi r} \cdot 1$$

At $\phi = 90°$, $(y - z$ plane$)$, $\underline{E}_\psi$ has only $\hat{a}_\phi$ direction.

$\underline{E}_\psi \rightsquigarrow \underline{E}_\phi$ polarization



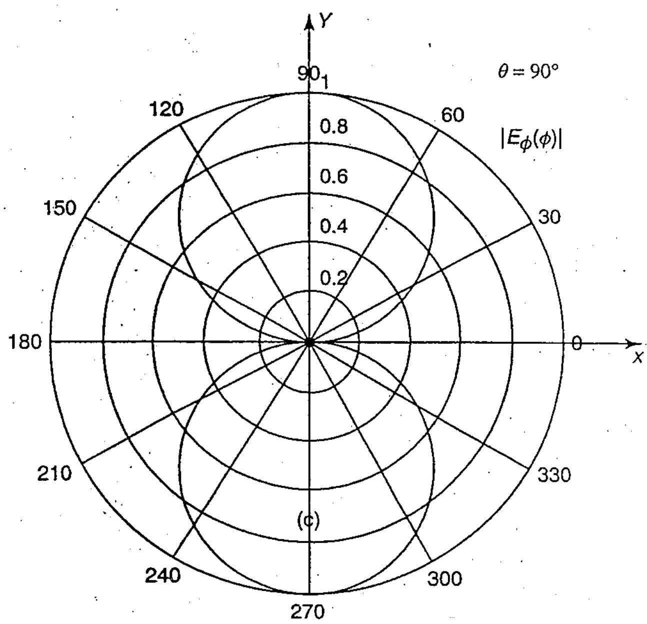
(b)

c. $\theta = 90°$ $(x - y \text{ plane})$

$$E_\psi = j\eta \frac{kI_0 le^{-jkr}}{4\pi r} \sqrt{1 - \cos^2 \phi} = j\eta \frac{kI_0 le^{-jkr}}{4\pi r} \cdot \sin \phi$$

At $\theta = 90°.(x, y)$, $\underline{E}_\psi$ has only $\hat{a}_\phi$ direction.

$\underline{E}_\psi \rightsquigarrow \underline{E}_\phi$ polarization



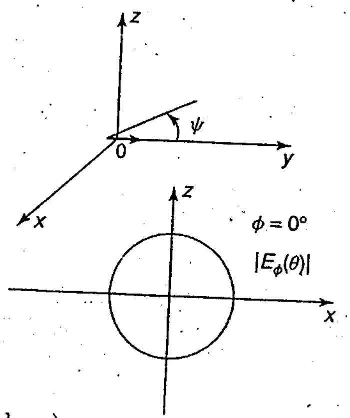
(c)

4-6. a. $\phi = 0°$ ($x - z$ plane)

$$E_\psi = j\eta \frac{kI_0 e^{-jkr}}{4\pi r} \cdot 1$$

At $\phi = 0°$, $\underline{E}_\psi$ direction has only $\hat{a}_\phi$ component

$\underline{E}_\psi \rightsquigarrow \underline{E}_\phi$ polarization



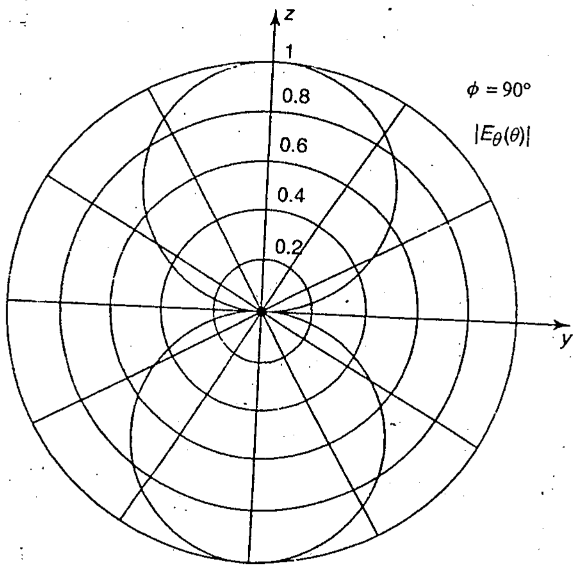
b. $\phi = 90°$ ($y - z$ plane)

$$E_\psi = j\eta \frac{k_0 I_0 l e^{-jkr}}{4\pi r} \cdot \sqrt{1 - \sin^2\theta}$$

$$= j\eta \frac{k_0 I_0 l e^{-jkr}}{4\pi r} \cdot \quad \cos\theta.$$

At $\phi = 90°$, $\underline{E}_\psi$ direction has only $\hat{a}_\theta$ component

$\underline{E}_\psi \rightsquigarrow E_\theta$ polarization

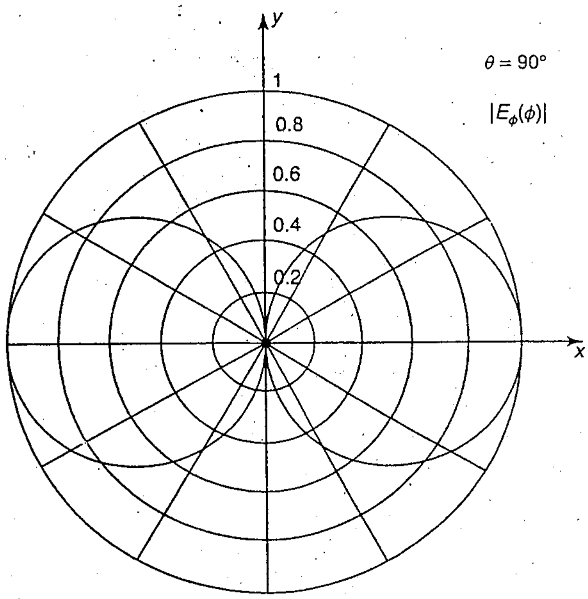


c. $\theta = 90^\circ$ (x, y) plane.

$$E_\psi = j\eta \frac{k_0 I_0 l e^{-jkr}}{4\pi r} \cdot \cos\phi$$

At $\theta = 90^\circ$, $\underline{E}_\psi$ direction has only $\hat{a}_\phi$ component

$\underline{E}_\psi \rightsquigarrow E_\phi$ polarization



4-7. $E_\theta = -j\frac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\cos\theta\cos\phi, E_\phi = -j\frac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\sin\phi$

(a) $\phi = 0$: $E_\theta = -j\frac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\cos\theta, E_\phi = 0$ (same as in Problem 4-5)

(b) $\phi = 90^\circ$: $E_\theta = 0, E_\phi = -j\frac{\omega\mu I_0 l e^{-jkr}}{4\pi r}$ (same as in Problem 4-5)

(c) $\theta = 90^\circ$: $E_\theta = 0, E_\phi = -j\frac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\sin\phi$ (same as in Problem 4-5)

4-8. From Example 4.5

$$E_\theta = -j\frac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\cos\theta\sin\phi$$

$$E_\phi = -j\frac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\cos\phi$$

(a) $\phi = 0$: $E_\theta = 0; E_\phi = -j\dfrac{\omega\mu I_0 l e^{-jkr}}{4\pi r}$ (same as in Problem 4-6)

(b) $\phi = 90°$: $E_\theta = -j\dfrac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\cos\theta$ (same as in Problem 4-6)

(c) $\theta = 90°$: $E_\theta = 0; E_\phi = -j\dfrac{\omega\mu I_0 l e^{-jkr}}{4\pi r}\cos\phi$

4-9. (a) Using (4-26a)–(4-26c) and the duality of Table 3.2, the fields of an infinitesimal magnetic dipole of length l and magnetic current I_m are given by

$$E_r = E_\theta = H_\phi = 0$$

$$E_\phi = -j\frac{kI_m l}{4\pi r}\sin\theta\left[1 + \frac{1}{jkr}\right]e^{-jkr}$$

$$H_r = \frac{I_m l\cos\theta}{2\pi\eta r^2}\left[1 + \frac{1}{jkr}\right]e^{-jkr}$$

$$H_\theta = j\frac{kI_m l}{4\pi\eta r}\sin\theta\left[1 + \frac{1}{jkr} - \frac{1}{(kr)^2}\right]e^{-jkr}$$

(b) Since the pattern of the magnetic dipole is the same as that of the electric, the directivities are also identical and equal to

$$D_0 = \frac{3}{2}(\text{dimensionless}) = 1.761 \text{ dB}$$

4-10. (a) When the element is placed along the x-axis

$$\sin\psi = \sqrt{1 - \cos^2\psi} = \sqrt{1 - |\hat{a}_x \cdot \hat{a}_r|^2}$$
$$= \sqrt{1 - |\hat{a}_x \cdot (\hat{a}_x \sin\theta\cos\phi + \hat{a}_y \sin\theta\sin\phi + \hat{a}_z\cos\theta)|^2}$$

and the fields can be written as

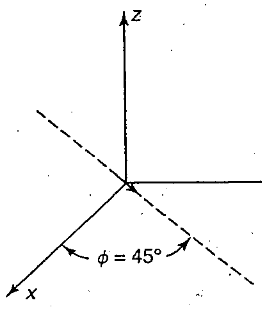
$$E_\chi = -j \cdot \frac{kI_m l e^{-jkr}}{4\pi r}\sqrt{1 - \sin^2\theta\cos^2\phi} = -j \cdot \frac{kI_m l e^{-jkr}}{4\pi r}\sin\psi$$
$$H_\psi = -\frac{E_\chi}{\eta}$$

(b) In a similar manner, when the element is placed along the y-axis

$$\sin\psi = \sqrt{1 - \cos^2\psi} = \sqrt{1 - |\hat{a}_y \cdot \hat{a}_r|^2} = \sqrt{1 - \sin^2\theta\sin^2\phi}$$

and the fields can be written as

$$E_\chi = -j \cdot \frac{kI_m l e^{-jkr}}{4\pi r}\sin\psi = -j \cdot \frac{kI_m l e^{-jkr}}{4\pi r}\sqrt{1 - \sin^2\theta\sin^2\phi}$$
$$H_\psi = -\frac{E_\chi}{\eta}$$



4-11. $E_\psi = j\eta \dfrac{kI_0 l}{4\pi r} e^{-jkr} \sin\psi$

$H_\chi = j\dfrac{kI_0 l}{4\pi r} e^{-jkr} \sin\psi$

Convert ψ to spherical coordinates

$$\sin\psi = 1 - \cos^2\psi = \sqrt{1 - \left(\frac{\hat{a}x + \hat{a}_y}{\sqrt{2}} \cdot \hat{a}_r\right)^2}$$

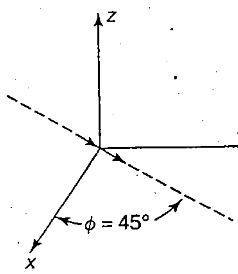
$$\frac{\hat{a}_x + \hat{a}_y}{\sqrt{2}} \cdot \hat{a}_r = \left(\frac{\hat{a}_x}{\sqrt{2}} + \frac{\hat{a}_y}{\sqrt{2}}\right) \cdot (\hat{a}_x \sin\theta\cos\phi + \hat{a}_y \sin\theta\sin\phi + \hat{a}_z\cos\theta)$$

$$= \frac{\sin\theta\cos\phi}{\sqrt{2}} + \frac{\sin\theta\sin\phi}{\sqrt{2}} = \frac{1}{\sqrt{2}}\sin\theta(\cos\phi + \sin\phi)$$

Thus

$$E_\psi = j\eta\frac{kI_0 l}{4\pi r} e^{-jkr}\sqrt{1 - \tfrac{1}{2}[\sin^2\theta(\cos\phi + \sin\phi)^2]}$$

$$H_\chi = j\frac{kI_0 l}{4\pi r} e^{-jkr}\sqrt{1 - \tfrac{1}{2}[\sin^2\theta(\cos\phi + \sin\phi)^2]}$$



4-12. $H_\psi = j\dfrac{kI_m l}{4\pi\eta r}e^{-jkr}\sin\psi$

$E_\chi = -j\dfrac{kI_m l}{4\pi r}e^{-jkr}\sin\psi$

Convert ψ to spherical coordinates

$$\sin\psi = \frac{1}{\sqrt{2}}\sin\theta(\sin\phi+\cos\phi)$$

Thus

$$H_\psi = j\frac{kI_m l}{4\pi\eta r}e^{-jkr}\sqrt{1-\tfrac{1}{2}[\sin^2\theta(\cos\phi+\sin\phi)^2]}$$
$$E_\chi = -j\frac{kI_m l}{4\pi r}e^{-jkr}\sqrt{1-\tfrac{1}{2}[\sin^2\theta(\cos\phi+\sin\phi)^2]}$$

4-13.

$$\underline{E} = \frac{1}{j\omega\epsilon}\nabla\times\underline{H} \quad \text{where } H_r = H_\theta = 0, H_\phi = j\frac{kI_0 l\sin\theta}{4\pi r}\left[1+\frac{1}{jkr}\right]e^{-jkr}$$

Since $\underline{H}$ is not a function of ϕ

$$\underline{E} = \frac{1}{j\omega\epsilon}\nabla\times\underline{H} = \frac{1}{j\omega\epsilon}\left\{\hat{a}_r\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}+(H_\phi\sin\theta)-\hat{a}_\theta\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi)+\hat{a}_\phi(0)\right\}$$

which reduces using the H_ϕ from above to

$$E_r = \eta\frac{I_0 l\cos\theta}{2\pi r^2}\left[1+\frac{1}{jkr}\right]e^{-jkr}$$
$$E_\theta = j\eta\frac{kI_0 l\sin\theta}{4\pi r}\left[1+\frac{1}{jkr}-\frac{1}{(kr)^2}\right]e^{-jkr}$$
$$E_\phi = 0$$

4-14. $\underline{W}_{\text{ave}} = \frac{1}{2}\text{Re}[\underline{E}\times\underline{H}^*] = \frac{1}{2}\text{Re}[\hat{a}_\theta E_\theta\times\hat{a}_\phi H_\phi^*]$

$$\underline{W}_{\text{ave}} = \hat{a}_r W_r = \frac{1}{2}\text{Re}\left[\hat{a}_\theta E_\theta\times\hat{a}_\phi\frac{E_\theta^*}{\eta}\right] = \hat{a}_r\frac{1}{2\eta}\text{Re}(|E_\theta|^2) = \hat{a}_r\frac{|E_\theta|^2}{2\eta}$$
$$W_r = \left[\frac{\eta}{2}\left|\frac{kI_0 l}{4\pi}\right|^2\right]\frac{\sin^2\theta}{r^2} = W_0\frac{\sin^2\theta}{r^2}, \quad \text{where } W_0 = \frac{\eta}{2}\left|\frac{kI_0 l}{4\pi}\right|^2$$
$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi}\underline{W}_{\text{ave}}\hat{a}_r r^2\sin\theta\,d\theta\,d\phi = 2\pi W_0\int_0^{\pi}\sin^3\theta\,d\theta = 2\pi W_0\left(\frac{4}{3}\right)$$
$$P_{\text{rad}} = \frac{8\pi}{3}W_0 = \eta\left(\frac{\pi}{3}\right)\left|\frac{I_0 l}{\lambda}\right|^2$$

4-15. $\underline{A} = \hat{a}_z A_z = \hat{a}_z \frac{\mu I_0 l}{4\pi r} e^{-jkr} \Rightarrow A_z = \frac{\mu_0 I_0 l}{4\pi r} e^{-jkr}$

using (4-6a)–(4-6c)

$$A_r = A_z \cos\theta = \frac{\mu I_0 l e^{-jkr}}{4\pi r} \cos\theta = A_r'(\theta) \frac{e^{-jkr}}{r} \quad \Rightarrow A_r' = \frac{\mu I_0 l}{4\pi} \cos\theta$$
$$A_\theta = -A_z \sin\theta = \frac{-\mu I_0 l e^{-jkr}}{4\pi r} \sin\theta = A_\theta'(\theta) \frac{e^{-jkr}}{r} \Rightarrow A_\theta' = \frac{-\mu I_0 l}{4\pi} \sin\theta$$
$$A_\phi = 0 \qquad \Rightarrow A_\phi' = 0$$

Substituting these into (3-57) and (3-57a) reduces to

$$E_r = 0$$
$$E_\theta = -j\omega \frac{e^{-jkr}}{r} A_\theta' = j \frac{\omega \mu I_0 l e^{-jkr}}{4\pi r} \sin\theta = j\eta \frac{k I_0 l e^{-jkr}}{4\pi r} \sin\theta$$
$$E_\phi = -j\omega \frac{e^{-jkr}}{r} A_\phi' = 0$$
$$H_r = 0$$
$$H_\theta = j \frac{\omega}{\eta} \frac{e^{-jkr}}{r} A_\phi' = 0$$
$$H_\phi = -j \frac{\omega}{\eta} \frac{e^{-jkr}}{r} A_\theta' = j \frac{\omega \mu I_0 l e^{-jkr}}{4\pi \eta r} \sin\theta = j \frac{k I_0 l e^{-jkr}}{4\pi r} \sin\theta$$

which are identical to (4-26a)–(4-26c)

4-16. $R = [r^2 + (-2rz'\cos\theta + z'^2)]^{1/2} = r\left[1 + \left(\frac{-2rz'\cos\theta + z'^2}{r^2}\right)\right]^{1/2}$

Using the binomial expansion of

$$(a+b)^n = \frac{a^n b^0}{0!} + \frac{n a^{n-1} b^1}{1!} + (n)(n-1)\frac{a^{n-2} \cdot b^2}{2!} + (n)(n-1)(n-2)\frac{a^{n-3} b^3}{3!} + \cdots$$

it can be shown by letting

$$a = r^2$$
$$b = (-2rz'\cos\theta + z'^2)$$
$$n = \tfrac{1}{2}$$

that

$$R = r - z'\cos\theta + \frac{1}{r}\left(\frac{z'^2}{2}\sin^2\theta\right) + \frac{1}{r^2}\left(\frac{z'^3}{2}\cos\theta \cdot \sin^2\theta\right)$$
$$+ \frac{1}{r^3}\left[\frac{z'^4}{8}(-1 + 6\cos^2\theta - 5\cos^4\theta)\right] + \cdots$$

Therefore the fifth term of (4-41) is

$$\frac{1}{r^3}\left[\frac{z'^4}{8}(-1+6\cos^2\theta-5\cos^4\theta)\right]$$

4-17. For maximum phase error of $\pi/8$ radians

$$0.62\sqrt{D^3/\lambda}\leqslant r\leqslant 2D^2/\lambda$$

(a) For a maximum phase error of $\pi/16$ radians

$$\sqrt{2(0.385)}\sqrt{D^3/\lambda}\leqslant r\leqslant 4D^2/\lambda$$
$$0.8775\sqrt{D^3/\lambda}\leqslant r\leqslant 4D^2/\lambda$$

(b) For a maximum phase error of $\pi/4$ radians

$$\sqrt{\frac{0.385}{2}}\cdot\sqrt{D^3/\lambda}\leqslant r\leqslant D^2/\lambda$$
$$0.43875\sqrt{D^3/\lambda}\leqslant r\leqslant D^2/\lambda$$

(c) For a maximum phase error of 18° radians

$$18^\circ \rightsquigarrow \frac{\pi}{10}\text{ radians}$$

$$\sqrt{1.25(0.385)}\sqrt{D^3/\lambda}\leqslant r\leqslant (1.25)\cdot 2D^2/\lambda$$
$$0.6937\sqrt{D^3/\lambda}\leqslant r\leqslant 2.5D^2/\lambda$$

(d) For a maximum phase error of 15° radians

$$15^\circ \rightsquigarrow \frac{\pi}{12}\text{ radians.}$$

$$\sqrt{1.5(0.385)}\sqrt{D^3/\lambda}\leqslant r\leqslant (1.5)\cdot 2\cdot D^2/\lambda$$
$$0.7599\sqrt{D^3/\lambda}\leqslant r\leqslant 3\cdot D^2/\lambda$$

4-18. $l=5\lambda_0 \Rightarrow z'|_{\max}=2.5\lambda$

a. Far-Field (Fraunhofer)

$$r=\frac{2l^2}{\lambda}=\frac{2(5\lambda)^2}{\lambda}=\frac{2(25\lambda^2)}{\lambda}=50\lambda$$

$$\Delta\phi_e=\frac{k}{r}\left(\frac{z'^2}{2}\sin^2\theta\right)\Bigg|_{\substack{\theta=30^\circ,z'=2.5\lambda,\\ r=50\lambda}}=\frac{\frac{2\pi}{\lambda}}{50\lambda}\left[\frac{(2.5)^2\lambda^2}{2}\frac{1}{4}\right]=0.0982\text{ rads}=5.6250^\circ$$

b. Fresnel

$$r = 0.62\sqrt{l^3/\lambda} = 0.62\sqrt{(5\lambda)^3/\lambda} = 0.62 \cdot \lambda \cdot \sqrt{125} = 6.9318\lambda$$

$$\Delta\phi_e = \frac{k}{r^2}\left(\frac{z'^3}{2}\cos\theta\sin^2\theta\right)\Bigg|_{\substack{\theta=30^\circ\\ z'=2.5\lambda\\ r=6.9318\lambda}} = \frac{2\pi}{\lambda}\frac{(2.5\lambda)^3}{(6.9318\lambda)^2\cdot 2}(\cos 30^\circ)(\sin 30^\circ)^2$$

$$\Delta\phi_e = \frac{\pi(2.5)^3}{(6.9318)^2}(0.866)(0.5)^2 = 0.2212 \text{ rads} = 12.6724^\circ$$

4-19. $$\underline{A} = \hat{a}_z\frac{\mu I_0}{4\pi}\int_0^l e^{-jkz'}\frac{e^{-jkR}}{R}\,dz' \cong \hat{a}_z\frac{\mu I_0}{4\pi r}e^{-jkr}\int_0^l e^{-jk(1-\cos\theta)z'}\,dz'$$

$$A_z \simeq \frac{\mu I_0 e^{-jkr}}{4\pi r}\int_0^l \frac{e^{-jk(1-\cos\theta)z'}\,d[-jk(1-\cos\theta)z']}{-jk(1-\cos\theta)}$$

$$A_z \simeq \frac{\mu I_0 e^{-jkr}}{4\pi r}\left[\frac{e^{-jk(1-\cos\theta)z'}}{-jk(1-\cos\theta)}\right]_0^l = \frac{\mu I_0 l e^{-jkr}}{4\pi r}e^{-jz}\frac{\sin(z)}{z}$$

where $z = \frac{kl}{2}(1-\cos\theta)$

(a) $$\left.\begin{array}{l} A_r = A_z\cos\theta \\ A_\theta = -A_z\sin\theta \\ A_\phi = 0 \end{array}\right\} \Rightarrow \text{For far-field} \Rightarrow \left\{\begin{array}{l} E_\theta \simeq -j\omega A_\theta \\ E_\phi \simeq -j\omega A_\phi \\ E_r \simeq 0 \end{array}\right.$$

Therefore

$$E_r \simeq 0 \simeq H_r, \quad E_\theta \simeq j\frac{\omega\mu I_0 l e^{-jkr}}{4\pi r}e^{-jz}\frac{\sin(z)}{z}\sin\theta$$

$$E_\phi = 0 = H_\theta, \quad H_\phi \simeq \frac{E_\theta}{\eta}$$

(b) $$\underline{W}_{\text{ave}} = \underline{W}_{\text{rad}} = \tfrac{1}{2}\text{Re}[\underline{E}\times\underline{H}^*] = \frac{1}{2\eta}|E_\theta|^2$$

$$= \frac{1}{2\eta}\left|\frac{\omega\mu I_0 l}{4\pi r}\cdot\frac{\sin(z)}{z}\cdot\sin\theta\right|^2$$

4-20. (a) $$\underline{A} = \frac{\mu}{4\pi}\int_{-\infty}^{\infty}\underline{I}(z')\frac{e^{-jkR}}{R}\,dz' = \hat{a}_z\frac{\mu I_0}{4\pi}\int_{-\infty}^{\infty}\frac{e^{-jkR}}{R}\,dz'$$

where

$$R = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\big|_{x'=y'=0} = \sqrt{x^2+y^2+(z-z')^2}$$

Making a change of variable of the form,

$$z - z' = -p, \ dz' = dp$$

reduces the potential to

$$A_z = \frac{\mu I_0}{4\pi}\int_{-\infty}^{\infty} \frac{e^{-jk\sqrt{\rho^2+p^2}}}{\sqrt{\rho^2+p^2}}\,dp \quad \text{where } \rho^2 = x^2+y^2$$

Using

$$\int_{-\infty}^{\infty} \frac{e^{-j\beta\sqrt{b^2+t^2}}}{\sqrt{b^2+t^2}}\,dt = -j\pi H_0{}^{(2)}(b\beta)$$

We can write the potential as

$$A_z = -j\frac{\mu I_0}{4} H_0{}^{(2)}(k\rho) = -j\frac{\mu I_0}{4} H_0{}^{(2)}(k\sqrt{x^2+y^2})$$

(b) $\underline{H} = \frac{1}{\mu}\nabla \times \underline{A}$ and $\underline{E} = \frac{1}{j\omega\epsilon}\nabla \times \underline{H}$

Since $A_\rho = A_\phi = 0$, in cylindrical coordinates

$$\underline{H} = \frac{1}{\mu}\nabla \times \underline{A} = \frac{1}{\mu}\left(-\hat{a}_\phi \frac{\partial A_z}{\partial \rho}\right) = \hat{a}_\phi j\frac{I_0}{4}\frac{\partial}{\partial\rho}H_0{}^{(2)}(k\rho)$$

Using Equation (V-19), we can write the $\underline{H}$-field as

$$\underline{H} = \hat{a}_\phi H_\phi = -\hat{a}_\phi j\frac{kI_0}{4} H_1{}^{(2)}(k\rho)$$

where $H_1{}^{(2)}(k\rho)$ is the Hankel function of the second kind of order one and argument $k\rho$.

The electric field can be obtained using

$$\underline{E} = \frac{1}{j\omega\epsilon}\nabla\times\underline{H} = \hat{a}_z\frac{1}{j\omega\epsilon}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\phi)\right] = \hat{a}_z\frac{1}{j\omega\epsilon}\left(\frac{\partial H_\phi}{\partial\rho} + \frac{H_\phi}{\rho}\right)$$
$$= \hat{a}_z\frac{1}{j\omega\epsilon}\left[-j\frac{kI_0}{4}\frac{\partial}{\partial\rho}H_1{}^{(2)}(k\rho) - j\frac{kI_0}{4\rho}H_1{}^{(2)}(k\rho)\right]$$

Since $\frac{\partial}{\partial\rho}H_1{}^{(2)}(k\rho) = kH_0{}^{(2)}(k\rho) - \frac{1}{\rho}H_1{}^{(2)}(k\rho)$ by using V-18

then

$$\underline{E} = \hat{a}_z\left[-\frac{kI_0}{4\,\omega\epsilon}kH_0{}^{(2)}(k\rho)\right] = -\hat{a}_z\eta\frac{I_0 k}{4}H_0{}^{(2)}(k\rho)$$

4-21. $P_{\text{rad}} = \eta \frac{|I_0|^2}{4\pi} I_{\text{int}}$

where

$$I_{\text{int}} = \int_0^{\pi} \frac{\left[\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)\right]^2}{\sin\theta} d\theta$$

which can also be written as

$$I_{\text{int}} = 2\int_0^{\pi/2} \frac{\left[\cos^2\left(\frac{kl}{2}\cos\theta\right) + \cos^2\left(\frac{kl}{2}\right) - 2\cos\left(\frac{kl}{2}\cos\theta\right)\cos\left(\frac{kl}{2}\right)\right]}{\sin\theta} d\theta$$

Letting $\begin{cases} l\cos\theta = u \\ -\sin\theta d\theta = du \Rightarrow d\theta = -\frac{du}{\sin\theta} \end{cases}$

reduces I_{int} to

$$I_{\text{int}} = 2\int_0^1 \frac{\left[\cos^2\left(\frac{kl}{2}u\right) + \cos^2\left(\frac{kl}{2}\right) - 2\cos\left(\frac{kl}{2}u\right)\cos\left(\frac{kl}{2}\right)\right]}{1-u^2} du$$

$$= \int_{-1}^1 \frac{\left[\cos^2\left(\frac{kl}{2}u\right) + \cos^2\left(\frac{kl}{2}\right) - 2\cos\left(\frac{kl}{2}u\right)\cos\left(\frac{kl}{2}\right)\right]}{(1+u)} du$$

$$= \tfrac{1}{2}\int_{-1}^1 \frac{[1+\cos(klu) + 1 + \cos(kl)}{(1+u)} du - \int_{-1}^1 \frac{\cos\left[\frac{kl}{2}(1+u)\right] + \cos\left[\frac{kl}{2}(1-u)\right]}{1+u} du$$

Making another change of variable of the form

$$(1+u)kl = v \Rightarrow du = \frac{dv}{kl}$$

we can write that

$$I_{\text{int}} = \tfrac{1}{2}\int_0^{2kl} \frac{2+\cos(kl)+\cos(klv)}{v} dv - \int_0^{kl} \frac{\cos v}{v} dv - \int_{-1}^1 \frac{\cos\left[\frac{kl}{2}(1-u)\right]}{1+u} du.$$

$$= \int_0^{kl} \frac{1+\cos(kl)-\cos(v)}{v} dv + \tfrac{1}{2}\int_0^{2kl} \frac{-\cos(kl)+\cos(v-kl)}{v} dv$$

$$- \int_0^1 \frac{\cos[kl(1-v)]}{v} dv$$

provided $v = \frac{1+u}{2}$

If $z = klv$,

$$I_{\text{int}} = \int_0^{kl} \frac{1+\cos(kl) - \cos(v)}{v}\, dv$$

$$+ \frac{1}{2}\int_0^{2kl} \frac{-\cos(kl) + \cos(v)\cos(kl) + \sin(v)\sin(kl)}{v}\, dv$$

$$- \int_0^{kl} \frac{\cos(kl)\cos(z) + \sin(kl)\sin(z)}{z}\, dz$$

$$I_{\text{int}} = [1+\cos(kl)] \int_0^{kl} \frac{1-\cos v}{v}\, dv - 2\int_0^{kl} \frac{\sin(v)\sin(kl)}{v}\, dv$$

$$+ \sin(kl)\int_0^{2kl} \frac{\sin v}{v}\, dv - \cos(kl)\int_0^{2kl} \frac{1-\cos v}{v}\, dv$$

which reduces to

$$I_{\text{int}} = \Big\{C + \ln(kl) - C_i(kl) + \tfrac{1}{2}\sin(kl)[S_i(2kl) - 2S_i(kl)]$$

$$+\tfrac{1}{2}\cos(kl)\left[C + \ln\left(\frac{kl}{2}\right) + C_i(2kl) - 2C_i(kl)\right]\Big\}$$

where $C = 0.5772$

and

~~$P_{\text{rad}} = \eta \frac{|I_0|^2}{4\pi} I_{\text{int}}$ is identical to (4 − 68)~~

From (4-88)

$$P_{\text{rad}} = \eta \frac{|I_0|^2}{4\pi} \int_0^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\, d\theta$$

Letting

$$\left.\begin{array}{l} u = \cos\theta \\ du = -\sin\theta\, d\theta \end{array}\right\} \Rightarrow \sin^2\theta = 1 - \cos^2\theta = 1 - u^2.$$

We can write

$$P_{\text{rad}} = -\eta \frac{|I_0|^2}{2\pi} \int_1^0 \frac{\cos^2\left(\frac{\pi}{2}u\right)}{1-u^2}\, du = \eta \frac{|I_0|^2}{2\pi} \int_0^1 \frac{\cos^2\left(\frac{\pi}{2}u\right)}{1-u^2}\, du$$

which can also be written as

$$P_{\text{rad}} = \eta \frac{|I_0|^2}{4\pi} \left[\int_0^1 \frac{\cos^2\left(\frac{\pi}{2}u\right)}{1-u}\, du + \int_0^1 \frac{\cos^2\left(\frac{\pi}{2}u\right)}{1+u}\, du\right]$$

Making another change of variable of the form

$$\left.\begin{array}{l} v = 1-u \\ dv = -du \end{array}\right\} \text{ for the first integral, } \left.\begin{array}{l} v = 1+u \\ dv = du \end{array}\right\} \text{ for the second integral}$$

We can write P_{rad} as

$$P_{\text{rad}} = \eta\frac{|I_0|^2}{4\pi}\left\{\int_0^1 \frac{\sin^2\left(\frac{\pi}{2}v\right)}{v}\,dv + \int_1^2 \frac{\sin^2\left(\frac{\pi}{2}v\right)}{v}\,dv\right\} = \eta\frac{|I_0|^2}{4\pi}\int_0^2 \frac{\sin^2\left(\frac{\pi}{2}v\right)}{v}\,dv$$

Using the half-angle identity $< \sin^2\left(\frac{\pi}{2}v\right) = \frac{1-\cos(\pi v)}{2} >$ reduces P_{rad} to

$$P_{\text{rad}} = \eta\frac{|I_0|^2}{8\pi}\int_0^2 \frac{[1-\cos(\pi v)]}{v}\,dv$$

By letting $y = \pi v,\ dy = \pi\,dv$

we can write P_{rad} as

$$P_{\text{rad}} = \eta\frac{|I_0|^2}{8\pi}\int_0^{2\pi}\left[\frac{1-\cos(y)}{y}\right]dy = \eta\frac{|I_0|^2}{8\pi}C_{\text{in}}(2\pi)$$

4-22. (a)

$$I_z(z') = \begin{cases} I_0\left(1+\frac{2}{l}z'\right), & \frac{-l}{2} < z' < 0 \\ I_0\left(1-\frac{2}{l}z'\right), & 0 < z' < l/2 \end{cases}$$

$$\underline{A}(\underline{r}) \cong \hat{a}_z\frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\int_{-l/2}^{l/2} I_z(z')e^{jk\hat{a}_r\cdot\underline{r}'}\,dz'$$

$$= \hat{a}_z\frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\int_{-l/2}^{l/2}\left(1-2\frac{|z'|}{l}\right)e^{jkz'\cos\theta}\,dz'$$

$$= \hat{a}_z\frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\left\{l\frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{kl}{2}\cos\theta\right)} - 2\int_{-l/2}^{l/2}\frac{|z'|}{l}e^{jkz'\cos\theta}\,dz'\right\}$$

$$\int_{-l/2}^{l/2}\frac{|z'|}{l}e^{jkz'\cos\theta}\,dz' = \int_0^{l/2}\frac{z'}{l}e^{jkz'\cos\theta}\,dz' - \int_{-l/2}^{0}\frac{z'}{l}e^{jkz'\cos\theta}\,dz'\Bigg\}$$

$$= \int_0^{l/2}\frac{z'}{l}e^{jkz'\cos\theta}\,dz' + \int_0^{l/2}\frac{z'}{l}e^{-jkz'\cos\theta}\,dz'$$

$$= 2\int_0^{l/2}\frac{z'}{l}\cos[kz'\cos\theta]\,dz' = \frac{l}{2}\int_0^1 \xi\cos\left[\frac{kl}{2}\xi\cos\theta\right]d\xi$$

$$= \frac{l}{2}\left\{\frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{\frac{kl}{2}\cos\theta} + \frac{\cos\left(\frac{kl}{2}\cos\theta\right)-1}{\left(\frac{kl}{2}\cos\theta\right)^2}\right\}$$

$$\therefore\ \underline{A}(\vec{r}) = \hat{a}_z \frac{\mu l}{4\pi}\frac{e^{-jkr}}{r}\left\{\frac{1-\cos\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{kl}{2}\cos\theta\right)^2}\right\}$$

$$A_\theta = \hat{a}_\theta \cdot \underline{A} = -\frac{\mu l}{4\pi}\frac{e^{-jkr}}{r}\sin\theta\left\{\frac{1-\cos\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{kl}{2}\cos\theta\right)^2}\right\}$$

$$A_\phi = \hat{a}_\phi \cdot \underline{A} = 0$$

In the far-zone

$$E_r \simeq 0$$

$$E_\theta \simeq j\omega\mu\frac{l}{4\pi}\frac{e^{-jkr}}{r}\sin\theta\left\{\frac{1-\cos\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{kl}{2}\cos\theta\right)^2}\right\}$$

$$E_\phi \simeq 0$$
$$H_r \simeq 0$$
$$H_\theta \simeq 0$$
$$H_\phi \simeq E_{\theta/\eta}$$

(b) From (4-58a).

$$E_\theta = j\eta\frac{ke^{-jkr}}{4\pi r}\sin\vartheta\left[\int_{-l/2}^{l/2} I(z')e^{jkz'\cos\theta}\,dz'\right]$$

$$E_\theta = j\eta\frac{ke^{-jkr}}{4\pi r}\sin\theta\cdot I_0\cdot\int_{-l/2}^{l/2}\cos\left(\frac{\pi z'}{l}\right)e^{jkz'\cos\vartheta}\,dz'$$

let $a = j^{k\cos\theta}$ and $b = \frac{\pi}{l}$, use following integral formula

$$\int \cos bze^{az}\,dz = \frac{e^{az}(a\cos bz + b\sin bz)}{a^2+b^2}$$

$$E_\theta = j\eta\frac{ke^{-jkr}}{4\pi r}\sin\theta\ I_0\cdot\left\{\frac{e^{jkz'\cdot\cos\theta}}{\left(\frac{\pi}{l}\right)^2 - k^2\cos^2\theta}\left[j^{k\cos\theta}\cos\frac{\pi z'}{l}+\frac{\pi}{l}\sin\frac{\pi z'}{l}\right]\right\}_{-l/2}^{l/2}$$

$$= j\eta\frac{ke^{-jkr}}{4\pi r}\sin\theta\cdot I_0\left[\frac{e^{jkl/2}\cos\theta}{\left(\frac{\pi}{l}\right)^2 - k^2\cos^2\theta}\frac{\pi}{l}+\frac{e^{-jkl/2}\cos\theta}{\left(\frac{\pi}{l}\right)^2 - k^2\cos^2\theta}\frac{\pi}{l}\right]$$

$$E_\theta = j\eta\frac{I_0 k e^{-jkr}}{4\pi r}\sin\theta\cdot\frac{\pi}{l}\cdot\frac{2\cos\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{\pi}{l}\right)^2 - k^2\cos^2\theta} = j\eta\frac{I_0 e^{-jkr}}{2\pi r}\cdot\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

$$H_\phi = j\frac{I_0 k e^{-jkr}}{4\pi r}\sin\theta\cdot\frac{\pi}{l}\cdot\frac{2\cos\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{\pi}{l}\right)^2 - k^2\cos^2\theta} = j\frac{I_0 e^{-jkr}}{2\pi r}\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

(c) $E_\theta = j\eta\frac{ke^{-jkr}}{4\pi r}\sin\theta[\int_{-l/2}^{l/2} I_0\cos^2\left(\frac{\pi}{l}z'\right)e^{jkz'\cos\theta}\,dz'$

let $a = j^{k\cos\theta}$ and $b = \frac{\pi}{l}$, use the following integral formula

$$\int\cos^2 bz\cdot e^{az}dz = \frac{e^{az}}{2a} + \frac{e^{az}}{a^2+4b^2}\left(\frac{a}{2}\cos 2bz + b\sin 2bz\right)$$

$$E_\theta = j\eta\frac{ke^{-jkr}}{4\pi r}\sin\theta\cdot I_0\left\{\frac{e^{jkz'\cos\theta}}{2j^{k\cos\theta}} + \frac{e^{jkz'\cos\theta}}{\left(\frac{2\pi}{l}\right)^2 - k^2\cos^2\theta}\left(\frac{jk\cos\theta}{2}\cdot\cos\frac{2\pi}{l}z' + \frac{\pi}{l}\sin\frac{2\pi}{l}z'\right)\right\}_{-l/2}^{l/2}$$

$$= j\eta\frac{ke^{-jkr}}{4\pi r}\sin\theta\cdot I_0\left\{\frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{k\cos\theta} + k\cos\theta\frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{2\pi}{l}\right)^2 - k^2\cos^2\theta}\right\}$$

$$H_\phi = j\frac{ke^{-jkr}}{4\pi r}\sin\theta\ I_0\left\{\frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{k\cos\theta} + k\cos\theta\frac{\sin\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{2\pi}{l}\right)^2 - k^2\cos^2\theta}\right\}$$

4-23. $\text{VSWR} = \frac{1+|\Gamma|}{1-|\Gamma|}$, $\Gamma = \frac{R_{\text{in}} - Z_0}{R_{\text{in}} + Z_0}$, $R_{\text{in}} = \frac{R_r}{\sin^2\left(\frac{kl}{2}\right)}$, $Z_0 = 50$

(a) $l = \lambda/4$, $kl/2 = \pi/4$, $kl = \pi/2$, $2kl = \pi$

$$R_r = 60\left\{C + \ln(\pi/2) - C_i(\pi/2) + \tfrac{1}{2}\sin\left(\frac{\pi}{2}\right)\left[S_i(\pi) - 2S_i\left(\frac{\pi}{2}\right)\right]\right\}$$

$$R_r = 60\left\{0.5772 + 0.45158 - 0.470 + \tfrac{1}{2}[1.85 - 2(1.3698)]\right\} = 6.8388$$

$$R_{\text{in}} = \frac{R_r}{\sin^2\left(\frac{kl}{2}\right)} = \frac{6.8388}{\sin^2\left(\frac{\pi}{4}\right)} = 13.6776$$

$$\Gamma = \frac{13.6776 - 50}{13.6776 + 50} = -0.5704 \Rightarrow \text{VSWR} = \frac{1 + 0.5704}{1 - 0.5704} = 3.6555$$

(b) $l = \lambda/2 : kl/2 = \pi/2, kl = \pi, 2kl = 2\pi$

$$R_r = 60\left\{C + \ln(\pi) - C_i(\pi) + \tfrac{1}{2}\cos(\pi)\left[C + \ln\left(\frac{\pi}{2}\right) + C_i(2\pi) - 2C_i(\pi)\right]\right\}$$

$$= 60\left\{0.5772 + 1.14473 - 0.059 - \tfrac{1}{2}[0.5772 + 0.45158 - 0.0227 - 2(0.059)]\right\}$$

$$R_r = 73.13 \Rightarrow R_{\text{in}} = \frac{R_r}{\sin^2\left(\frac{kl}{2}\right)} = \frac{73.13}{\sin^2\left(\frac{\pi}{2}\right)} = 73.13$$

$$\Gamma = \frac{73.13 - 50}{73.13 + 50} = 0.18785 \Rightarrow \text{VSWR} = \frac{1 + 0.18785}{1 - 0.18785} = 1.4626$$

(c) $l = 3\lambda/4; kl/2 = 3\pi/4, kl = 3\pi/2, 2kl = 3\pi$

$$R_r = 60\left\{0.5772 + \ln\left(\frac{3\pi}{2}\right) - C_i\left(\frac{3\pi}{2}\right) + \tfrac{1}{2}\sin\left(\frac{3\pi}{2}\right)\left[S_i(3\pi) - 2S_i\left(\frac{3\pi}{2}\right)\right]\right\}$$

$$= 60\left\{0.5772 + 1.5502 - (-0.19839) - \tfrac{1}{2}[1.67473 - 2(1.611)]\right\}$$

$$R_r = 185.965 \Rightarrow R_{\text{in}} = \frac{185.965}{\sin^2(3\pi/4)} = 371.93$$

$$\Gamma = \frac{371.93 - 50}{371.93 + 50} = 0.7630 \Rightarrow \text{VSWR} = \frac{1 + 0.7630}{1 - 0.7630} = 7.4386$$

(d) $l = \lambda; kl/2 = \pi, kl = 2\pi, 2kl = 4\pi$

$$R_r = 60\left\{0.5772 + \ln(2\pi) - C_i(2\pi) + \tfrac{1}{2}\cos(2\pi)[0.5772 + \ln(\pi) + C_i(4\pi) - 2C_i(2\pi)]\right\}$$

$$= 60\left\{0.5772 + 1.8378 - (-0.0227) + \tfrac{1}{2}(1)[0.5772 + 1.14473 - 0.006 - 2(-0.0227)]\right\}$$

$$R_r = 199.099 \Rightarrow R_{\text{in}} = \frac{199.099}{\sin^2(\pi)} = \infty$$

$$\Gamma = \frac{\infty - 50}{\infty + 50} = \frac{1 - 50/\infty}{1 + 50/\infty} = 1 \Rightarrow \text{VSWR} = \infty$$

4-24. $R_r = 80\pi^2\left(\frac{l}{\lambda}\right)^2, a = 10^{-4}\lambda, f = 10 \text{ MHz}, b = 5.7 \times 10^7 \text{ s/m}$

$$R_L = R_{hf} = \frac{l}{p}\sqrt{\frac{\omega\mu}{2\sigma}} = \frac{l}{C}\sqrt{\frac{\omega\mu_0}{2\sigma}} = \frac{l}{2\pi a}\sqrt{\frac{\omega\mu_0}{2\sigma}} = \frac{l}{2\pi \times 10^{-4}\lambda}\sqrt{\frac{2\pi \times 10^7(4\pi \times 10^{-7})}{2 \cdot (5.7 \times 10^7)}}$$

$$R_L = R_{hf} = 1.3245\left(\frac{l}{\lambda}\right), \quad e_{cd} = \frac{R_r}{R_L + R_r}$$

(a) $l = \lambda/50; R_r = 80\pi^2 \left(\frac{\lambda}{50\lambda}\right)^2 = 0.316$ ohms

$$R_L = R_{hf} = 1.3245 \left(\frac{1}{50}\right) = 0.02649$$

$$e_{cd} = \frac{R_r}{R_L + R_r} \times 100 = \frac{0.316 \times 100}{0.02649 + 0.316} = 92.26\%$$

(b) $l = \lambda/4$; From Prob. 4-23 $R_r = 6.8388$

$$R_L = R_{hf} = \frac{1.3245}{4} = 0.3311$$

$$e_{cd} = \frac{6.8388 \times 100}{6.8388 + 0.3311} = 95.38\%$$

(c) $l = \lambda/2$; From Prob. 4-23, $R_r = 73.13$

$$R_L = R_{hf} = \frac{1.3245}{2} = 0.66225$$

$$e_{cd} = \frac{73.13 \times 100}{73.13 + 0.66225} = 99.10\%$$

(d) $l = \lambda$; From Prob 4-23, $R_r = 199.099$

$$R_L = R_{hf} = 1.3245$$

$$e_{cd} = \frac{199.099}{199.099 + 1.3245} \times 100 = 99.34\%$$

4-25. $H_\theta = j\frac{ke^{-jkr}}{4\pi r \cdot \eta} \cdot \sin\theta \left[\int_{-l/2}^{l/2} I_m \cdot \cos\left(\frac{\pi}{l}z'\right) e^{jkz'\cos\theta}\, dz'\right]$

$$H_\theta = j\frac{k \cdot I_m e^{-jkr}}{4\pi r \cdot \eta} \cdot \sin\theta \int_{-l/2}^{l/2} \cos\left(\frac{\pi}{l}z'\right) e^{jkz'\cos\theta}\, dz'$$

Using the same formula in Problem 4-22(b).

$$H_\theta = j\frac{I_m \cdot k \cdot e^{-jkr}}{\eta 4\pi r} \cdot \sin\theta \cdot \frac{\pi}{l} \cdot \frac{2\cos\left(\frac{kl}{2}\cos\theta\right)}{\left(\frac{\pi}{l}\right)^2 - k^2 \cdot \cos^2\theta}$$

$$= j\frac{I_m k \cdot e^{-jkr}}{\eta 4\pi r} \cdot \frac{l}{k} \cdot \frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

$$= j\frac{I_m e^{-jkr}}{\eta 2\pi r} \cdot \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

$$E_\phi = -\eta H_\theta = -j\frac{I_m e^{-jkr}}{2\pi r} \cdot \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

4-26. (a) $$\text{VSWR} = \frac{1+|\Gamma|}{1-|\Gamma|} \Rightarrow |\Gamma| = \frac{\text{VSWR}-1}{\text{VSWR}+1} = \left|\frac{2-1}{2+1}\right| = \left|\frac{1}{3}\right|$$

$$|\Gamma| = \left|\frac{1}{3}\right| = \left|\frac{Z_{in}-Z_c}{Z_{in}+Z_c}\right| = \left|\frac{Z_{in}/Z_c-1}{Z_{in}/Z_c+1}\right| = \begin{cases} \left|\frac{2-1}{2+1}\right| \Rightarrow \frac{Z_{in}}{Z_c} = 2 \\ \left|\frac{\frac{1}{2}-1}{\frac{1}{2}+1}\right| \Rightarrow \frac{Z_{in}}{Z_c} = \frac{1}{2} \end{cases}$$

Largest

$$\frac{Z_{in}}{Z_c} = 2 \Rightarrow Z_{in} = 2Z_c = 100$$

(b)
$$R_{in} = 11.14G^{4.17} \qquad \lambda/2 < l < 2\lambda/\pi$$
$$100 = 11.14G^{4.17} \qquad \pi/2 < kl/2 < 2$$
$$\frac{100}{11.14} = G^{4.17}, \quad 8.9767 = G^{4.17}$$
$$\log_{10}(8.9767) = 4.17\log_{10}(G), \quad 0.953 = 4.17\log_{10} G$$
$$0.2286 = \log_{10} G, \quad G = 10^{0.2286} = 1.6928 = \frac{kl}{2} = 96.99^\circ$$
$$kl = 2(1.6928), \quad l = \frac{2(1.6928)\lambda}{2\pi} = \frac{1.6928}{\pi}\lambda = 0.5388\lambda$$
$$l = 0.5388\lambda$$

(c) $$R_{in} = \frac{R_r}{\sin^2\left(\frac{kl}{2}\right)} \Rightarrow R_r = R_{in}\sin^2\left(\frac{kl}{2}\right) = 100\sin^2(96.99^\circ)$$
$$R_r = 100(0.9926)^2 = 100(0.9852) = 98.52 \text{ ohms}$$
$$R_r = 98.52 \text{ ohms}$$

4-27. $$W_{av} = \frac{1}{2\eta}(|E_\theta|^2 + |E_\phi|^2) = \frac{1}{2\eta}\left[\frac{\omega^2\mu^2\sin^2\theta}{16\pi^2r^2}I_0{}^2(k^2A_1{}^2 + 4A_2{}^2)\right]$$
$$P_{rad} = \frac{1}{2\eta}\frac{\omega^2\mu^2I_0{}^2}{16\pi^2}\left[\int_0^{2\pi}\int_0^{\pi}\sin^3\theta\, d\theta\, d\phi\right][k^2A_1{}^2 + 4A_2{}^2]$$
$$P_{rad} = \frac{\omega^2\mu^2I_0{}^2(k^2A_1{}^2+4A_2{}^2)}{12\pi\eta}\left(\int_0^{2\pi}\int_0^{\pi}\sin^3\theta\, d\theta\, d\phi = \frac{8\pi}{3}\right)$$
$$\Rightarrow R_{rad} = \frac{2P_{rad}}{I_0{}^2} = \frac{\omega^2\mu^2(k^2A_1{}^2+4A_2{}^2)}{6\pi\eta}$$

Elliptical polarization since

$$\vec{E}(t) = \frac{-\omega\mu k\sin\theta}{4\pi r}I_0 \cdot A_1 \cdot \sin(\omega t - kr)\hat{a}_\theta + \frac{\omega\mu k\sin\theta}{2\pi r}I_0 \cdot A_2\cos(\omega t - kr)\hat{a}_\phi$$

4-28. Dipole with $l = \lambda/2$

$$\underline{E}^a \simeq \hat{a}_\theta j\eta \frac{I_0 e^{-jkr}}{2\pi r}\left\{\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right\}$$

$$\simeq -\hat{a}_\theta j\eta \frac{I_0 k e^{-jkr}}{2\pi r}\left\{\frac{-\cos\left(\frac{\pi}{2}\cos\theta\right)}{k\sin\theta}\right\}$$

$$\simeq -\hat{a}_\theta j\eta \frac{I_0 k e^{-jkr}}{2\pi r}\left\{-\frac{\lambda}{2\pi}\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right\}$$

$$\underline{E}^a \simeq -j\eta \frac{I_0 k e^{-jkr}}{4\pi r}\left\{-\hat{a}_\theta \frac{\lambda}{\pi}\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right\}$$

(a) $\underline{l}_e(\theta) = -\hat{a}_\theta \frac{\lambda}{\pi}\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$

(b) $|\underline{l}_e(\theta)|_{\max} = \left|-\hat{a}_\theta \frac{\lambda}{\pi}\cdot\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right|_{\max}^{\theta=90^\circ} = \frac{\lambda}{\pi} = 0.3183\lambda$

(c) $\frac{|l_e(\theta)|_{\max}}{l=\lambda/2} = \frac{\lambda/\pi}{\lambda/2} = \frac{2}{\pi} = 0.6366$

which is 63.66% of l.

(d) $V_{oc} = |\underline{l}_e \cdot \underline{E}^i|_{\theta=90^\circ} = \left|-\hat{a}_\theta \frac{\lambda}{\pi}\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\cdot -\hat{a}_\theta\frac{10^{-3}}{\lambda}V\right|_{\theta=90^\circ}$

$$= \frac{\lambda}{\pi}\left(\frac{10^{-3}}{\lambda}\right) = \frac{10^{-3}}{\pi} = 3.183\times10^{-4}\ \text{Volts}$$

4-29. $\lambda/2$ dipole $\Rightarrow (P_{rad} = P_{in} = 1$ watt, $D_0 = 1.643 = 2.1564$ dB$)$

$Z_{in} = 73 + j42.5$, $f = 1{,}900$ MHz $\Rightarrow \lambda = 3\times10^8/1.9\times10^9 = 0.15789$ meters

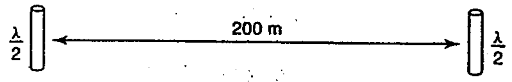
(a) $U_0 = \frac{P_{rad}}{4\pi} = \frac{1}{4\pi} = 0.07958$ watts/sterad

$U_{dipole} = U_0 D_0 = 0.07958(1.643) = 0.130745$ watts/unit solid angle (sterad)

(b) $W_{dipole} = \frac{U_{dipole}}{r^2} = \frac{0.130745}{(5\times10^3)^2} = 5.229\times10^{-9}$ watts/m^2

$W_0 = \frac{U_0}{r^2} = \frac{0.07958}{(5\times10^3)^2} = 3.183\times10^{-9}$ watts/m^2

4-30.



$\theta = 90°$, $\phi = 40°$

At $f = 300$ MHz, $\lambda = \dfrac{c}{f} = 1$ m

$$\Rightarrow \frac{2D^2}{\lambda} = \frac{2\left(\frac{\lambda}{2}\right)^2}{\lambda} 0.5 \text{ m}$$

$r = 200 \text{ m} \gg 0.5$ m

$$P_r = \left(\frac{\lambda}{4\pi r}\right)^2 G_{0t}G_{0r} = \left(\frac{\lambda}{4\pi r}\right)^2 D_{0t}D_{0r}$$

for lossless antenna.

Now since $D_{0t} = D_{0r} = 1.643$ for $\dfrac{\lambda}{2}$ dipole.

$$P_r = \left(\frac{1}{4\pi \cdot 200}\right)^2 (1.643)(1.643) \quad W = 0.2 \text{ mW}$$

4-31. The time average power density $\left(W_{av} = \frac{1}{2}\frac{|E|^2}{\eta}\right)$

$$W_{av} = \eta\frac{|I_0|^2}{8\pi^2 r^2}\left[\frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}\right], \quad P_{rad} = \eta\frac{|I_0|^2}{4\pi}\int_0^{\pi}\frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta}\,d\theta$$

$$P_{rad} = \tfrac{1}{2}R_{rad}|I_0|^2, \quad R_{rad} = \frac{\eta}{4\pi}[r + \ln(2\pi) - C_i(2\pi)] = 30[0.5772 + 1.838 + 0.02]$$

$$R_{rad} = 73.0523.$$

$P_{rad} = (0.5 \cdot 100) = 50$ watts. $\cdot\ 50 = \frac{1}{2}(73.0523)|I_0|^2$

$|I_0|^2 = 1.36888$

At $r = 500$ m, $\theta = 60°$, $\phi = 0°$

$$W_{av} = 120\pi \cdot \frac{1.36888}{8\pi^2 \cdot (500)^2} \cdot \left[\frac{\cos^2\left(\frac{\pi}{2}\cos 60°\right)}{\sin^2 60°}\right]$$

$$= 15 \cdot \frac{1.36888}{\pi \cdot 25 \times 10^4} \cdot (0.6667)$$

$$= 1.743 \times 10^{-5} \text{ watts/m}^2.$$

4-32. $l = \lambda/20 \Rightarrow$ triangular current distribution; $a = \lambda/400, f = 30$ MHz $\Rightarrow \lambda = 0.1$ meters

(a) $R_r = R_{\text{in}} = 20\pi^2 \left(\frac{l}{\lambda}\right) = 20\pi^2 \left(\frac{1}{20}\right)^2 = 0.4935$ ohms

$$X_{\text{in}} = -j120 \frac{\left[\ln\left(\frac{l}{2a}\right) - 1\right]}{\tan\left(\frac{\pi l}{\lambda}\right)} = -j120 \frac{\left[\ln\left(\frac{\lambda}{20}\frac{1}{2}\frac{400}{\lambda}\right) - 1\right]}{\tan\left(\frac{\pi}{\lambda}\frac{\lambda}{20}\right)} = -j120 \frac{[\ln(20) - 1]}{\tan(\pi/20)}$$

$$= -j986.935$$

$Z_{\text{in}} = 0.4935 - j986.935$ (capacitive)

(b) $e_{cd} = \dfrac{R_r}{R_r + R_L}$

Since element is PEC $\Rightarrow \sigma = \infty \Rightarrow R_L = 0$

$e_{cd} = \dfrac{R_r}{R_r} = 1 = 100\%$

(c) Must use an inductor in series to resonate the element with a reactance of

$$X_L = \omega L = 2\pi f L = 2\pi(30 \times 10^6)L = 986.35$$

$$L = \frac{986.35}{2\pi(30 \times 10^6)} = 5.236 \times 10^{-6} \text{ henries}$$

$$\boxed{L = 5.236 \times 10^{-6} \text{ henries}}$$

4-33. $Z_a = 73 + j42.5, Z_c = 75, f = 100$ MHz

a. $\Gamma = \dfrac{Z_a - Z_c}{Z_a + Z_c} = \dfrac{73 + j42.5 - 75}{73 + j42.5 + 75} = \dfrac{-2 + j42.5}{148 + j42.5} = \dfrac{42.547\underline{|92.694}}{153.981\underline{|16.02}}$

$\Gamma = 0.2763\underline{|76.674} \Rightarrow |\Gamma| = 0.2763, \underline{|\phi} = 76.674° = 1.338$(rads)

b. $\text{VSWR} = \dfrac{1 + |\Gamma|}{1 - |\Gamma|} = \dfrac{1 + 0.2763}{1 - 0.2763} = \dfrac{1.2763}{0.2763} = \boxed{1.76358}$

c. $Z_a = 73 + j42.5$

Need a **capacitor in** series to resonate.

$X_c = 42.5$

d. $X_c = \dfrac{1}{\omega C} = \dfrac{1}{2\pi f C} = 42.5 \Rightarrow C = \dfrac{1}{2\pi f(42.5)}$

$C = \dfrac{1}{2\pi(42.5)(10^8)} = \boxed{0.00374 \times 10^{-8}} = \boxed{37.4 \times 10^{-12} \text{ farads}}$

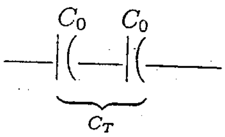
e. $$Z_{\text{in}} = Z_a - jX_c = 73 + j42.5 - j42.5 = 73$$
$$Z_{\text{in}} = 73$$
$$\Gamma = \frac{Z_{\text{in}} - Z_c}{Z_{\text{in}} + Z_c} = \frac{73 - 75}{73 + 75} = \frac{-2}{148} = -0.0135$$
$$\Gamma = -0.0135 \Rightarrow |\Gamma| = 0.0135$$
$$\text{VSWR} = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.0135}{1 - 0.0135} = 1.027$$
$$\boxed{\text{VSWR} = 1.027}$$

4-34. $\lambda/2$ dipole $\Rightarrow Z_{\text{in}} = 73 + j42.5$, $f = 1.9 \times 10^9$ Hz

(a) $$|\Gamma| = \left|\frac{Z_{\text{in}} - Z_c}{Z_{\text{in}} + Z_c}\right| = \left|\frac{73 + j42.5 - 50}{73 + j42.5 + 50}\right| = \left|\frac{23 + j42.5}{123 + j42.5}\right| = \frac{48.324}{130.1355} = 0.371$$
$$\text{VSWR} = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.371}{1 - 0.371} = 2.17965$$

(b) Capacitance $\Rightarrow X_T = \dfrac{1}{\omega C_T} = 42.5$
$$\Rightarrow C_T = \frac{1}{42.5\,\omega} = \frac{1}{42.5(2\pi f)} = \frac{1}{42.5(2\pi \times 1.9 \times 10^9)}$$
$$C_T = 1.971 \times 10^{-12} f$$

(c) $C_0 = 2C_T = 2(1.971 \times 10^{-12}) = 3.942 \times 10^{-12} f$



$$\frac{1}{C_T} = \frac{1}{C_0} + \frac{1}{C_0} = \frac{Z}{C_0}$$
$$C_0 = 2C_T$$

(d) $$|\Gamma| = \left|\frac{Z_{\text{in}} - Z_c}{Z_{\text{in}} + Z_c}\right| = \frac{73 - 50}{73 + 50} = \frac{23}{123} = 0.18699$$
$$\text{VSWR} = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.18699}{1 - 0.18699} = 1.46$$

4-35. (a) $$I_{\text{in}} = I_0 \sin\left[k\left(\frac{l}{2} + |z|\right)\right]$$
$$l = \lambda/4,\ z = \lambda/8$$
$$I_{\text{in}} = I_0 \cdot \sin\left[k\left(\frac{\lambda}{4} \pm \frac{\lambda}{8}\right)\right] = I_0 \sin\left[k\frac{\lambda}{8}\right] = I_0 \sin\left[\frac{2\pi}{\lambda}\frac{\lambda}{8}\right]$$
$$= I_0 \sin\left(\frac{\pi}{4}\right)$$
$$I_{\text{in}} = 0.707 I_0$$

$$R_{in} = \left(\frac{I_0}{I_{in}}\right)^2 \cdot R_r = \left(\frac{I_0}{0.707 I_0}\right)^2 \cdot R_r = 2R_r = 2(73) = 146$$

$$X_{in} = \left(\frac{I_0}{I_{in}}\right)^2 \cdot X_m = \left(\frac{I_0}{0.707 I_0}\right)^2 \cdot X_m = 2X_m = 2(42.5) = 85$$

$$Z_{in} = R_{in} + jX_{in} = 146 + j85$$

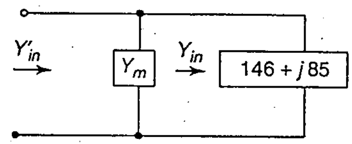
(b) $$Y_{in} = \frac{1}{146 + j85} \cdot \frac{146 - j85}{146 - j85} = \frac{146 - j85}{168.941} = (5.115 - j2.978) \times 10^{-3}$$

$$Y_m = +j2.978 \times 10^{-3} \Rightarrow X_m = \frac{1}{Y_m} = -j335.776 \quad \text{(Capacitive)}$$

(c) $$Y'_{in} = 5.115 \times 10^{-3} \Rightarrow Z_{in} = 5.115 \times 10^{-3} = 195.503 \text{ ohms}$$

$$|\Gamma| = \left|\frac{195.503 - 300}{195.503 + 300}\right| = \frac{104.4966}{495.503} = 0.21$$

$$\text{VSWR} = (1 + 0.211)/(1 - 0.211) = 1.5346$$



4-36. $l = \lambda/2, \quad Z_c = 50$ ohms

$$Z_{in} = 73 + j42.5, \quad Y_{in} = \frac{1}{Z_{in}} = \frac{1}{73 + j42.5} \cdot \frac{73 - j42.5}{73 - j42.5}$$

$$Y_{in} = 0.01023 - j0.0059563 = (10.23 - j5.9563) \times 10^{-3} = G_{in} - jB_{in}$$

$$B_{in} = \omega C_{in} = 2\pi f C_{in} \Rightarrow C_{in} = \frac{B_{in}}{2\pi f} = \frac{5.9563 \times 10^{-3}}{2\pi \cdot (10 \times 10^8)} = 0.94797 \times 10^{-12}$$

$$\therefore\ C_{in} = 0.94797 \text{ pF}$$

$$G_{in} = 10.23 \times 10^{-3}$$

$$R_{in} = \frac{1}{G_{in}} = 97.75, \quad \Gamma_{in} = \frac{R_{in} - Z_c}{R_{in} + Z_c} = \frac{97.75 - 50}{97.75 + 50} = 0.3232$$

$$\text{VSWR} = \frac{1 + |\Gamma_{in}|}{1 - |\Gamma_{in}|} = \frac{1 + 0.3232}{1 - 0.3232} = 1.955$$

4-37.

$$\underline{E} = -\hat{a}_\theta j\,\frac{\omega\mu b I_0 e^{-jkr}}{4\pi r}\cdot\frac{\sin\left(\frac{kb}{2}\cos\theta\right)}{\frac{kb}{2}\cos\theta}\Bigg|_{\theta=90^\circ} = -\hat{a}_\theta\cdot j\cdot\frac{\omega\mu b I_0 e^{-jkr}}{4\pi r}\cdot(1)$$

$$\underline{E}|_{\theta=90^\circ} = -\hat{a}_\theta\cdot j\frac{\omega\mu I_0 e^{-jkr}}{4\pi r}b = -j\frac{\omega\mu I_0 e^{-jkr}}{4\pi r}\cdot l_e(\theta)$$

$$l_e(\theta) = \hat{a}_\theta\cdot b$$

$$\underline{E}^{\text{inc}}|_{\theta=90^\circ} = \hat{a}_\theta j\eta\,\frac{kI_0 l e^{-jkr}}{4\pi r}\sin\theta\Big|_{\theta=90^\circ} = \hat{a}_\theta\cdot j\eta\cdot\frac{kI_0 l e^{-jkr}}{4\pi r}$$

$$P = \frac{|\underline{l}_e(\theta)\cdot\underline{E}^{\text{inc}}|^2}{|l_e(\theta)|^2\cdot|\underline{E}^{\text{inc}}|^2} = \frac{\left|\eta\frac{bkI_0 l}{4\pi r}\right|^2}{|b|^2\cdot\left|\eta\frac{kI_0 l}{4\pi r}\right|^2} = 1.$$

$$P(\text{dB}) = 10\log_{10}(1) = 0\text{ dB}$$

4-38.

$$V_1 = 4e^{j20^\circ} = C[\hat{a}_y]\cdot\left[\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right],\text{ at } z=0$$

$$V_1 = 4e^{j20^\circ} = j\cdot C\cdot\frac{1}{\sqrt{2}} \Rightarrow C = -j4\sqrt{2}e^{j20^\circ},\ C = 4\sqrt{2}e^{-j70^\circ}$$

$$V_2 = (4\sqrt{2}e^{-j70^\circ})[10(2\hat{a}_x + \hat{a}_y e^{j30^\circ})]\cdot\left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right) = 40\sqrt{2}e^{-j70^\circ}\left[\frac{2+je^{j30^\circ}}{\sqrt{2}}\right]$$

$$= 40e^{-j70^\circ}[2 + j(\cos 30^\circ + j\sin 30^\circ)]$$

$$= 40e^{-j70^\circ}[1.5 + j0.866] = 40e^{-j70^\circ}[1.73e^{j30^\circ}]$$

$$V_2 = 70e^{-j40^\circ} = 53.6 - j45^\circ$$

4-39. $l = 3$ cm, $\lambda = 5$ cm, $I = 10e^{j60^\circ}$

$$r > \frac{2D^2}{\lambda} = \frac{2\times 3^2}{5} = \frac{18}{5} = 3.6\text{ cm} \Rightarrow 10\text{ cm is in the far field.}$$

$$\frac{l}{\lambda} = \frac{3}{5} = 0.6 \Rightarrow \text{length of dipole is finite, } \frac{kl}{2} = \pi\cdot\frac{l}{\lambda} = 0.6\pi$$

$$E_\theta \simeq j\eta\frac{I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin\theta}\right] = j\eta\frac{I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos(0.6\pi\cos\theta) + 0.309}{\sin\theta}\right]$$

$$H_\phi \simeq \frac{E_\theta}{\eta},\ \left(\leftarrow \frac{\cos(0.6\pi\cdot\cos 45^\circ) + 0.309}{\sin 45^\circ}\Bigg|_{\theta=45^\circ} = 0.7703\right)$$

$$e^{-jkr} \Rightarrow kr = \frac{2\pi}{\lambda}r = \frac{2\pi}{5}\cdot 10 = 4\pi = 12.5663 \text{ rad}$$

$$\Rightarrow E_\theta = j^{120\pi}\cdot\frac{I_0 e^{j60}\cdot e^{-j4\pi}}{2\pi(0.1 \text{ m})}\cdot(0.7703) = 4620e^{j11.52}$$

$$|E_\theta| = 4620 \text{ v/m}, |H_\phi| = \frac{4620}{120\pi} = 12.25 \text{ amperes/meter}$$

4-40. Using equation (4-79)

$$R_{\text{in}} = \frac{R_r}{\sin^2\left(\frac{kl}{2}\right)} = \frac{120 \text{ ohms}}{\sin^2(0.6\pi)} = 132.668 \text{ ohms}$$

4-41. $\frac{kl}{2} = \frac{3\pi}{4}$, $kl = \frac{3\pi}{2}$, $2kl = 3\pi$

(a) Using (8-60a), (8-60b)

$$R_r = 185.808, X_r = 190.7967$$

(b) Using (8-61a), (8-61b)

$$R_{\text{in}} = \frac{185.808}{\sin^2(3/4\pi)} = 371.617, X_{\text{in}} = \frac{190.7967}{\sin^2(3/4\cdot\pi)} = 385.5936$$

(c)
$$\Gamma = \frac{371.617-300}{371.617+300} = 0.10663,$$

$$\text{VSWR} = \frac{1+0.10663}{1-0.10663} = 1.2387$$

4-42. $l = 0.625\lambda$,

a. Using (8-60a), (8-60b)

$$R_r = 131.9415, \quad X_r = 146.131638$$

b. Using (8-61a), (8-61b)

$$R_{\text{in}} = 154.579, \quad X_{\text{in}} = 171.203$$

c. $\Gamma = \frac{154.579-300}{154.579+300} = -0.3199, \quad \text{VSWR} = \frac{1+|0.3199|}{1-1-0.31991} = 1.9407$

4-43. a. $l = 200$ m, $a = 1$ m, $f = 150$ $kHz \to \lambda = 2000$ meters. Using (11-37).

$$Z_{\text{in}} \simeq 20\pi^2\left(\frac{l}{\lambda}\right)^2 - j120\frac{\left[\ln\left(\frac{l}{2a}\right)-1\right]}{\tan\left(\pi\frac{l}{\lambda}\right)} \simeq 20\pi^2\cdot\left(\frac{1}{10}\right)^2 - j120\cdot\frac{[\ln(100)-1]}{\tan(\pi/10)}$$

$$Z_{\text{input}} = 2 + Z_{\text{in}} = 2 + 1.9739 + j1377.07$$
$$Z_{\text{input}} = 3.9739 + j1377.07$$

b. Radiation efficiency $= 100 \cdot \dfrac{R_r}{R_L + R_r} = 100 \cdot \dfrac{1.9739}{3.9739} = 49.67\%$

c. $\text{RPF} = \dfrac{R_r}{|\text{Im}(Z_{\text{input}})|} = \dfrac{1.9739}{1.377 \times 10^{+3}} = 1.4335 \times 10^{-3}$

d. $X = -\text{Im}(Z_{\text{input}}) = -1377.07$

$$n = \sqrt{\frac{R_r + R_L}{Z_0}} = \sqrt{\frac{3.9739}{50}} = 0.282$$

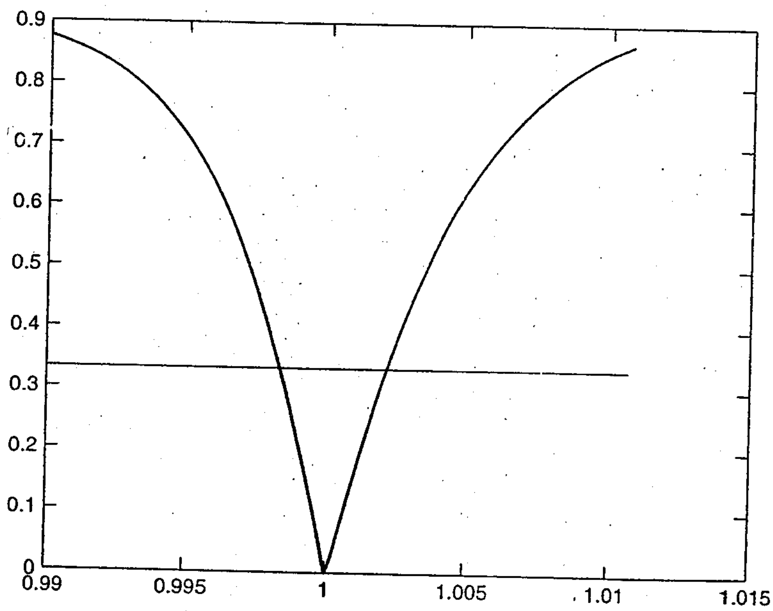
e. The answer to this part was found by manually entering values of X until $|\Gamma| = 0.333$ was obtained.

The values obtained are

$$X_1 = 0.99803$$
$$X_2 = 1.00198$$

The corresponding percent bandwidth is

$$\text{BW} = (X_2 - X_1) \times 100\% = 0.395\%$$



4-44. $\underline{E}_w = (2\hat{a}_x - j\hat{a}_y)E_0 e^{+jkz} = \underbrace{\left(\frac{2\hat{a}_x - j\hat{a}_y}{\sqrt{5}}\right)}_{\hat{\rho}_w} \sqrt{5}E_0 e^{+jkz}$

(a) $\hat{\rho}_w = \left(\frac{2\hat{a}_x - j\hat{a}_y}{\sqrt{5}}\right)$

(b) $\hat{\rho}_a = \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)$ or $\hat{\rho}_a = \left(\frac{-j\hat{a}_x + \hat{a}_y}{\sqrt{2}}\right)$

(c) 1. Elliptical, AR = 2
 2. CCW

(d) 1. Circular, AR = 1
 2. CCW

(e) $\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\left(\frac{2\hat{a}_x - j\hat{a}_y}{\sqrt{5}}\right) \cdot \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)\right|^2 = \left|\frac{2 - j^2 1}{\sqrt{10}}\right|^2 = \left|\frac{2+1}{\sqrt{10}}\right|^2$

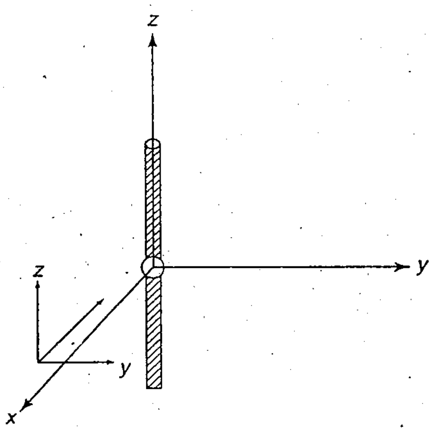
$= \frac{9}{10} = -0.4576 \text{ dB}$

or

$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\left(\frac{2\hat{a}_x - j\hat{a}_y}{\sqrt{5}}\right) \cdot \left(\frac{-j\hat{a}_x + \hat{a}_y}{\sqrt{2}}\right)\right|^2 = \left|\frac{-j2 - j}{\sqrt{10}}\right|^2 = \left|\frac{-j3}{\sqrt{10}}\right|^2$

$= \frac{9}{10} = -0.4576 \text{ dB}$

4-45. $W_i = 2\ \mu\text{w/m}^2 = 2 \times 10^{-6}\ \text{w/m}^2$



(a) $\underline{E}_W^L = (3\hat{a}_z + j\hat{a}_y)E_0 e^{+jkx}$

$\underline{E}_W^L = \left(\frac{3\hat{a}_z + j\hat{a}_y}{\sqrt{10}}\right) 10E_0 e^{+jkx}$

$\hat{\rho}_W = \left(\frac{3\hat{a}_z + j\hat{a}_y}{\sqrt{10}}\right)$

Elliptical CCW

AR = 3/1 = 3

(b) $\underline{E}_a = \hat{a}_\theta j\eta \frac{I_0 e^{-jky}}{2\pi r} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$

$= \hat{a}_\theta E_0 \left.\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right|_{\theta=\pi/2}$

$\underline{E}_a = \underbrace{\hat{a}_\theta}_{\hat{\rho}_a} E_0$

$\hat{\rho}_a = \hat{a}_\theta$ **Linear**

$\hat{\rho}_a = [\hat{a}_x \cos\theta\cos\phi + \hat{a}_y \cos\theta\sin\phi - \hat{a}_z \sin\theta]_{\theta=90^\circ}$

$\hat{\rho}_a = -\hat{a}_z$

(c) $\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\left(\dfrac{3\hat{a}_x + j\hat{a}_y}{\sqrt{10}}\right) \cdot (-\hat{a}_x)\right|^2 = \dfrac{9}{10} = 0.9 = -0.4576 \text{ dB}$

$\text{PLF} = -0.4576 \text{ dB} = 0.9$

(d) $\lambda = \dfrac{3 \times 10^8}{300 \times 10^6} = 1 \text{ m}$

$$A_{em} = \frac{\lambda^2}{4\pi} D_0 = \frac{1}{4\pi} D_0 = \frac{1.643}{4\pi} = 0.1307 \text{ m}^2$$

$$P_L = A_{em} W_i(\text{PLF}) = 0.1307(2 \times 10^{-6})(0.9) = (0.2614)(0.9) \times 10^{-6}$$

$$P_L = 0.2353 \times 10^{-6} \text{ watts}$$

4-46. $E_\theta = j\eta \dfrac{kI_0 l e^{-jkr}}{2\pi r} \sin\theta \cdot \cos(kh\cos\theta); 0 \leqslant \theta \leqslant \pi/2, 0 \leqslant \phi \leqslant 2\pi$

$$\underline{W}_{\text{ave}} = \tfrac{1}{2}\text{Re}[\underline{E} \times \underline{H}^*] = \frac{\hat{a}_r}{2\eta}|E_\theta|^2 = \hat{a}_r \frac{\eta}{2}\left|\frac{kI_0 l}{2\pi r}\right|^2 \sin^2\theta \cdot \cos^2(kh\cos\theta)$$

$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi/2} \underline{W}_{\text{ave}} \cdot \hat{a}_r r^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{\eta}{2}\left|\frac{kI_0 l}{2\pi}\right|^2 \int_0^{2\pi}\int_0^{\pi/2} \sin^3\theta \cos^2(kh\cos\theta) \, d\theta \, d\phi$$

$$= \frac{\eta}{\pi}\left|\frac{kI_0 l}{2}\right|^2 \cdot \int_0^{\pi/2} \sin^3\theta \cos^2(kh\cos\theta) \, d\theta$$

$$= \frac{\eta}{\pi}\left|\frac{kI_0 l}{2}\right|^2 \int_0^{\pi/2} \sin^3\theta \left[\frac{1 + \cos(2kh\cos\theta)}{2}\right] d\theta$$

$$= \frac{\eta}{2\pi}\left|\frac{kI_0 l}{2}\right|^2 \left\{\int_0^{\pi/2} \sin^3\theta \, d\theta + \int_0^{\pi/2} \sin^3\theta \cdot \cos(2kh\cos\theta) \, d\theta\right\}$$

$$P_{\text{rad}} = \frac{\eta}{2\pi}\left|\frac{kI_0 l}{2}\right|^2 \{I_1 + I_2\}$$

where $I_1 = \displaystyle\int_0^{\pi/2} \sin^3\theta \, d\theta = -\frac{1}{3}\cos\theta(\sin^2\theta + 2)\Big|_0^{\pi/2} = \frac{2}{3}$

$$I_2 = \int_0^{\pi/2} \sin^3\theta \cos(kh\cos\theta) \, d\theta = \int_0^{\pi/2} \sin^2\theta \cdot \cos(kh\cos\theta) \sin\theta \, d\theta$$

Let $u = \sin^2\theta$ $\qquad v = -\dfrac{1}{2kh}\sin(2kh\cos\theta)$

$du = 2\sin\theta\cos\theta \, d\theta \qquad dv = -\dfrac{\cos(2kh\cos\theta)}{2kh} \cdot d(2kh\cos\theta)$

Thus

$$I_2 = -\frac{\sin^2\theta}{2kh} \cdot \sin(2kh\cos\theta)\Big|_0^{\pi/2} + \frac{2}{2kh}\int_0^{\pi/2} \cos\theta \cdot \sin(2kh\cos\theta) \sin\theta \, d\theta$$

Let $u = \cos\theta \qquad dv = -\frac{1}{2kh}\sin(2kh\cos\theta)\, d(2kh\cos\theta)$

$du = -\sin\theta\, d\theta \quad v = \frac{1}{2kh}\cos(2kh\cos\theta)$

$$I_2 = 0 + \frac{2}{2kh}\left\{\frac{\cos\theta}{2kh}\cos(2kh\cos\theta)\Big|_0^{\pi/2} + \frac{1}{2kh}\int_0^{\pi/2}\cos(2kh\cos\theta)\sin\theta\, d\theta\right\}$$

$$= \frac{2}{2kh}\left\{-\frac{1}{2kh}\cos(2kh) - \frac{1}{(2kh)^2}\sin(2kh\cos\theta)\Big|_0^{\pi/2}\right\} = 2\left\{-\frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3}\right\}$$

Therefore

$$P_{\text{rad}} = \frac{\eta}{2\pi}\left|\frac{kI_0 l}{2}\right|^2 \cdot \{I_1 + I_2\} = \pi\eta\left|\frac{I_0 l}{\lambda}\right|^2 \cdot \left[\frac{1}{3} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3}\right]$$

4-47. $E_\theta = C_1 \sin\theta\cos(kh\cos\theta)$, where $C_1 = j\eta\frac{kI_0 l e^{-jkr}}{2\pi r}$

a. $E_\theta|_{\theta=30^\circ} = C_1\sin\theta\cdot\cos(kh\cos\theta)|_{\theta=30^\circ} = 0 \Rightarrow \cos(kh\cos\theta)|_{\theta=30^\circ} = 0$

$kh\cos(30^\circ) = \frac{2\pi}{\lambda}h(0.867) = \cos^{-1}(0) = \frac{\pi}{2} \Rightarrow h = \frac{1}{4(0.867)}\cdot\lambda = 0.288\lambda$

b. $D_0 = \frac{2}{\left[\frac{1}{3} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3}\right]}, 2kh = 2\cdot\left(\frac{2\pi}{\lambda}\right)\cdot(0.288\lambda) = 3.632$

$D_0 = \frac{2}{\left[\frac{1}{3} - \frac{\cos(3.632)}{(3.632)^2} + \frac{\sin(3.632)}{(3.632)^3}\right]} = \frac{2}{\left[\frac{1}{3} + 0.06689 - 0.00983\right]}$

$= 5.12 = 7.1$ dB

c. $R_r = 2\pi\eta\left(\frac{l}{\lambda}\right)^2\cdot\left[\frac{1}{3} - \frac{\cos(3.632)}{(3.632)^2} + \frac{\sin(3.632)}{(3.632)^3}\right]$

$= 2\pi(377)\left(\frac{1}{50}\right)^2\cdot[0.39] = 0.37$ ohms

4-48. $E_\theta = C_1\sin\theta\cdot\cos(kh\cos\theta)$, where $C_1 = j\eta\frac{kI_0 l e^{-jkr}}{2\pi r}$

$E_\theta|_{h=2\lambda} = C_1\cdot\sin\theta_n\cdot\cos(kh\cos\theta_n)|_{h=2\lambda} = 0 \Rightarrow \sin\theta_n = 0, \cos(kh\cos\theta_n)|_{h=2\lambda} = 0$

$\sin\theta_n = 0 \Rightarrow \theta_n = 0^\circ$

$\cos(kh\cos\theta_n)|_{h=2\lambda} = \cos(4\pi\cos\theta_n) = 0 \Rightarrow 4\pi\cos\theta_n = \cos^{-1}(0) = \pm\left(\frac{2n+1}{2}\right)\pi,$

$n = 0, 1, 2, \ldots$

$\theta_n = \cos^{-1}[\pm(2n+1)/8], n = 0, 1, 2, 3, 4, \ldots$

$$\left.\begin{aligned} n = 0:\quad & \theta_0 = \cos^{-1}\left(\pm\tfrac{1}{8}\right) = 82.82^\circ \\ n = 1;\quad & \theta_1 = \cos^{-1}\left(\pm\tfrac{3}{8}\right) = 67.98^\circ \\ n = 2;\quad & \theta_2 = \cos^{-1}\left(\pm\tfrac{5}{8}\right) = 51.32^\circ \\ n = 3;\quad & \theta_3 = \cos^{-1}\left(\pm\tfrac{7}{8}\right) = 28.96^\circ \end{aligned}\right\} \begin{aligned} &\text{for } 0^\circ \leqslant \theta \leqslant 90^\circ \\ &(\text{for } 90^\circ \leqslant \theta \leqslant 180^\circ, \text{the field is zero}) \end{aligned}$$

$n = 4;\quad \theta_4 = \cos^{-1}\left(\pm\frac{9}{8}\right) =$ Does not exist. The same holds for $n \geq 5$.

Therefore where the field vanishes for $0^\circ \leqslant \theta \leqslant 90^\circ$, are

$$\theta = 0^\circ, 28.96^\circ, 51.32^\circ, 67.98^\circ, \text{ and } 82.82^\circ$$

4-49. $E_\theta = C_1 \cdot \sin\theta \cdot \cos(kh\cos\theta)$, where $C_1 = j\eta \dfrac{kI_0 e^{-jkr} \cdot l}{2\pi r}$

$E_\theta|_{\theta=60^\circ} = C_1 \cdot \sin(60^\circ) \cdot \cos(kh_n \cos(60^\circ)) = 0 \Rightarrow \cos(kh_n\cos(60^\circ)) = 0$

$kh_n \cos(60^\circ) = kh_n\left(\frac{1}{2}\right) = \dfrac{\pi}{\lambda}h_n = \cos^{-1}(0) = \pm\left(\dfrac{2n+1}{2}\right)\pi, n = 0, 1, 2, 3, \ldots$

Choosing the positive values

$$h_n = \left(\frac{2n+1}{2}\right)\lambda, n = 0, 1, 2, 3, \ldots$$

$$h_n = 0.5\lambda, 1.5\lambda, 2.5\lambda, 3.5\lambda, 4.5\lambda$$

4-50. E_θ(4-99) $\simeq C \cdot \sin\theta \cdot [2\cos(kh\cos\theta)] \Rightarrow \text{AF} = 2[\cos(kh\cos\theta)]_{\max} = \pm 2$

$\cos(kh\cos\theta_m) = \pm 1$

a. $kh\cos\theta_m = \cos^{-1}(\pm 1) \Rightarrow \theta_m = \dfrac{1}{kh}\cos^{-1}(\pm 1) = \pm\dfrac{m\pi}{kh} = \dfrac{\pm m\pi}{\frac{2\pi}{\lambda}\left(\frac{3\lambda}{2}\right)}$

$= \pm\dfrac{m}{3}, m = 0, 1, 2, \ldots$

$\theta_m = \cos^{-1}(\pm m/3); m = 0, 1, 2, 3, \ldots$

$m = 0;\ \theta_0 = \cos^{-1}(\pm 0) = 90^\circ$

$m = 1;\ \theta_1 = \cos^{-1}(\pm 1/3) = \begin{cases} \cos^{-1}(1/3) = 70.5288^\circ \\ \cos^{-1}(-1/3) = 109.47/2^\circ (\Rightarrow \text{Below Ground Plane}) \end{cases}$

$m = 2;\ \theta_2 = \cos^{-1}(\pm 2/3) = \begin{cases} \cos^{-1}(2/3) = 48.1897^\circ \\ \cos^{-1}(-2/3) = 131.8103^\circ (\Rightarrow \text{Below Ground Plane}) \end{cases}$

$m = 3$: $\theta_3 = \cos^{-1}(\pm 1) = \begin{cases} \cos^{-1}(1) = 0^\circ \\ \cos(-1) = 180^\circ (\Rightarrow \text{Below Ground Plane}) \end{cases}$

$m = 4$: $\theta_4 = \cos^{-1}(\pm 4/3) \Rightarrow$ does not exist

b. $E_{\theta m} = C \cdot \sin\theta[2 \cdot \cos(kh\cos\theta)]_{\max} = \pm 2C$, where $\theta = 90°$

c. $\frac{E_\theta}{E_{\theta m}} = \sin\theta \cdot \cos(kh\cos\theta)$

$\theta = 0°$: $\frac{E_\theta}{E_{\theta m}} = 0 \Rightarrow \frac{E_\theta}{E_{\theta m}} = 20\log_{10}(0) = -\infty$ dB

$\theta = 48.1897°$: $\frac{E_\theta}{E_{\theta m}} = \sin\theta \cdot \cos(kh\cos\theta)\big|_{h=\frac{3\lambda}{2}} = 0.7454 \Rightarrow \frac{E_\theta}{E_{\theta m}}$

$= 20\log_{10}(0.7454)$

$$\frac{E_\theta}{E_{\theta m}} = -2.55 \text{ dB}$$

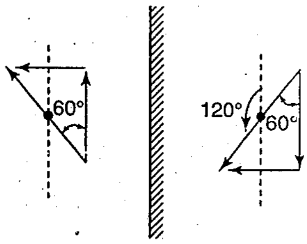
$\theta = 70.5288°$: $E_\theta/E_{\theta m} = \sin\theta \cdot \cos(kh\cos\theta)\big|_{h=\frac{3\lambda}{2}} = 0.9428 \Rightarrow \frac{E_\theta}{E_{\theta m}}$

$= 20\log_{10}(0.9428)$

$$\frac{E_\theta}{E_{\theta m}} = -0.5115 \text{ dB}$$

$\theta = 90°$: $E_\theta/E_{\theta m} = \sin\theta \cdot \cos(kh\cos\theta)\big|_{h=\frac{3\lambda}{2}} = 1 \Rightarrow \frac{E_\theta}{E_{\theta m}} = 20\log_{10}(1) = 0$ dB

4-51.



4-52. $E_\theta \simeq j\eta \frac{kI_0 l e^{-jkr}}{4\pi r} \sin\theta \cdot [2\cos(kh\cos\theta)]$

$|\text{AF}|_{\max} = |\cos(kh\cos\theta)|_{\max} = 1$ when $kh\cos\theta_{\max} = \pi$

$kh\cos\theta_{\max} = \pi,\ kh\cos(60°) = \pi,$

$\frac{2\pi}{\lambda} \cdot h \cdot \left(\frac{1}{2}\right) = \pi,\ h = \lambda$

No matter what the height is when $\theta = 90°$, it is a maximum.

So you always have a maximum at $\theta = 90°$. If you want a maximum at $\theta = 60°$, then $kh\cos\theta = n\pi, (n = 1, 2, 3, \ldots)$ leads to a maximum at $\theta = 60°$.

$n = 1$: $kh\cos\theta\big|_{\max} = \pi, h = \lambda$ leads to maxima at $\theta = 90°, 60°$

If you check closely, it also leads to a maximum at $\theta = 0°$.

So you cannot only have one maximum at $\theta = 60°$

4-53. $E_\theta \sim C_1 \cdot \sin\theta \cdot \cos(kh\cos\theta)\Big|_{\theta=80°} = 0$

$$\cos(kh\cos\theta)\Big|_{\theta=80°} = 0, kh\cos\theta\Big|_{\theta=80°} = \frac{\pi}{2}, \frac{2\pi}{\lambda}h\cos'\theta\Big|_{\theta=80°} = \frac{\pi}{2}$$

$$h = \frac{\lambda}{4\cos\theta}\Big|_{\theta=80°} = \frac{\lambda}{4(0.1736)} = \frac{\lambda}{0.6946} = 1.4397\cdot\lambda$$
$$h = 1.4397\lambda, \lambda = \frac{3\times10^8}{50\times10^6} = \frac{30\times10^7}{5\times10^7} = 6 \text{ meters}$$
$$h = 1.4397\cdot\lambda = 1.4397\cdot(6) = 8.6382 \text{ m}$$
$$h = 8.6382 \text{ meters}$$

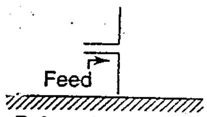
4-54. a. $Z_{im}(l=\lambda/2)\big|_{\text{above ground plane}} = \frac{1}{2}Z_{im}(l=\lambda)\big|_{\text{free space}} \simeq \frac{1}{2}(R_{im}+jX_{im})\big|_{l=\lambda}$

From Problem 4-23 ⇒ $R_{im} = R_r = 199.099$

From Figure 4.20 ⇒ $X_{im}\left(l=\frac{\lambda}{2}\right)\Big|_{\text{above ground plane}} \simeq 62.5$

Therefore

$$Z_{im}(l=\lambda/2)\big|_{\text{above ground plane}} = \frac{199.099}{2} + j62.5 = 99.5495 + j625$$



b. $$Z_{\text{in}} = \frac{Z_{im}}{\sin^2\left(\frac{kl}{2}\right)} = \frac{99.5495 + j62.5}{\sin^2(\pi)} = \infty$$

c. $$\Gamma = \frac{Z_{\text{in}} - Z_c}{Z_{\text{in}} + Z_c} = \frac{\infty - 50}{\infty + 50} = \frac{1 - 50/\infty}{1 + 50/\infty} = 1$$

$$\text{VSWR} = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+1}{1-1} = \infty$$

X_{im} can also be obtained using (8-60b). For $l = \lambda \Rightarrow kl = 2\pi,\ 2kl = 4\pi$. Thus

$$\begin{aligned}
&X_{im}(l = \lambda/2)\big|_{\text{above ground plane}} = \tfrac{1}{2}X_{im}(l = \lambda)\big|_{\text{free space}} \\
&= \frac{\eta}{8\pi}\{2S_i(kl) + \cos(kl)[2S_i(kl) - S_i(2kl)]\} \\
&= \frac{120\pi}{80\pi}\{2S_i(2\pi) + \cos(2\pi)[2S_i(2\pi) - S_i(4\pi)]\} \\
&= 15\{2(1.418) + [2(1.418) - 1.492]\} = 62.7
\end{aligned}$$

4-55. $\text{AF} = \cos(kh\cos\theta), f = 1\text{ GHz} \Rightarrow \lambda = \dfrac{3\times10^8}{1\times10^9} = 0.3$ meters

(a)
$$|(\text{AF})|_{\theta_n=30^\circ} = |\cos(kh\cos30^\circ)| = |\cos(0.866kh)| = 0$$
$$\Rightarrow 0.866kh = \cos^{-1}(0) = \frac{n\pi}{2}, n = 1,2,3,\ldots$$

$$h_1 = \frac{n\pi/2}{0.866\ k} = \frac{n\pi/2}{0.866 2\pi/\lambda} = \frac{n\lambda}{0.866(4\pi)} = \left.\frac{n(3)}{4(0.866\pi)}\right|_{n=1} = 0.0866 \text{ meters}$$

$h_1 = 0.0866$ meters

(b) 1.
$$|\cos(kh\cos\theta)|_{h=0.3\ m=1\lambda} = \left|\cos\left(\frac{2\pi}{\lambda}\lambda\cos\theta_n\right)\right| = |\cos(2\pi\cos\theta_n)| = 0$$
$$2\pi\cos\theta_n = \cos^{-1}(0) = \frac{n\pi}{2} \Rightarrow n = 1,3,5,\ldots$$

$$\theta_n = \cos^{-1}\left(\frac{n\pi/2}{2\pi}\right) = \cos^{-1}\left(\frac{n}{4}\right), n = 1,3,5,\ldots$$

$$\begin{aligned}
\mathbf{n = 1:}\quad & \theta_1 = \cos^{-1}\left(\tfrac{1}{4}\right) = 75.52^\circ \\
\mathbf{n = 3:}\quad & \theta_3 = \cos^{-1}\left(\tfrac{3}{4}\right) = 41.41^\circ \\
\mathbf{n = 5:}\quad & \theta_5 = \cos^{-1}\left(\tfrac{5}{4}\right) = \text{does not exist}
\end{aligned}$$

2.
$$|\cos(kh\cos\theta_m)|_{h=0.3\ m=\lambda} = \left|\cos\left(\frac{2\pi}{\lambda}\lambda\cos\theta_m\right)\right| = |\cos(2\pi\cos\theta_m)| = 1$$
$$2\pi\cos\theta_m = \cos^{-1}(1) = m\pi, \quad m = 0,1,2,3,\ldots$$

$$\theta_m = \cos^{-1}\left(\frac{m\pi}{2\pi}\right) = \cos^{-1}\left(\frac{m}{z}\right)$$

$$\begin{aligned}
\mathbf{m = 0:}\quad & \theta_0 = \cos^{-1}(0) = 90^\circ \\
\mathbf{m = 1:}\quad & \theta_1 = \cos^{-1}\left(\tfrac{1}{2}\right) = 60^\circ \\
\mathbf{m = 2:}\quad & \theta_2 = \cos^{-1}(1) = 0^\circ \\
\mathbf{m = 3:}\quad & \theta_3 = \cos^{-1}\left(\tfrac{3}{2}\right) = \text{does not exist}
\end{aligned}$$

4-56. $f = 200$ MHZ $\Rightarrow \lambda = \dfrac{3 \times 10^8}{2 \times 10^8} = 1.5$ meters

$$E_\theta(\text{normalized}) = \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\cos(kh\cos\theta).$$

Since $\dfrac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$ has a null only toward $\theta = 0°$, the only way to place a null toward $\theta = 60°$ will be through $\cos(kh\cos\theta)$.

$$|\cos(kh\cos\theta)|_{\theta=\theta_h=60°} = |\cos(kh\cos\theta_n)| = |\cos(kh\cos 60°)| = 0$$

$$\left|\cos\left(\frac{2\pi}{\lambda}h\tfrac{1}{2}\right)\right| = \left|\cos\left(\frac{\pi h}{\lambda}\right)\right| = 0$$

$$\frac{\pi h}{\lambda} = \cos^{-1}(0) = \frac{n\pi}{2}, \quad n = 1, 3, 5, \ldots$$

$$\boxed{h = \frac{n\pi}{2}\left(\frac{\lambda}{\pi}\right) = \frac{n\lambda}{2}, \quad n = 1, 3, 5, \ldots}$$

a. $h|_{n=1} = h_1 = \dfrac{\lambda}{2} = \dfrac{3}{2}\left(\frac{1}{2}\right) = \dfrac{3}{4} = \boxed{0.75 \text{ meters}}$

b. $h|_{n=3} = h_3 = \dfrac{3\lambda}{2} = \boxed{2.25 \text{ meters}}$

c. $h|_{n=5} = h_5 = \dfrac{5\lambda}{2} = \boxed{3.75 \text{ meters}}$

4-57. $G_0(\text{dB}) = 10\log_{10} G_0(\text{dimensionless}) \Rightarrow 16 = 10\log_{10} G_0$

$\Rightarrow G_0(\text{dimensionless}) = 10^{1.6} = 39.81$

$P_{rad} = e_0 P_{in} = (1)(8) = 8$ watts

$$W_0 = \frac{P_{rad}}{4\pi r^2} = \frac{8}{4\pi(100 \times 100)^2} = \frac{8}{4\pi \times 10^8} = \frac{2}{\pi} \times 10^{-8} = 0.6366 \times 10^{-8}$$

$= 6.366 \times 10^{-9}$ watts/cm^2

$W_{max} = W_0 G_0(\text{dimensionless}) = 39.81(6.366 \times 10^{-9})$

$= 2.534 \times 10^{-7} = 0.2534 \times 10^{-6}$ Watts/cm^2

4-58. $l = \lambda/4, f = 1.9$ GHz, $W_i = 10^{-6}$ W/m$^2 \Rightarrow \lambda = \dfrac{3 \times 10^6}{1.9 \times 10^9} = 0.15789$ m

a. The power pattern of a $\lambda/4$ monopole *above* a PEC is equivalent to that of a $\lambda/2$ dipole in free space. Since the same power radiated by the monopole above the PEC is concentrated only in the *upper* hemisphere, instead over the entire free space, its radiation intensity will be *twice* as strong/intense as that of the $\lambda/2$ dipole radiating in free space. Since the directivity is given by

$$D_0 = \frac{4\pi U_{max}}{P_{rad}}$$

The U_{max} of the monopole will be twice that of the dipole, or

$$D_0(l = \lambda/4) = 2(1.643) = \boxed{3.286 = 5.17 \text{ dB}}$$

Using the computer program directivity it gives

$$D_0(l = \lambda/4) = \boxed{3.3365 = 5.2329 \text{ dB}}$$

b. $A_{em} = \dfrac{\lambda^2}{4\pi} D_0 = \dfrac{(0.15789)^2}{4\pi}(3.286) = 6.52 \times 10^{-3} \text{ m}^2$

$P_L = A_{em} W_i = 6.52 \times 10^{-3}(10^{-6}) = 6.52 \times 10^{-9}$

$\boxed{P_L = 6.52 \times 10^{-9} \text{ watts}}$

4-59. $f = 900$ MHz, $P_{rad} = 1{,}000$ watts

a. *Isotropic*

$$W_{r0} \le \frac{P_{rad}}{4\pi r^2}$$

$$r^2 \ge \frac{P_{rad}}{4\pi W_{r0}} = \frac{1{,}000}{4\pi(10)} = \frac{100}{4\pi} = 7.9558$$

$$\boxed{r \ge 2.821 \text{ meters}}$$

b. **$\lambda/4$ monopole**

$$D_0(\text{monopole}) = 2(1.643) = 3.286$$

$$W_{rad} \le D_0\, W_{r0} = D_0 \frac{P_{rad}}{4\pi r^2}$$

$$r^2 \ge D_0 \frac{P_{rad}}{4\pi W_{rad}} = 3.286\left(\frac{1{,}000}{4\pi(10)}\right) = 26.1492$$

$$\boxed{r \ge 5.114 \text{ meters}}$$

4-60. Using the coordinate system of Figure 4.24 the total field is given by (4-116) or

$$E_\psi = j\eta \frac{kI_0 l e^{-jkr}}{4\pi r} \sqrt{1 - \sin^2\theta \cdot \sin^2\phi}\,[2j \sin(kh \cos\theta)],\ 0 \leqslant \theta \leqslant \pi, 0 \leqslant \phi \leqslant 2\pi$$

However if we rotate the axes so that the z axis is parallel to the axis of the element and y is vertical to the ground, the total E-field can be written as

$$E_\theta = j\eta \frac{kI_0 e^{-jkr}}{4\pi r} \cdot \sin\theta \cdot [2j\sin(kh\sin\theta\cdot\sin\phi)], \quad \text{and}$$

$$P_{\text{rad}} = \int_0^\pi \int_0^\pi \underline{W}_{\text{ave}} \cdot \hat{a}_r r^2 \sin^2\theta \, d\theta \, d\phi = \frac{1}{2\eta}\int_0^\pi \int_0^\pi |E_\theta|^2 r^2 \sin\theta \, d\theta \, d\phi$$

$$P_{\text{rad}} = \frac{\eta}{2}\left|\frac{kI_0 l}{2\pi}\right|^2 \int_0^\pi \int_0^\pi \sin^3\theta \cdot \sin^2(kh\sin\theta\sin\phi)\, d\theta\, d\phi = \frac{\eta}{2}\left|\frac{kI_0 l}{2\pi}\right|^2 I$$

$$I = \int_0^\pi \sin^3\theta \left\{\int_0^\pi \sin^2(kh\sin\theta\sin\phi)\, d\phi\right\} d\theta = \int_0^\pi \sin^3\theta [I_1]\, d\theta$$

$$\text{where } I_1 = \int_0^\pi \sin^2(kh\sin\theta\sin\phi)\, d\phi = \tfrac{1}{2}\left\{\int_0^\pi d\phi - \int_0^\pi \cos(2kh\sin\theta\cdot\sin\phi\, d\phi\right\}$$

$$= \tfrac{1}{2}\left\{\pi - \int_0^\pi \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right) d\phi\right\}, \text{where } y = 2kh\sin\theta\sin\phi$$

$$= \frac{\pi}{2} - \tfrac{1}{2}\left\{\pi - \tfrac{1}{2}\int_0^\pi y^2\, d\phi + \frac{1}{(2\times 2)!}\int_0^\pi y^4\, d\phi - \frac{1}{(2\times 3)!}\int_0^\pi y^6\, d\phi + \cdots\right\}$$

$$I_1 = \tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\int_0^\pi \frac{(y)^{2n}}{2n!}\, d\phi = \tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2\alpha)^{2n}}{2n!}\int_0^\pi \sin^{2n}\phi\, d\phi$$

$$I_1 = \tfrac{1}{2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2\alpha)^{2n}}{2n!}\left[2\int_0^{\pi/2}\sin^{2n}\phi\, d\phi\right]$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2\alpha)^{2n}}{2n!}\int_0^{\pi/2}\sin^{2n}\phi\, d\phi$$

From Mathematical Handbook of Formulas and Tables $\left(\begin{matrix}\leftarrow \alpha = kh\sin\theta \\ y = 2\alpha\sin\phi\end{matrix}\right)$ Schaum's Outline Series, pg. 96 Equation 15-30.

$$\int_0^{\pi/2}\sin^{2n}x\, dx = \frac{1\cdot 3\cdot 5\cdot\cdots(2n-3)(2n-1)}{2\cdot 4\cdot 6\cdot\cdots(2n-2)(2n)}\frac{\pi}{2}, n = 1,2,3,4,\ldots$$

$$\text{Thus } I_1 = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2kh\sin\theta)^{2n}}{(2n)!}\cdot\frac{\pi}{2}\cdot A_{2n}, \text{ where } A_{2n} = \frac{1\cdot 3\cdot 5\cdots(2n-3)(2n-1)}{2\cdot 4\cdot 6\cdots(2n-2)(2n)}$$

and

$$I = \int_0^\pi \sin^3\theta[I_1]\, d\theta = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(2kh)^{2n}}{(2n)!}\left(\frac{\pi}{2}\right)A_{2n}\int_0^\pi \sin^{2n+3}(\theta)\, d\theta$$

$$= \pi\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{(2kh)^{2n}}{(2n)!}\cdot A_{2n}\int_0^{\pi/2}\sin^{2n+3}\theta\, d\theta$$

Using Series equation of the previous reference, or

$$\int_0^{\pi/2} (\sin x)^{2n+3}\, dx = \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n-2)(2n)(2n+2)}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)(2n+1)(2n+3)}, n = 1, 2, 3, \ldots$$

We can write that

$$I = \pi \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2kh)^{2n}}{(2n)!} (A_{2n})(A_{2n+3}),$$

$$A_{2n+3} = \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n-2)(2n)(2n+2)}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)(2n+1)(2n+3)}$$

However

$$A_{2n}\cdot A_{2n+3} = \frac{1\cdot 3\cdot 5\cdots(2n-3)(2n-1)\ 2\cdot 4\cdot 6\cdot\cdots\cdot(2n-2)(2n)(2n+2)}{2\cdot 4\cdot 6\cdots(2n-2)(2n)\ 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)(2n+1)(2n+3)}$$
$$= (2n+2)/[(2n+1)(2n+3)]$$

Therefore

$$I = \pi \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2kh)^{2n}}{(2n)!} \frac{(2n+2)}{(2n+1)(2n+3)}$$
$$= \pi \left[\frac{(2kh)^2}{2!}\frac{4}{3\cdot 5} - \frac{(2kh)^4}{4!}\frac{6}{5\cdot 7} + \frac{(2kh)^6}{6!}\frac{8}{7\cdot 9} - \cdots \right.$$
$$\left. + (-1)^{n+1} \frac{(2kh)^{2n}}{(2n)!} \frac{(2n+2)}{(2n+1)(2n+3)} \right]$$

which when expanded can be written as

$$I = \pi \left\{ \frac{2}{3} - \left[\frac{2}{3} + (2kh)^2 \left(-\frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} \right) + (2kh)^4 \left(\frac{1}{5!} - \frac{1}{6!} + \frac{1}{7!} \right) + \cdots \right.\right.$$
$$\left.\left. \pm (2kh)^{2n} \left(\frac{1}{(2n+1)!} - \frac{1}{(2n+2)!} + \frac{1}{(2n+3)!} \right) \right] \right\}$$

Recombining appropriate terms, we have that

$$I = \pi \left\{ \frac{2}{3} - \frac{1}{(2kh)} \left[(2kh) + \sum_{n=1}^{\infty} (-1)^n \frac{(2kh)^{2n+1}}{(2n+1)!} \right] \right.$$
$$- \frac{1}{(2kh)^2} \left[1 - \frac{(2kh)^2}{2!} + \sum_{n=2}^{\infty} (-1)^n \frac{(2kh)^{2n}}{(2n)!} \right]$$
$$\left. + \frac{1}{(2kh)} \left[(2kh) - \frac{(2kh)^3}{3!} + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2kh)^{2n+3}}{(2n+3)!} \right] \right\}$$

which reduces when expanded to

$$I = \pi \left[\frac{2}{3} - \frac{\sin(2kh)}{(2kh)} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3} \right]$$

Therefore the radiated power can be written as

$$P_{\text{rad}} = \frac{\eta}{2} \left| \frac{kI_0 l}{2\pi} \right|^2 I = \eta \frac{\pi}{2} \cdot \left| \frac{I_0 l}{\lambda} \right|^2 \left[\frac{2}{3} - \frac{\sin(2kh)}{(2kh)} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3} \right]$$

4-61. $E_\psi = C_2 \cdot \sqrt{1 - \sin^2\theta \sin^2\phi} \cdot [\sin(kh\cos\theta)]$. $C_2 = -\eta \dfrac{kI_0 l e^{-jkr}}{2\pi r}$

a. $E_\psi(\phi = 90°)\Big|_{\theta=45°} = C_2 \cdot \cos\theta \cdot \sin(kh\cos\theta)\Big|_{\theta=45°} = C_2 \cdot \cos(45°) \sin\left(\dfrac{kh}{\sqrt{2}}\right) = 0$

$\dfrac{kh}{\sqrt{2}} = \sin^{-1}(0) = \pm n\pi, n = 0, 1, 2, 3, \ldots$

Choosing the positive values and excluding the $n = 0$ value, we have the smallest height of $(n = 1)$

$$h = \frac{\sqrt{2}\pi}{k} = \frac{\sqrt{2}\pi}{2\pi}\lambda = \frac{\lambda}{\sqrt{2}} = 0.707\lambda$$

b. $h = \dfrac{\lambda}{\sqrt{2}} \Rightarrow 2kh = 2\left(\dfrac{2\pi}{\lambda}\right)\dfrac{\lambda}{\sqrt{2}} = 2\sqrt{2}\pi = 8.88576$

1. $R_r = 120\pi^2 \left(\dfrac{1}{50}\right)^2 \left[\dfrac{2}{3} - \dfrac{\sin(2\sqrt{2}\pi)}{2\sqrt{2}\pi} - \dfrac{\cos(2\sqrt{2}\pi)}{(2\sqrt{2}\pi)^2} + \dfrac{\sin(2\sqrt{2}\pi)}{(2\sqrt{2}\pi)^3} \right]$

$R_r = 120\pi^2 \left(\dfrac{1}{50}\right)^2 \left[\dfrac{2}{3} - 0.057765 + 0.0108694 + 0.0007316 \right] = 0.294$

2. $kh = \sqrt{2}\pi$

$D_g = \dfrac{4(-0.9639)^2}{\left[\frac{2}{3} - 0.57765 + 0.0108694 + 0.0007316\right]} = \dfrac{4(0.9291)}{0.6205} = 5.9893$

$D_g = 5.9893 = 7.774$ dB

4-62. $E_\psi(\phi = 90°) = C_2 \cdot \cos\theta \cdot \sin(kh\cos\theta), C_2 = -\eta \dfrac{kI_0 l e^{-jkr}}{2\pi r}$

$E_\psi(\phi = 90°)|_{h=0.707\lambda} = C_2 \cdot \cos\theta_n \cdot \sin(0.707\lambda k \cos\theta_n) = 0$

$$\cos\theta_n = 0 \Rightarrow \theta_n = \cos^{-1}(0) = 90°$$

$$\sin(0.707\lambda k\cos\theta_n) = \sin(1.414\pi\cos\theta_n) = 0 \Rightarrow 1.414\pi\cos\theta_n = \sin^{-1}(0)$$

$$1.414\pi\cos\theta_n = \sin^{-1}(0) = \pm n\pi, n = 0, 1, 2, 3, \ldots$$

$$\theta_n = \cos^{-1}\left(\pm\frac{n}{1.414}\right), n = 0, 1, 2, 3, \ldots$$

$$\left.\begin{array}{ll} n=0: & \theta_n=\cos^{-1}(0)=90^\circ \\ n=\pm 1: & \theta_n=\cos^{-1}\left(\pm\dfrac{1}{1.414}\right)=45^\circ \end{array}\right\} \text{ for } 0^\circ \leqslant \theta \leqslant 90^\circ$$

$$n=\pm 2: \quad \theta_n=\cos^{-1}\left(\pm\frac{2}{1.414}\right)=\text{Does not exist.}$$

The same holds for $n \geq 3$.

4-63. Since the horizontal dipole is placed a distance of 2λ above the PEC, then its image must also be a distance of 2λ below the PEC. This makes the separation between the actual source and its image to be 4λ. Since the minimum far-field distance is equal to

$$r = 2D^2/\lambda$$

where D is the large distance, which in this case is the hypotenuse, or

$$D = \sqrt{(4\lambda)^2 + (\lambda/50)^2} = 4.00005\lambda \simeq 4\lambda$$

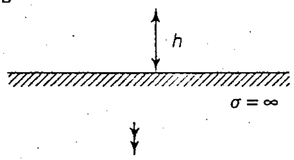
then

$$r = 2(4\lambda)^2/\lambda = 32\lambda$$

Since λ at 300 MHz the wavelength is 1 meter, then

$$r = 32\lambda|_{\lambda=1} = 32 \text{ meters}$$

4-64. $H_\theta{}^d = j\dfrac{kI_m l e^{-jkr_1}}{\eta \cdot 4\pi r} \cdot \sin\theta_1$

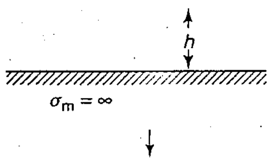


$$H_\theta{}^r = -j\frac{kI_m l e^{-jkr_2}}{\eta \cdot 4\pi r} \cdot \sin\theta_2,$$

$$\left.\begin{array}{l} r_1 = r - h\cos\theta \\ r_2 = r + h\cos\theta \end{array}\right\} \text{phase}, \quad \left.\begin{array}{l} r = r_1 = r_2, \text{ for amplitude} \\ \theta_1 = \theta_2 = \theta. \end{array}\right\} \Rightarrow \text{Far Field}$$

$$H_\theta = j\frac{kI_m l e^{-jkr}}{\eta \cdot 4\pi r} \cdot \sin\theta[2j\sin(kh\cos\theta)]$$

4-65. $E_\theta{}^d = j\eta \cdot \dfrac{kI_0 l e^{-jkr_1}}{4\pi r_1} \cdot \sin\theta_1$



$$E_\theta{}^r = -j\eta \cdot \frac{kI_0 l e^{-jkr_2}}{4\pi r_2} \cdot \sin\theta_2$$

Far field: $\left.\begin{array}{l} r_1 = r - h\cos\theta \\ r_2 = r + h\cos\theta \end{array}\right\}$ phase

$(r = r_1 = r_2, \ \theta = \theta_1 = \theta_2)$ amplitude

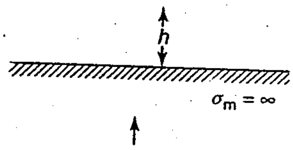
$$E_\theta = j\eta\frac{kI_0 l e^{-jkr}}{4\pi r}\sin\theta[e^{jkh\cdot\cos\theta} - e^{-jkh\cos\theta}]$$

$$E_\theta = j\eta\frac{kI_0 l e^{-jkr}}{4\pi r} \cdot \sin\theta \cdot [2j\sin(kh\cos\theta)]$$

4-66. $H_\theta{}^{\text{total}} = H_\theta^d + H_\theta^r$

$$H_\theta{}^d = j\frac{kI_m l e^{-jkr_1}}{\eta \cdot 4\pi r_1} \cdot \sin\theta_1$$

$$H_\theta{}^r = j\frac{kI_m l e^{-jkr_2}}{\eta \cdot 4\pi r_2} \cdot \sin\theta_2$$



For Far field. $\left.\begin{matrix} r_1 \simeq r - h\cos\theta \\ r_2 \simeq r + h\cos\theta \end{matrix}\right\}$ phase, $(r_1 = r_2 = r) \rightarrow$ Amplitude

$$\theta = \theta_1 = \theta_2$$

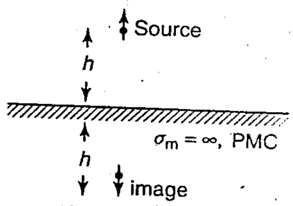
$$H_\theta^{\text{total}} = j\frac{kI_m l e^{-jkr}}{\eta \cdot 4\pi r} \cdot \sin\theta \cdot [2\cos(kh\cos\theta)]$$

4-67. a. $E_\theta \simeq j\eta\frac{kl}{4\pi}I_0\frac{e^{-jkr}}{r}\sin\theta[j2\sin(kh\cos\theta)]$

$\sin(kh\cos 60°) = 0 \rightarrow kh_n\cos 60° = n\pi, n = 1, 2, 3, \ldots$

$$h_n = \frac{n\pi}{k\cos 60°} = \frac{n\lambda}{2 \cdot \cos 60°} = n\lambda$$

smallest $h \rightarrow n = 1 \rightarrow h = \lambda$



b. $$W_{av} \simeq \frac{|E_0|^2}{2\eta} \simeq \frac{\eta(kl)^2}{32 \cdot \pi^2} \cdot |I_0|^2 \cdot \frac{(kl)^2}{r^2} \cdot \sin^2\theta[4\sin^2(kh\cos\theta)]$$

$$U(\theta,\phi) = \lim_{r\to\infty} r^2\, W_{av} = \frac{\eta}{2} \cdot \left(\frac{l}{\lambda}\right)^2 \cdot |I_0|^2 \cdot \sin^2\theta \cdot \sin^2(kh\cos\theta)$$

$$= \int_0^{2\pi}\int_0^{\pi/2} U(\theta,\phi)\sin\theta\, d\theta\, d\phi$$

$$= \pi\eta\left(\frac{l}{\lambda}\right)^2 \cdot |I_0|^2 \cdot \int_0^{\pi/2}\sin^3\theta \cdot \sin^2(kh\cos\theta)\, d\theta$$

$$= \pi\eta \cdot \left(\frac{l}{\lambda}\right)^2 \cdot |I_0|^2 \cdot \left\{\frac{1}{3} + \frac{\cos(2kh)}{(2kh)^2} - \frac{\sin(2kh)}{(2kh)^3}\right\}$$

$$P_{rad} \underset{\substack{\uparrow \\ kh=2\pi}}{=} \pi\eta \cdot \left(\frac{l}{\lambda}\right)^2 \cdot |I_0|^2 \cdot \left\{\frac{1}{3} + \frac{1}{(4\pi)^2}\right\} = \pi\eta\left(\frac{l}{\lambda}\right)^2 |I_0|^2\{0.3397\}$$

1. $$D_g(\theta = 45°, \phi) = \frac{4\pi U(\theta = 45°, \phi)}{P_{rad}} = \frac{2\sin^2(45°)\sin^2(2\pi\cos 45°)}{0.3397}$$

$$= 2.74 = 4.37 \text{ dB}$$

2. $$R_r = \frac{2P_{rad}}{|I_0|^2} = 2 \cdot \pi \cdot \eta\left(\frac{l}{\lambda}\right)^2 \cdot \{0.3397\}$$

$$\frac{R_r}{\eta} = 2\pi \times 10^{-4} \times 0.3397 = 2.13 \times 10^{-4}.$$

4-68. Since $d \ll a$

$$\tan\psi \simeq \frac{h_1'}{d_1} \simeq \frac{h_2'}{d_2} = \frac{h_2'}{d-d_1} \Rightarrow h_1'(d-d_1) = d_1 h_2'$$

$$d_1(+h_1' + h_2') = h_1'\, d \Rightarrow d_1 = \frac{h_1'\, d}{h_1' + h_2'} = \frac{5(20 \times 10^3)}{5 + 1{,}000} = 99.5 \text{ meters}$$

$$\psi = \tan^{-1}\left(\frac{h_1'}{d}\right) = \tan^{-1}\left(\frac{5}{99.5}\right) = 2.87669^\circ$$

$$\frac{\sigma}{\omega\epsilon} = \frac{10^{-2}}{2\pi \times 10^9(5 \times 10^{-9}/(36\pi))} = \frac{18}{5} \times 10^{-2} = 3.6 \times 10^{-2} \ll 1$$

Therefore the earth is a good dielectric $\Rightarrow \eta_1 \simeq \sqrt{\frac{\mu_1}{\epsilon_1}}, \beta_1 = \omega\sqrt{\mu_1\epsilon_1}$

The divergence factor is equal to ($a = 5280$ miles $= 8.497368 \times 10^6$ m)

$$D \simeq \left[1 + 2\frac{h_1' h_2'}{a d \tan^3\psi}\right]^{-1/2} = \left[1 + \frac{2(5)(1{,}000)}{8.497368 \times 10^6 \times 2 \times 10^4 (0.05)^3}\right]^{-1/2}$$

$$= (1 + 0.000463)^{-1/2}$$

$$= 0.99977$$

and the reflection coefficient equal to

$$R_v = \frac{\eta_0 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_0 \cos\theta_i + \eta_1 \cos\theta_t}, \text{ where } \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}},\ \eta_1 = \sqrt{\frac{j\omega\mu_0}{\sigma_1 + j\omega\epsilon_1}} \simeq \sqrt{\frac{\mu_0}{\epsilon_1}}$$

$$\gamma_0 \sin\theta_i = \gamma_1 \sin\theta_t \Rightarrow \beta_0 \sin\theta_i = \beta_1 \sin\theta_t \Rightarrow \sin\theta_t = \frac{\beta_0}{\beta_1}\sin\theta_i = \sqrt{\frac{\epsilon_0}{\epsilon_1}}\sin\theta_i$$

$$\cos\theta_t = \sqrt{1 - \sin^2\theta_t} = \sqrt{1 - \frac{\epsilon_0}{\epsilon_1}\sin^2\theta_i} = \sqrt{1 - \frac{\sin^2\theta_i}{\epsilon_r}}$$

Therefore

$$R_v = \frac{\cos\theta_i - \frac{\eta_1}{\eta_0}\cos\theta_t}{\cos\theta_i + \frac{\eta_1}{\eta_0}\cos\theta_t} = \frac{\cos\theta_i - \frac{1}{\sqrt{\epsilon_r}}\sqrt{1 - \sin^2\theta_i/\epsilon_r}}{\cos\theta_i + \frac{1}{\sqrt{\epsilon_r}}\sqrt{1 - \sin^2\theta_i/\epsilon_r}} = \frac{\epsilon_r\cos\theta_i - \sqrt{\epsilon_r - \sin^2\theta_i}}{\epsilon_r\cos\theta_i + \sqrt{\epsilon_r - \sin^2\theta_i}}$$

$$\theta_i = 90 - \psi = 90 - 2.87669^\circ = 87.12331^\circ \Rightarrow \sin\theta_i = 0.9987, \cos\theta_i = 0.0502$$

$$\text{Thus } R_v = \frac{5(0.0502) - \sqrt{5 - (0.9987)^2}}{5(0.0502) + \sqrt{5 - (0.9987)^2}} = \frac{-1.749649}{2.251649} = -0.777$$

$$E_\theta \simeq j\eta\frac{I_0 e^{-jkr}}{2\pi r} \cdot \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \cdot [e^{jkh_1'\cos\theta} + DR_v e^{-jkh_1'\cos\theta}]_{\theta=\theta_i \simeq 87.12331^\circ}$$

$$r \simeq \sqrt{d^2 + (h_2' - h_1')^2} = \sqrt{(20{,}000)^2 + (1{,}000 - 99.5)^2} = 20{,}020.26 \text{ m} = 66{,}734.207\lambda$$

$$h_1' = 5 \text{ m} = 16.667\lambda, h_2' = 1{,}000 \text{ m} = 3{,}333.3333\lambda$$

$$E_\theta = j \cdot 120\pi \cdot \frac{I_0 e^{-j\frac{2\pi}{\lambda}(66{,}734.207\lambda)}}{2\pi(20{,}020.26)} \cdot \frac{\cos\left(\frac{\pi}{2}\cos(87.12^\circ)\right)}{\sin(87.12^\circ)}[e^{jkh_1'\cos\theta_i} + DR_v e^{-jkh_1'\cos\theta_i}]$$

$$e^{-j\frac{2\pi}{\lambda}(66{,}734.207\lambda)} = e^{-j2\pi(0.207)} = e^{-j1.3} = \cos(74.52^\circ) - j\sin(74.52^\circ)$$
$$= 0.2669 - j0.9637 = 1\angle -74.52^\circ$$

$$\cos\left[\frac{\pi}{2}\cos(87.12^\circ)\right] = 0.996887, \sin(87.12^\circ) = 0.99874$$

$$e^{jkh_1'\cos\theta_i} = e^{j\frac{2\pi}{\lambda}(16.667\lambda)(0.0502)} = e^{j2\pi(16.667)(0.0502)} = e^{j5.257}$$
$$= \cos(301.2^\circ) + j\sin(301.2^\circ) = 1\angle 301.2^\circ = 0.5181 - j0.8553$$
$$e^{-jkh_1'\cos\theta_i} = 1\angle -301.2^\circ = 0.5181 + j0.8553$$
$$DR_v e^{-jkh_1'\cos\theta_i} = 0.99977(-0.777)[0.5181 + j0.8553] = -(0.4025 + j0.6644)$$

Thus

$$e^{jkh_1'\cos\theta_i} + R_v D e^{-jkh_1'\cos\theta_i} = (0.5181 - j0.8553) - (0.4025 + j0.6644)$$
$$= 0.1156 - j1.5197 = 1.5241\angle -85.65^\circ$$

Therefore

$$E_\theta \simeq (1\angle 90^\circ)(120\pi)\frac{I_0(1\angle -74.52^\circ)(0.996887)}{2\pi(0.99874)(20{,}020.26)}(1.5241\angle -85.65^\circ)$$
$$E_\theta \simeq 4.5592 \times 10^{-3} \cdot I_0\angle -70.17$$

or

$$|E_\theta| = 4.5592 \times 10^{-3}|I_0| \quad \text{Volts/m}$$

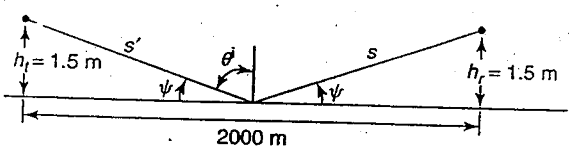
4-69. From calibration;

$$\frac{P_r}{P_t} = \frac{C_1}{R^2} \rightarrow C_1 = \frac{P_r}{P_t}R^2 = \frac{10 \times 10^{-6}}{5} \times (10 \times 10^3)^2 = 200 \text{ m}^2$$

on asteroid

$$\frac{P_r}{P_t} = \frac{C_1}{R^2}|1 + DR_v e^{-j2kht\cos\theta}|^2$$

Approximate geometry;



$$\psi = \tan^{-1}\left(\frac{1.5}{1000}\right) \simeq 1.5 \times 10^{-3} = 0.086°$$

$$\theta^i = \frac{\pi}{2} - \psi; \cos\theta^i = \sin\psi \simeq 1.5 \times 10^{-3}$$

$$\sin\theta^t = \frac{\sin\theta^i}{3} \simeq \frac{1}{3}; \cos\theta^t \simeq \frac{2\sqrt{2}}{3}$$

$$R_v = \frac{\frac{\eta_0}{\eta_1}\cos\theta^i - \cos\theta^t}{\frac{\eta_0}{\eta_1}\cos\theta^i + \cos\theta^t} = \frac{3(1.5 \times 10^{-3}) - \frac{2\sqrt{2}}{3}}{3(1.5 \times 10^{-3}) + \frac{2\sqrt{2}}{3}} = -0.9905$$

$$s' \simeq s \simeq 1000 \text{ m}; a = 10^6 \text{ m}$$

$$D \simeq \left[1 + 2\frac{ss'}{ad\tan\psi}\right]^{-1/2} \simeq \left[1 + 2\frac{(1000)(1000)}{10^6(2000)1.5 \times 10^{-3}}\right]^{-1/2} = 0.7746$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{300 \times 10^6} = 1 \text{ m}$$

$$\begin{aligned} |1 + DR_v e^{-j2kht\cos\theta}|^2 &\simeq |1 - (0.7746)(0.9905)e^{-j4\pi\frac{ht}{\lambda}}|^2 \\ &= |1 - (0.7746)(0.9905)e^{-j4\pi}|^2 \\ &= 0.0541772 \end{aligned}$$

$$P_r = \frac{200}{(2 \times 10^3)^2} \cdot (0.0541772)(5) = 1.3544 \times 10^{-5} \text{ W} = 13.5 \text{ µW}$$

4-70. $P_{\text{rad}} = 10$ Watt, $r = 3.7 \times 10^7$ m, $D_0 = 50$ dB $\Rightarrow 10^5$

a. $$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{4\pi \cdot r^2|E|^2}{2\eta \cdot 10} = 10^5, \left(\text{Since } U_{\max} = \frac{r^2 E_{\max}^2}{2\eta}; \eta = 120\pi\right)$$

$$\Rightarrow E^2 = \frac{10^5 \times 2 \times 120\pi \times 10}{4\pi(3.7 \times 10^7)^2} = 4.4 \times 10^{-8}$$

$$E = 2 \times 10^{-4} \text{ v/m}$$

b. Use Friis Transmission

$$\frac{P_r}{P_t} = \left(\frac{\lambda}{4\pi R}\right)^2 \cdot G_{0t} \cdot G_{0v} = \left(\frac{\lambda}{4\pi R}\right)^2 \cdot D_{0t} \cdot D_{0r}$$

(Since we assume 100% efficiency)

At 10 GHz, $\lambda = 0.03$ m

$$\frac{P_r}{10} = \left[\frac{0.03}{4\pi \cdot (3.7 \times 10^7)}\right]^2 \cdot (10{,}000)(1.643)$$

$$P_r = 6.84 \times 10^{-15}$$

$$P_{\text{received}} = \frac{V^2}{8R_{\text{in}}}, \text{Since } R_r = 73 = R_{\text{in}} \text{ for } \lambda/2 \text{ dipole then}$$

$$V = \sqrt{8(P_{\text{received}})(R_{\text{in}})} = 2\mu V$$

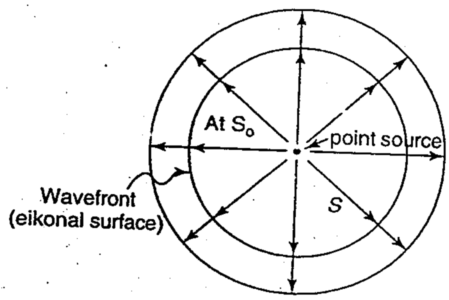
4-71. $S_0\, dA_0 = S dA, \quad \dfrac{S}{S_0} = \dfrac{dA_0}{dA}$

far zone $S = \frac{1}{2\eta}|\underline{E}|^2$

$$\frac{|\underline{E}|}{|\underline{E}_0|} = \sqrt{\frac{dA_0}{dA}}$$

For spherical wave: $\dfrac{|\underline{E}|}{|\underline{E}_0|} = \sqrt{\dfrac{S_0^2}{(S+S_0)^2}} = \dfrac{S_0}{S+S_0}$

For plane wave: $|\underline{E}|/|\underline{E}_0| = 1$.



In general, it can be shown that for a wave front eikonal surface we have

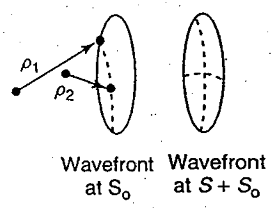
$\dfrac{|\underline{E}|}{|\underline{E}_0|} = \sqrt{\dfrac{\rho_1\rho_2}{(\rho_1+s)(\rho_2+s)}}$; ρ_1 and ρ_2 are radii of curvature of wavefront.

e.g. spherical wave: $\rho_1 = \rho_2 = S_0$

$$\frac{|\underline{E}|}{|\underline{E}_0|} = \sqrt{\frac{{S_0}^2}{(S+S_0)^2}} = \frac{S_0}{S+S_0}$$

plane wave: $\rho_1 = \rho_2 = \infty$

$$\frac{|\underline{E}|}{|\underline{E}_0|} = 1.$$

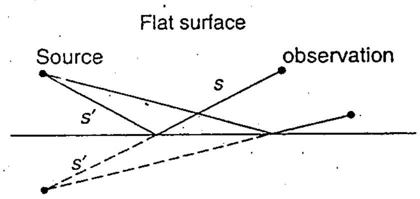


When the wave front is reflected from a surface we have

$$\frac{|\underline{E}|}{|\underline{E}_0|} = \sqrt{\frac{{\rho_1}^r \cdot {\rho_2}^r}{({\rho_1}^r + s)({\rho_2}^r + s)}} = \sqrt{\frac{1}{\left(1 + \frac{s}{{\rho_1}^r}\right)\left(1 + \frac{s}{{\rho_2}^r}\right)}}$$

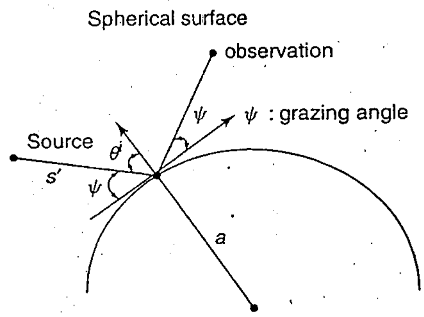
$\underline{E}$ is field at observation point.

$\underline{E}_0$ is field at reflection point.



Radius of curvature of wavefront not changed by reflection.

$$ {\rho_1}^r = {\rho_2}^r = \rho_1 = \rho_2 = s' \quad \frac{|\underline{E}|}{|\underline{E}_0|} = \frac{s'}{s' + s} = \frac{1}{1 + s/s'}$$



$$\frac{1}{{\rho_1}^r} = \frac{1}{s'} + \frac{1}{f_1}; \frac{1}{{\rho_2}^r} = \frac{1}{s'} + \frac{1}{f_2}$$

In physics, we always used $f_1 = f_2 = a/2$. This is not valid here because that f was valid for near normal incidence; we have near grazing incidence

$$f_1 = \frac{a\cos\theta^i}{2} \text{ (perpendicular to the plane of incidence = elevation plane)}$$

$$f_2 = \frac{a}{2\cos\theta_i}\text{: (parallel to the plane of incidence = azimuthal plane)}$$

Thus $\dfrac{1}{\rho_1{}^r} = \dfrac{1}{s'} + \dfrac{2}{a\cos\theta_i}$; $\frac{1}{\rho_2{}^r} = \frac{1}{s'} + \dfrac{2\cos\theta^i}{a}$

$$\frac{|E|}{|E_0|} = \sqrt{\frac{1}{\left\{1 + s\left(\frac{1}{s'} + \frac{2}{a\cos\theta_i}\right)\right\}\left\{1 + s\left(\frac{1}{s'} + \frac{2\cos\theta^i}{a}\right)\right\}}}$$

$$= \frac{1}{\sqrt{1 + \frac{s}{s'} + \frac{2s}{a\cos\theta^i}} \cdot \sqrt{1 + \frac{s}{s'} + \frac{2s\cos\theta_i}{a}}}$$

$$= \frac{1}{\left(1 + \frac{s}{s'}\right)} \frac{1}{\sqrt{1 + \frac{\frac{2ss'}{a\cos\theta_i}}{s+s'}}\sqrt{1 + \frac{\frac{2ss'\cos\theta_i}{a}}{s+s'}}}$$

$\cos\theta_i = \cos\left(\frac{\pi}{2} - \psi\right) = \sin\psi$

$$\frac{|E|}{|E_0|} = \frac{1}{\left(1 + \frac{s}{s'}\right)} \frac{1}{\sqrt{1 + \frac{2ss'}{a(s+s')\sin\psi}}\sqrt{1 + \frac{2ss'\sin\psi}{a(s+s')}}}$$

$$\simeq \left[1 + \frac{2ss'}{a(s+s')\sin\psi}\right]^{-1/2} \cdot \frac{1}{(1 + s/s')}$$

near grazing neglect divergence in azimuthal plane.

CHAPTER 5

5-1. From (5-17) $\Rightarrow \underline{A} = \hat{a}_\phi A_\phi(r,\theta) = \hat{a}_\phi j\frac{k\mu a^2 I_0 \sin\theta}{4r}\left[1+\frac{1}{jkr}\right]e^{-jkr}$

(a) Using (3-2a) and (VII-26)

$$\underline{H} = \frac{1}{\mu}\nabla\times\underline{A} = \frac{1}{\mu}\left\{\hat{a}_r\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \cancel{\frac{\partial A_\theta}{\partial\phi}}^{\,0}\right]\right.$$
$$\left. + \hat{a}_\theta\frac{1}{r}\left[\frac{1}{\sin\theta}\cancel{\frac{\partial A_r}{\partial\phi}}^{\,0} - \frac{\partial}{\partial r}(rA_\phi)\right] + \hat{a}_\phi\frac{1}{r}\left[\frac{\partial}{\partial r}\cancel{(rA_\theta)}^{\,0} - \cancel{\frac{\partial A_r}{\partial\theta}}^{\,0}\right]\right\}$$

which reduces to

$$\underline{H} = \frac{1}{\mu}\left\{\hat{a}_r\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \hat{a}_\theta\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right\}$$

Using the A_ϕ from above

$$\underline{H} = \frac{1}{\mu}\left\{\hat{a}_r\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[j\frac{k\mu a^2 I_0\sin\theta}{4r}\left(1+\frac{1}{jkr}\right)e^{-jkr}\right]\right.$$
$$\left. - \hat{a}_\theta\frac{1}{r}\frac{\partial}{\partial r}\left[j\frac{k\mu a^2 I_0\sin\theta}{4}\left(1+\frac{1}{jkr}\right)e^{-jkr}\right]\right\}$$

which can be written as

$$H_r = j\frac{ka^2 I_0\cos\theta}{2r^2}\left(1+\frac{1}{jkr}\right)e^{-jkr}$$
$$H_\theta = -\frac{(ka)^2 I_0\sin\theta}{4r}\left[1+\frac{1}{jkr}-\frac{1}{(kr)^2}\right]e^{-jkr}$$
$$H_\phi = 0$$

(b) Using Equation (3-10) with $\underline{J} = 0$ along with the $\underline{H}$-field components from above

$$\underline{E} = \frac{1}{j\omega\epsilon}\nabla\times\underline{H} = \frac{1}{j\omega\epsilon}\left\{\hat{a}_r(0) + \hat{a}_\theta(0) + \hat{a}_\phi\frac{1}{r}\left[\frac{\partial}{\partial r}(rH_\theta) - \frac{\partial H_r}{\partial\theta}\right]\right\}$$

which reduces to

$$E_r = 0$$
$$E_\theta = 0$$
$$E_\phi = \eta \frac{(ka)^2 I_0 \sin\theta}{4r}\left[1+\frac{1}{jkr}\right]e^{-jkr}$$

The same expressions can be obtained using (3-15) with the A_ϕ from part a.

5-2. According to the duality theorem and the dual quantities as outlined in Table 3.2

Electric Dipole		Magnetic Dipole
$\underline{E}$	⇔	$\underline{H}$
$\underline{H}$	⇔	$-\underline{E}$
$\underline{I}_e$	⇔	$\underline{I}_m$
ϵ	⇔	μ
μ	⇔	ϵ
κ	⇔	κ
η	⇔	$1/\eta$
$1/\eta$	⇔	η

Thus applying the above to the fields of an electric dipole, as given by (4-8a)–(4-10c), we obtain the fields of a magnetic dipole given by

$$E_r = 0$$
$$E_\theta = 0$$
$$E_\phi = -j\frac{kI_m l \sin\theta}{4\pi r}\left(1+\frac{1}{jkr}\right)e^{-jkr}$$
$$H_r = \frac{1}{\eta}\frac{I_m l\cos\theta}{2\pi r^2}\left(1+\frac{1}{jkr}\right)e^{-jkr}$$
$$H_\theta = j\frac{1}{\eta}\frac{kI_m l\sin\theta}{4\pi r}\left[1+\frac{1}{jkr}-\frac{1}{(kr)^2}\right]e^{-jkr}$$
$$H_\phi = 0$$

which are identical to (5-20a)–(5-20d)

5-3. $a = \lambda/30$, $b = \lambda/1{,}000 = 10^{-3}\lambda$, $f = 10$ MHz $\Rightarrow \lambda = 30$ meters, $\sigma = 5.7\times 10^7$ s/m

(a) $R_r = 20\pi^2\left(\frac{C}{\lambda}\right)^4 = 20\pi^2\left(\frac{2\pi a}{\lambda}\right)^4 = 20\pi^2\left(\frac{2\pi}{30}\right)^4$

$= 20\pi^2(0.2094)^4 = 0.3798$ ohms

(b) $$R_L = R_{hf} = \frac{C}{P}\sqrt{\frac{\omega\mu_0}{2\sigma}} = \frac{2\pi a}{2\pi b}\sqrt{\frac{2\pi f\mu_0}{2\sigma}}$$
$$= \frac{a}{b}\sqrt{\frac{\pi f\mu_0}{\sigma}} = \frac{\lambda/30}{\lambda/1{,}000}\sqrt{\frac{\pi(10^7)4\pi\times10^{-7}}{5.7\times10^7}} = 0.02774$$
$$R_L = R_{hf} = 0.02774$$

(c) $$X_A = \omega L_A = 2\pi f L_A = 2\pi f\left\{\mu_0 a\left[\ln\left(8\frac{a}{b}\right)-2\right]\right\}$$
$$= 2\pi\times10^7\left\{4\pi\times10^{-7}\left(\frac{\lambda}{30}\right)\left[\ln\left(8\frac{1{,}000}{30}\right)-2\right]\right\}$$
$$X_A = 8\pi^2\left(\frac{30}{30}\right)[\ln(266.667)-2] = 8\pi^2(5.58599-2) = 283.139$$

$$X_i = \omega L_i = \omega\left[\frac{a}{wb}\sqrt{\frac{w\mu_0}{2\sigma}}\right] = \frac{a}{b}\sqrt{\frac{2\pi f\mu_0}{2\sigma}} = \frac{a}{b}\sqrt{\frac{\pi f\mu_0}{\sigma}}$$
$$= \frac{\lambda/30}{\lambda/1{,}000}\sqrt{\frac{\pi(10^7)4\pi\times10^{-7}}{5.7\times10^7}}$$
$$X_i = \frac{1{,}000}{30}(2\pi)\times10^4\times10^{-7}\frac{1}{\sqrt{57}} = 0.02774$$
$$X_T = X_A + X_i = 283.139 + 0.02774 = 283.1667$$

(d) $$Z_{in} = (R_r + R_L) + j(X_A + X_i) = (0.3798 + 0.02774) + j(283.1667)$$
$$Z_{in} = 0.40754 + j283.1667$$

(e) $$e_{cd} = \frac{R_r}{R_r + R_L} = \frac{0.3798}{0.3798 + 0.02774} = 0.9319 = 93.19\%$$

5-4. The pattern of a small circular loop of uniform current is given by

$$E_{\phi n} \sim \sin\theta \Rightarrow \mathcal{U} \sim \sin^2\theta$$

which is omnidirectional.

(a) $$D_0 = \frac{4\pi U_{max}}{P_{rad}}$$
$$P_{rad} = \int_0^{2\pi}\int_0^{\pi} U\sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi}\sin^2\theta\sin\theta\, d\theta\, d\phi$$
$$= 2\pi\int_0^{\pi}\sin^3\theta\, d\theta$$
$$= 2\pi\left(\frac{4}{3}\right) = \frac{8\pi}{3}$$
$$D_0(\text{exact}) = \frac{4\pi(1)}{8\pi/3} = \frac{3}{2} = \boxed{1.5 = 1.761\text{ dB}}$$

(b) **Half-power beamwidth of $\sin^2\theta$ is**

$$\sin^2\theta_h = \tfrac{1}{2} \Rightarrow \sin\theta_h = 0.707 \Rightarrow \theta_h = 45^\circ$$
$$\Theta_H = 2\theta_h = 90^\circ = \text{HPBW}$$

$$D_0(\text{McDonald}) = \frac{101}{\text{HPBW(degrees)} - 0.0027\,[\text{HPBW (degrees)}]^2}$$
$$= \frac{101}{90 - 0.0027(90^\circ)^2} = \frac{101}{90 - 21.87} = 1.48246$$
$$D_0(\text{McDonald}) = \boxed{1.48246 = 1.7098 \text{ dB}}$$

(c)
$$D_0(\text{Pozar}) = -172.4 + 191\sqrt{0.818 + \frac{1}{\text{HPBW (degrees)}}}$$
$$= -172.4 + 191\sqrt{0.818 + \frac{1}{90}}$$
$$= -172.4 + 191(0.91055) = -172.4 + 173.916 = 1.5116$$
$$D_0(\text{Pozar}) = \boxed{1.516 = 1.807 \text{ dB}}$$

5-5. $C = \lambda/4 = 2\pi a \Rightarrow a = \lambda/8\pi < \lambda/6\pi \Rightarrow$ small loop

(a) $R_r = 20\pi^2 \left(\frac{C}{\lambda}\right)^4 N^2 = 20\pi^2 \left(\frac{1}{4}\right)^4 N^2 = \frac{20\pi^2}{256} N^2 = 300$

$$\Rightarrow N = \left(\frac{300(256)}{20\pi^2}\right)^{1/2} = 19.72 \simeq 20$$

(b) $R_{\text{in}} = R_r = \frac{20\pi^2}{256}(20)^2 = 308.425$ ohms

(c) $\Gamma = \frac{R_{\text{in}} - Z_c}{R_{\text{in}} + Z_c} = \frac{308.425 - 300}{308.425 + 300} = 0.01385$

(d) VSWR $= \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.01385}{1 - 0.01385} = 1.0281$

5-6. $\underline{E}_w^i = (\hat{a}_y + 2\hat{a}_z)e^{-jkx} = \left(\frac{\hat{a}_y + 2\hat{a}_z}{\sqrt{5}}\right)\sqrt{5}e^{-jkx}$

(a) Linear: Two components in phase.

(b) AR $= \infty$

(c) $\underline{E} = \hat{a}_\phi E_\phi = \hat{a}_\phi C \sin\theta, \quad \hat{a}_\phi = (-\hat{a}_x \sin\phi + \hat{a}_y \cos\phi)\big|_{\substack{\theta=\pi/2 \\ \phi=0}} = \hat{a}_y$

$\underline{E}\big|_{\substack{\theta=90^\circ \\ \phi=0^\circ}} = \hat{a}_y C \Rightarrow$ **Polarization** : Linear in y direction

(d) PLF $= \left|\left(\frac{\hat{a}_y + 2\hat{a}_z}{\sqrt{5}}\right) \cdot \hat{a}_y\right|^2 = \frac{1}{5} = -6.99$ dB

(e) $$f = 1 \text{ GHz} \Rightarrow \lambda = \frac{30 \times 10^9}{1 \times 10^9} = 30 \text{ cm}$$

$$A_{em} = \frac{\lambda^2}{4\pi} D_0 = \frac{\lambda^2}{4\pi}\left(\frac{3}{2}\right) = \frac{(30)^2}{4\pi}\left(\frac{3}{2}\right) = 107.4296 \text{ cm}^2$$

$$P_r = A_{em} W^i(\text{PLF}) = 107.4296(5 \times 10^{-3})\left(\frac{1}{5}\right) = 107.4296 \times 10^{-3}$$

$$P_r = 107.4296 \times 10^{-3} \text{ watts}$$

5-7. $$R_r(1 \text{ turn}) = 20\pi^2 \left(\frac{C}{\lambda}\right)^4 = 20\pi^2\left(\frac{1}{5}\right)^4 = 0.31583 \text{ ohms}$$

$$R_r(4 \text{ turn}) = N^2 R_r(1 \text{ turn}) = 4^2 \cdot (0.31583) = 5.0532 \text{ ohms}$$

$$R_L(1 \text{ turn}) = R_{\text{nf}}(1 \text{ turn}) = \frac{a}{b}\sqrt{\frac{\omega\mu_0}{26}} = \frac{1}{10\pi \times 10^{-3}}\sqrt{\frac{2\pi \times 10^7 \cdot (4\pi \cdot 10^{-7})}{2 \cdot (5.7 \times 10^7)}}$$

$$R_L = R_h = 0.0265$$

$$R_L(4 \text{ turn}) = R_{\text{omic}} = \frac{N_a}{b} R_s \left(\frac{R_p}{R_0} + 1\right)$$

$$R_s = \sqrt{\frac{\omega\mu_0}{2\sigma}} = \sqrt{\frac{2\pi \times 10^7 \times (4\pi \times 10^{-7})}{2(5.7 \times 10^7)}} = 8.3223 \times 10^{-4}$$

$$R_0 = \frac{NR_s}{2\pi b} = \frac{4 \cdot (8.3223 \times 10^{-4})}{2\pi(10^{-3})} = 0.5298$$

$$\frac{R_p}{R_0} \simeq 0.5 \text{ from Fig. 5.3}$$

Thus $$R_L = R_{\text{ohmic}} = \frac{4(8.3223 \times 10^{-4})}{4\pi \times 10^{-3}}(0.5 + 1) = 0.15724$$

and $$e_{cd} \text{ (1 turn)} = 100 \cdot R_r/(R_r + R_L) = \frac{0.3158 \times 100}{0.3158 + 0.0265} = 92.26 = 92.26\%$$

$$e_{cd} \text{ (4 turn)} = 100 \cdot R_r/(R_r + R_L) = \frac{5.0532(100)}{5.0532 + 0.15724} = 96.98\%$$

5-8. $$H_\theta = -\frac{\pi S I_0 e^{-jkr}}{\lambda^2 r}\sin\theta \quad \text{where } S = \pi a^2$$

$$E_\phi = -\eta H_\theta = \eta \frac{\pi S I_0 e^{-jkr}}{\lambda^2 r}\sin\theta$$

$$\underline{W}_{\text{ave}} = \tfrac{1}{2}\text{Re}(\underline{E} \times \underline{H}^*) = \tfrac{1}{2}\text{Re}(\hat{a}_\phi E_\phi \times \hat{a}_\theta H_\theta^*) = \tfrac{1}{2}\text{Re}(-\hat{a}_o \eta H_\theta \times \hat{a}_\theta H_\theta^*)$$

$$\underline{W}_{\text{ave}} = \hat{a}_r \tfrac{1}{2}\text{Re}(\eta |H_\theta|^2) = \hat{a}_r \frac{\eta}{2}|H_\theta|^2 = \hat{a}_r \frac{\eta}{2}\left|\frac{\pi S I_0}{\lambda^2}\right|\frac{\sin^2\theta}{r^2} = \hat{a}_r W_r$$

$$P_{\text{rad}} = \oiint_{S0} \underline{W}_{\text{ave}}\, ds = \int_0^{2\pi}\int_0^{\pi} \hat{a}_r W_r \cdot \hat{a}_r r^2 \sin\theta \; d\theta \; d\phi$$
$$= 2\pi \int_0^{\pi} W_r r^2 \sin\theta \; d\theta$$
$$= \pi\eta \left|\frac{\pi S I_0}{\lambda^2}\right|^2 \int_0^{\pi} \sin^3\theta \; d\theta = \frac{4\pi\eta}{3}\left|\frac{(\pi a)^2 I_0}{\lambda^2}\right|^2$$
$$= \eta\frac{\pi}{12}(ka)^4 |I_0|^2$$

5-9. $\underline{A} = \hat{a}_\phi j\dfrac{k\mu a^2 I_0 \sin\theta}{4r}\left(1+\dfrac{1}{jkr}\right)e^{-jkr} \simeq \hat{a}_\phi j\dfrac{k\mu a^2 I_0 e^{-jkr}}{4r}\sin\theta$

from equation (5-17) and $r \to$ large.
Using (30-58a)

$$E_r \simeq E_\theta \simeq 0$$
$$E_\phi \simeq -j\omega A_\phi = -j\omega\left(j\frac{k\mu a^2 I_0 e^{-jkr}}{4r}\sin\theta\right) = \eta\frac{\pi S I_0 e^{-jkr}}{\lambda^2 r}\sin\theta$$

where $S = \pi a^2$, $\eta = \sqrt{\mu/\epsilon}$
also using (3-58b)

$$H_r \simeq H_\phi \simeq 0$$
$$H_\theta \simeq j\frac{\omega}{\eta}A_\phi = j\frac{\omega}{\eta}\left(j\frac{\mu k a^2 I_0 e^{-jkr}}{4r}\sin\theta\right) = -\frac{\pi S I_0 e^{-jkr}}{\lambda^2 r}\sin\theta$$

5-10. $a = \lambda/8\pi$, $b = 10^{-4}\lambda/2\pi$, $\sigma = 5.7\times10^7$ s/m
Assuming uniform current

a. $R_r = 20\pi^2\left(\dfrac{C}{\lambda}\right)^4, \quad C = 2\pi a = 2\pi\left(\dfrac{\lambda}{8\pi}\right) = \dfrac{\lambda}{4}$ (5-24)

$$R_r = 20\pi^2\left(\frac{\lambda}{4\lambda}\right)^4 = 20\pi^2\left(\frac{1}{256}\right) = \frac{197.392}{256} = 0.771$$
$$R_L = R_{hf} = \frac{a}{b}\sqrt{\frac{w\mu_0}{2\sigma}} = \frac{\frac{\lambda}{8\pi}}{\frac{10^{-4}\lambda}{2\pi}}\sqrt{\frac{2\pi(10^8)4\pi\times10^{-7}}{2(5.7\times10^7)}}$$
$$R_L = \frac{10^4}{4}\frac{(2\pi)\times10^{-3}}{\sqrt{5.7}} = \frac{20\pi}{4\sqrt{5.7}} = \frac{5\pi}{\sqrt{5.7}} = \frac{15.708}{\sqrt{5.7}} = 6.5794$$
$$e_{cd} = \frac{R_r}{R_r+R_L} = \frac{0.771}{0.771+6.5794} = 0.10489 = 10.489\%$$

b. $D_0 = 3/2 = 1.5 = 1.761$ dB Uniform current

$G_0 = e_{cd} D_0 = 0.10489(1.5) = 0.15734$

$$\boxed{G_0 = 0.15734 = -8.03 \text{ dB}}$$

5-11. a. $R_r = 20 \cdot \pi^2 \cdot \left(\frac{C}{\lambda}\right)^4 = 20 \cdot \pi^2 \cdot \left(\frac{2\pi a}{\lambda}\right)^4, \lambda = \frac{3 \times 10^8}{10^7} = 30$ m

$0.73 = 20\pi^2 \left(\frac{2\pi a}{\lambda}\right)^4 \Rightarrow a = 0.03924\lambda = 1.177$ meters

b. $0.73N^2 = 300 \Rightarrow N = 20.272 \simeq 20$

$R_r(20 \text{ turns}) = 0.73(20)^2 = 292$

c. $P_L = A_{em} \cdot W_i \cdot e_o = \frac{\lambda^2}{4\pi} D_0 \epsilon_o (10^{-6}) = \frac{\lambda^2}{4\pi}\left(\frac{3}{2}\right)(1 - |\Gamma|^2) \cdot 10^{-6}$

$$= \frac{(30)^2}{4\pi} \cdot \left(\frac{3}{2}\right) \cdot \left(1 - \left|\frac{292 - 300}{292 + 300}\right|^2\right) \cdot 10^{-6} = 0.1074 \times 10^{-3} \text{ watts}$$

5-12. $a = \lambda/30, \quad b = \lambda/300, \quad 2C = \lambda/100 \Rightarrow C = \lambda/200, \quad N = 6,$
$f = 5 \times 10^7$ Hz

a. Since $a = \lambda/30 \ll \lambda$

$D_0 = 1.5 = 1.761$ dB

b. $R_r = 20\pi^2 \left(\frac{c}{\lambda}\right)^4 = 20\pi^2 \left(\frac{\pi}{15}\right)^4 = 20\pi^2(1.924 \times 10^{-3}) = 0.3798$ ohms

$C = 2\pi a = 2\pi \left(\frac{\lambda}{30}\right) = \frac{\pi}{15}\lambda$

$R_r(\text{single turn}) = 0.3798$ ohms

$R_r(6 \text{ turns}) = 13.673$ ohms

$$R_L = \frac{N_a}{b} \cdot R_s \cdot \left(\frac{R_p}{R_0} + 1\right)$$

$$c/b = \frac{\lambda/200}{\lambda/300} = \frac{3}{2} = 1.5 \Rightarrow \frac{R_p}{R_0} = 0.65$$

$$R_s = \sqrt{\frac{\omega\mu_0}{2\sigma}} = \sqrt{\frac{2\pi f(4\pi \times 10^{-7})}{2(5.7 \times 10^7)}} = \sqrt{\frac{4\pi^2 f}{5.7}} \times 10^{-7}$$

$$= \sqrt{\frac{4\pi^2(50 \times 10^6)}{5.7}} \times 10^{-7} = 2\pi\sqrt{\frac{50}{5.7}} \times 10^{-4} = 18.609 \times 10^{-4}$$

$$R_L = 6\left(\frac{\lambda/30}{\lambda/300}\right) \cdot 18.609 \times 10^{-4}(0.65 + 1)$$

$$= 6(10)(18.609)(1.65) \times 10^{-4} = 1{,}842.31 \times 10^{-4}$$

$R_L(6 \text{ turns}) = 0.184231$

$$\text{(Single) } R_L = \frac{\lambda/30}{\lambda/300}\sqrt{\frac{2\pi f(4\pi\times 10^{-7})}{2\cdot(5.7\times 10^7)}} = 10\sqrt{\frac{4\pi^2 f}{5.7}}\times 10^{-7}$$

$$= 2\pi(10)\sqrt{\frac{50}{5.7}}\times 10^{-4} = 186.0919\times 10^{-4}$$

$$\text{(6 turns) } R_L = 186.0919\cdot(6)(1.65)\times 10^{-4} = 1{,}842.31\times 10^{-4}$$

$$e_{cd} = \frac{R_r}{R_r+R_L} = \frac{13.673}{13.673+0.184231}\times 100 = 98.67\%$$

c. $$|\Gamma| = \left|\frac{(R_r+R_L)-50}{(R_r+R_L)+50}\right| = \left|\frac{13.857-50}{13.857+50}\right| = \left|\frac{-36.14277}{63.857}\right| = 0.566$$

$$e_r = (1-|\Gamma|^2)\times 100 = (1-|0.566|^2)\times 100 = (1-0.32)\times 100 = 68\%$$

d. $G_0 = e_{cd}D_0 = (0.9867)D_0 = (0.9867)(1.5)$

$G_0 = 1.48005$ (← Total Maximum gain does not include the reflection loss)

5-13. $f = 30\text{ MHz} \rightarrow \lambda = 10\text{ m},\ ka = \frac{2\pi}{10}(0.15) = 0.03\pi = 0.09425\text{ (rad)}$

$$R_r = N^2\frac{\pi\eta_0}{6}(ka)^4 = 64\times\frac{\pi^2\cdot 120}{6}\times(0.03\pi)^4 = 0.9968\ \Omega$$

$$\delta = \frac{1}{\sqrt{\pi f\mu_0\sigma}} = \frac{1}{\sqrt{\pi\times 30\times 10^6\times 4\pi\times 10^{-7}\times 5.7\times 10^7}}$$

$$= 1.217\times 10^{-5}\text{ m} \ll b$$

$$\text{1-turn: } R_L = \frac{a}{\sigma\cdot b\delta} = \frac{0.15}{5.7\times 10^7\times 0.001\times 1.217\times 10^{-5}} = 0.2162\ \Omega$$

$$\text{8-turn: } R_L = 8\times R_L(\text{1-turn})\times\left(\frac{R_p}{R_0}+1\right),\ c/b = 1.8 \Rightarrow \frac{R_p}{R_0} = 0.5$$

$$\therefore R_L = 8\times(0.2162)\times 1.5 = 2.594\ \Omega$$

$$e_{cd} = \frac{R_r}{R_r+R_L} = \frac{0.9968}{0.9968+2.594} = 0.278 = 27.8\%$$

5-14. Since the small circular loop area is parallel to the $y-z$ plane, its electrical equivalent is an **infinitestimal magnetic dipole** directed along the x-axis.

a. Thus, using the procedure of Example 4.5, we can write the electric and magnetic fields for the **infinitesimal electric dipole** of length l directed along the **x-axis** as

$$E_r \simeq 0 \qquad E_r \simeq 0 \qquad H_r \simeq 0$$

$$E_\theta \simeq -j\omega A_\theta \qquad E_\theta \simeq -j\frac{\omega\mu I_o l e^{-jkr}}{4\pi r}\cos\theta\cos\phi \qquad H_\phi \simeq \frac{E_\theta}{\eta}$$

$$E_\phi \simeq -j\omega A_\phi \qquad E_\phi \simeq -j\frac{\omega\mu I_o l e^{-jkr}}{4\pi r}\sin\phi \qquad H_\theta \simeq -\frac{E_\phi}{\eta}$$

Using duality and Table 3.2, the fields of an **x-directed infinitesimal magnetic dipole** of constant current I_m can be written as

$$H_r \simeq 0 \qquad E_r \simeq 0$$

$$H_\theta \simeq -j\frac{\omega\varepsilon I_m l e^{-jkr}}{4\pi r}\cos\theta\cos\phi \qquad E_\phi \simeq -\eta H_\theta = +j\eta\frac{\omega\varepsilon I_m l e^{-jkr}}{4\pi r}\cos\theta\cos\phi$$

$$H_\phi \simeq -j\frac{\omega\varepsilon I_m l e^{-jkr}}{4\pi r}\sin\phi \qquad E_\theta \simeq +\eta H_\phi = -j\eta\frac{\omega\varepsilon I_m l e^{-jkr}}{4\pi r}\sin\phi$$

Since the infinitesimal magnetic dipole directed along the x-aixs is equivalent to a small circular loop, with its area parallel to the $y-z$ plane, we can write the fields of the circular loop by making in the above equations the substitution

$$lI_m = jS\omega\mu I_o = j(\pi a^2)\omega\mu I_o$$

Thus the far-zone electric fields can be written as

$$E_r \simeq 0$$

$$E_\theta \simeq -j\eta\frac{\omega\varepsilon I_o(jS\omega\mu)e^{-jkr}}{4\pi r}\sin\phi = -j\eta\frac{\omega\varepsilon I_o(j\pi a^2\omega\mu)e^{-jkr}}{4\pi r}\sin\phi$$

$$\simeq \eta\frac{\omega^2\mu\varepsilon a^2 I_o e^{-jkr}}{4r}\sin\phi = \eta\frac{(ka)^2 I_o e^{-jkr}}{4r}\sin\phi$$

$$E_\phi \simeq +j\eta\frac{\omega\varepsilon I_o(jS\omega\mu)e^{-jkr}}{4\pi r}\cos\theta\cos\phi = +j\eta\frac{\omega\varepsilon I_o(j\pi a^2\omega\mu)e^{-jkr}}{4\pi r}\cos\theta\cos\phi$$

$$\simeq -\eta\frac{\omega^2\mu\varepsilon a^2 I_o e^{-jkr}}{4r}\cos\theta\cos\phi = -\eta\frac{(ka)^2 I_o e^{-jkr}}{4r}\cos\theta\cos\phi$$

while the far-zone magnetic fields can be expressed as

$$H_r \simeq 0; \qquad H_\theta \simeq -\frac{E_\phi}{\eta}; \qquad H_\phi \simeq \frac{E_\theta}{\eta}$$

b. Since the far-field pattern of the antenna is the same as that of a loop with an area parallel to the $x-y$ plane, or an infinitesimal magnetic dipole oriented along the x-axis, their directivities are the same. Thus $\mathbf{D_o = 3/2 = 1.5}$.

5-15. Using the results of Problem 5-14

a. $E_\chi \cong \dfrac{a^2\,\omega\mu k I_0 e^{-jkr}}{4r}\sqrt{1-|\hat{a}_y\cdot\hat{a}_r|^2} = \dfrac{a^2\,\omega\mu k I_0 e^{-jkr}}{4r}\sqrt{1-\sin^2\theta\sin^2\phi}$

$H_\psi \cong \dfrac{E_\chi}{\eta}$

b. Directivity $= D_0 = \frac{3}{2}$

5-16. Using the computer program of Chapter 5.

a. $a = \lambda/50 = 0.02\lambda$

$D_0 = 1.4988 = 1.7575$ dB, $R_r = 0.0$4[illegible] ohms

b. $a = \lambda/10 = 0.1\lambda$
$D_0 = 1.4699 = 1.6731$ dB, $R_r = 28.41$ ohms

c. $a = \lambda/4 = 0.25\lambda$
$D_0 = 1.2969 = 1.1291$ dB, $R_r = 723.938$ ohms

d. $a = \lambda/2 = 0.5\lambda$
$D_0 = 1.7968 = 2.5449$ dB, $R_r = 2{,}202.528$ ohms

5-17. According to (5-54b)

$$E_\phi \simeq \frac{aknI_0 e^{-jkr}}{2r} J_1(ka\sin\theta) \sim J_1(ka\sin\theta)$$

Therefore the nulls of the pattern occur when

$$J_1(ka\sin\theta_n) = 0 \Rightarrow ka\sin\theta_n = 0, 3.84, 7.01, 10.19, \ldots$$

Excluding $\theta = 0$

$$\theta_n = \begin{cases} \sin^{-1}\left(\frac{3.84}{ka}\right) = \sin^{-1}\left[\frac{3.84}{2\pi(1.25)}\right] = \sin^{-1}(0.4889) = 29.27^\circ \\ \sin^{-1}\left(\frac{7.01}{ka}\right) = \sin^{-1}\left[\frac{7.01}{2\pi(1.25)}\right] = \sin^{-1}(0.8925) = 63.19^\circ \end{cases}$$

5-18. Since $E_\phi \sim J_1(ka\sin\theta)$

(a) $E_\phi|_{\theta=0} = J_1(ka\sin\theta)|_{\theta=0} = J_1(0) = 0$
$E_\phi|_{\theta=\pi/2} = J_1(ka\sin\theta)|_{\theta=90^\circ} = J_1(ka) = 0 \Rightarrow ka = 3.84$

Thus $a = \dfrac{3.84}{k} = \dfrac{3.84\lambda}{2\pi} = 0.61115\lambda$

(b) Since $a = 0.61115\lambda > 0.5\lambda$, use large loop approximation. According to (5-63a)

$$R_r = 60\pi^2(C/\lambda) = 60\pi^2\left(\frac{2\pi a}{\lambda}\right) = 60\pi^2(2\pi(0.61115))$$
$$= 2{,}273.94$$

(c) The directivity is given by (5-63b), or

$$D_0 = 0.682\left(\frac{C}{\lambda}\right) = 0.682\left(\frac{2\pi a}{\lambda}\right) = 0.682(2\pi)(0.61115) = 2.619$$

5-19. $E_\phi \sim J_1(ka\sin\theta)$

a. $E_\phi|_{\theta=30^\circ} = J_1(ka\sin\theta)|_{\theta=30^\circ} = J_1\left(\dfrac{ka}{2}\right) = 0 \Rightarrow \dfrac{ka}{2} = 3.84$

From the Table for $J_1(x)$ in Appendix V. Thus

$$a = \frac{2(3.84)}{k} = \frac{2(3.84)}{2\pi}\lambda = 1.222\lambda$$

b. $E_\phi|_{\max} = E_\phi|_{ka\sin\theta=1.84} = J_1(1.84) = 0.58152 = -4.709 \text{ dB}$

$$E_\phi|_{\theta=90^\circ} = J_1(ka) = J_1\left[\frac{2\pi}{\lambda}(1.222\lambda)\right] = J_1(7.678) = 0.175 = -15.139 \text{ dB}$$

Thus

$$\Delta E = E_\phi|_{\theta=90^\circ} - E_\phi|_{\max} = -15.139 - (-4.709) = -10.43 \text{ dB}$$

5-20. $E_\phi \sim J_1(ka\sin\theta)$

a. According to the Table for $J_1(x)$ in Appendix V

$$J_1(x) = 0 \quad \text{when } x = 0, 3.84, 7.01, 10.19, \ldots$$

Since we want a null in the plane of the loop ($\theta = 0^\circ$) and two additional ones for $0^\circ \leq \theta \leq 90^\circ$, then

$$ka\sin\theta|_{\max} = ka\sin\theta|_{\theta=90^\circ} = ka = 7.01$$

Thus

$$a = \frac{7.01}{k} = \frac{7.01}{2\pi}\cdot\lambda = 1.1157\lambda$$

b. The nulls will occur at

$$\theta = 0^\circ \text{ and } 180^\circ$$
$$\theta = 90^\circ$$

and

$$ka\sin\theta|_{a=1.1157\lambda} = 3.84$$
$$\Rightarrow \theta = \sin^{-1}\left[\frac{3.84}{2\pi(1.1157)}\right] = 33.21^\circ$$
$$\text{and } \theta = 180^\circ - 33.21^\circ = 146.79^\circ$$

5-21. $\underline{E} = \hat{a}_\phi C_1 J_1(ka\sin\theta)$ where C_1 is a constant $\Rightarrow \hat{\rho}_w = \hat{a}_\phi$ and $\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = |\hat{a}_\phi \cdot \hat{\rho}_a|^2$

By inspection, the PLF is maximized if the probe antenna is also linearly polarized in the ϕ direction. This can be accomplished by using as a probe antenna another loop antenna so that

$$\hat{\rho}_a = \hat{a}_\phi \text{ and PLF} = |\hat{a}_\phi \cdot \hat{a}_\phi|^2 = 1.$$

It can also be accomplished by using a linear dipole as a probe antenna with its length parallel to the plane of the loop and tangent to its curvature. Some specific examples would be [using the transformation of VII-7b]

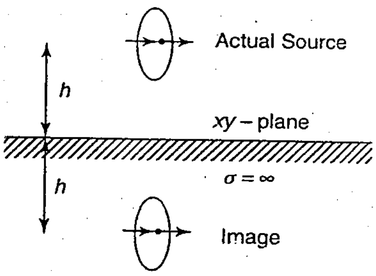
$$\hat{\rho}_a = \hat{a}_x|_{\phi=90^\circ} \Rightarrow \text{PLF} = |\hat{a}_\phi \cdot \hat{a}_x|_{\phi=90^\circ} = |\hat{a}_\phi \cdot (\hat{a}_\rho \cos\phi - \hat{a}_\phi \sin\phi)|^2_{\phi=90^\circ}$$
$$= |\hat{a}_\phi \cdot (-\hat{a}_\phi)|^2 = 1$$
$$\hat{\rho}_a = \hat{a}_y|_{\phi=0^\circ} \Rightarrow \text{PLF} = |\hat{a}_\phi \cdot \hat{a}_y|_{\phi=0^\circ} = |\hat{a}_\phi \cdot (\hat{a}_\rho \sin\phi + \hat{a}_\phi \cos\phi)|^2_{\phi=0^\circ}$$
$$= |\hat{a}_\phi \cdot \hat{a}_\phi|^2 = 1$$

and many others.

5-22. A very small loop of constant current is equivalent to a magnetic dipole. Since the loop is placed for both parts (a and b) perpendicular to the xy-plane (the plane of the loop is perpendicular to the xy-plane), the axis of the linear magnetic dipole will also be parallel to the xy-plane. Therefore according to Figure 4.12a, the image of the horizontal magnetic dipole will be as shown in this figure. In turn the array factor for both parts (a and b) of this problem will be the same as that of the vertical electric dipole of Figure 4.13 or

$$\text{AF} = 2 \cdot \cos(kh \cos\theta)$$

Since the actual source and the image are oriented in the same direction. Therefore according to (5-27a)–(5-27c)



a. Plane of the loop is parallel to the xz-plane

$$E_\chi = \eta \frac{(ka)^2 I_0 e^{-jkr}}{4r} \sin\psi(\text{AF}), \quad \sin\psi = \sqrt{1-\cos^2\psi} = \sqrt{1-|\hat{a}_y \cdot \hat{a}_r|^2}$$
$$= \sqrt{1-\sin^2\theta \sin^2\phi}$$
$$= \eta \frac{(ka)^2 I_0 e^{-jkr}}{4r} \sin\psi[2\cos(kh\cos\theta)]$$
$$E_\chi = \eta \frac{(ka)^2 I_0 e^{-jkr}}{2r} \cos(kh\cos\theta)\sqrt{1-\sin^2\theta\sin^2\phi}$$
$$H_\psi = -\frac{E_\chi}{\eta}$$

b. Plane of the loop is parallel to the yz-plane. The fields for this problem are the same as those in part a. above except that

$$\sin\psi = \sqrt{1-\cos^2\psi} = \sqrt{1-|\hat{a}_x \cdot \hat{a}_r|^2} = \sqrt{1-\sin^2\theta\cos^2\phi}$$

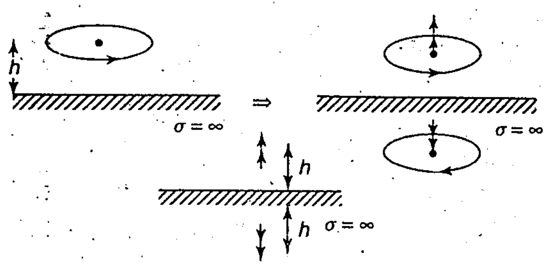
5-23. a. $$E_\phi = \eta \frac{(ka)^2 \cdot I_0 \cdot e^{-jkr}}{4r} \sin\theta$$

$$|\mathrm{AF}| = |2j \sin(kh \cos\theta)|$$

$$E_\phi = \eta \frac{\pi S I_0 e^{-jkr}}{\lambda^2 r} \cdot \sin\theta, S = \pi a^2$$

$$(E_\phi)_t = E_\phi(\mathrm{AF}) = \eta \frac{\pi S I_0 e^{-jkr}}{\lambda^2 r} \cdot \sin\theta \cdot [2j \sin(kh\cos\theta)],$$

$\leftarrow$ above ground plane total field.



b. $h = \lambda$, $kh = 2\pi$

$$\sin\theta \cdot [2j \sin(2\pi\cos\theta)] = 0,\ \sin(2\pi\cos\theta) = 0, 2\pi\cos\theta = n\pi.$$

$$n = 0, 1, 2$$

$$\theta_n = 0^\circ, \qquad \cos\theta_n = \frac{n}{2},\ n = 0, 1, 2. \Rightarrow \theta_n = 90^\circ,$$

$$\theta_0 = \cos^{-1}(0) = 90^\circ$$

$$\theta_1 = \cos^{-1}\left(\tfrac{1}{2}\right) = 60^\circ$$

$$\theta_2 = \cos^{-1}(1) = 0^\circ.$$

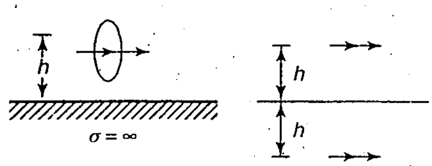
c. $$(E_\phi)_t = C \sin\theta \sin(kh\cos\theta)|_{\theta=60^\circ} = 0 = C \cdot \left(\frac{\sqrt{3}}{2}\right) \cdot \sin\left(\frac{2\pi}{\lambda} \cdot h \cdot \frac{1}{2}\right)$$

$$= C \cdot \frac{\sqrt{3}}{2} \cdot \sin\left(\frac{\pi h}{\lambda}\right)$$

$$\sin\left(\frac{\pi h}{\lambda}\right) = 0 \Rightarrow \frac{\pi h}{\lambda} = \sin^{-1}(0) = n\pi, n = 0, 1, 2, 3, \ldots$$

$$\frac{h}{\lambda} = \pm n \Rightarrow \text{physical nonzero height} \Rightarrow h = n\lambda, n = 1, 2, 3, \ldots$$

5-24. a. Array Factor $= 2\cos(kh\cos\theta)$



b. $AF = 2\cos(kh\cos\theta_n) = 0$

$$\Rightarrow kh\cos\theta_n = \cos^{-1}(0) = n\pi/2, n = \pm1, \pm3, \pm5, \ldots.$$

$$\theta_n = \cos^{-1}\left[\frac{n\pi/2}{kh}\right] = \cos^{-1}\left(\frac{\frac{n\pi}{2}}{\frac{2\pi}{\lambda}h}\right) = \cos^{-1}\left(\frac{n\lambda}{4h}\right)\Big|_{h=\lambda/2} = \cos^{-1}\left(\frac{n}{2}\right)$$

$$\theta_1 = \cos^{-1}\left(\pm\tfrac{1}{2}\right) = 60^\circ, \theta_3 = \cos^{-1}\left(\pm\tfrac{3}{2}\right) = \text{does not exist}$$

5-25. Since the small circular loop area is parallel to the $x-z$ plane, its electrical equivalent is an **infinitesimal magnetic dipole** directed along the y-axis placed a height h above the PEC. Also its image is at a depth h below the PEC interface. The image is in the same direction as the actual source (the same magnitude and phase).

a. Therefore its normalized array factor is

$$(AF)_n = \cos(kh\cos\theta)$$

whose maximum value is unity.

b. To find the two smallest heights, **other than** $h = 0$, where the maximum will be directed along $\theta = 0^\circ$, we set the normalized array factor to unity, or

$$[AF_n(\theta = 0^\circ]_{\max} = [\cos(kh\cos\theta)|_{\theta=0^\circ}]_{\max} = \cos(kh)|_{\max} = 1$$

$$kh = \cos^{-1}(1) = m\pi$$

$$h = \frac{m\pi}{k} = \frac{m\pi\lambda}{2\pi} = \frac{\lambda}{2}m, \quad m = 0, 1, 2, 3 \ldots$$

$$m = 0: \quad h = 0$$

$$\boxed{\begin{array}{ll} m = 1: & h = \frac{\lambda}{2} \\ m = 2: & h = \lambda \end{array}}$$

5-26. From Problem 5-16(a)

$$E_x|_{\substack{\phi=90^\circ\\ \theta=45^\circ}} = C_1\cos(kh\cos\theta)\sqrt{1-\sin^2\theta\sin^2\phi}\,\Big|_{\substack{\theta=45^\circ\\ \phi=90^\circ}}$$

$$= C_1\cos(kh\cos\theta)\cos(\theta)|_{\theta=45^\circ}$$

$$= 0.707\cdot C_1\cdot\cos\left(\frac{kh}{\sqrt{2}}\right) = 0 \Rightarrow \frac{kh}{\sqrt{2}} = \cos^{-1}(0) = \frac{\pi}{2}n, n = 1, 3, 5, \ldots$$

For the smallest height

$$\frac{kh}{\sqrt{2}} = \frac{\pi}{2} \qquad h = \frac{\sqrt{2}}{2}\frac{\pi}{k} = \frac{\sqrt{2}}{4}\lambda = 0.3535\ \lambda$$

5-27. a. $R_L = \frac{a}{b}\sqrt{\frac{\omega\mu_0}{2\sigma}} = \frac{1}{20\cdot(10^{-4})}\cdot\sqrt{\frac{2\pi(3\times10^8)(4\pi\times10^{-7})}{2\cdot5\cdot7\times10^7}} = \frac{\pi}{20}\sqrt{\frac{12}{5\cdot7}}\times 10$

$= 2.27915$ ohms

b. $R_r = 120\cdot\pi\cdot\left(\frac{2}{3}\pi\right)\cdot\left(\frac{kS}{\lambda}\right)^2 = 120\cdot\pi\left(\frac{2}{3}\pi\right)\left(\frac{2\pi^2}{(20)^2}\right)^2$

$= 80\cdot\frac{4\cdot\pi^6}{(400)^2} = 1.92278$ ohms

$\left(\leftarrow S = \pi\left(\frac{1}{20}\right)^2\right)$

c. inductive reactance $X_A = \omega L_A$

$L_A = \mu_0\cdot a\cdot\left[\ln\left(\frac{8a}{b}\right)-2\right] = 4\pi\times10^{-7}\cdot\left(\frac{\lambda}{20}\right)\cdot\left[\ln\left(\frac{1}{20}\cdot\frac{1}{10^{-4}}\right)-2\right]$

$= 2.648\times10^{-7}\left(\leftarrow a = \frac{\lambda_0}{20},\ b = 10^{-4}\lambda_0\right) \leftarrow \lambda = 1\text{ m},\ f = 3\times10^8$

$X_A = 2\cdot\pi\cdot f\cdot L_A = 2\cdot\pi\cdot(3\times10^8)\cdot(2.648\times10^{-7}) = 499.158$

$\therefore X_A \gg (R_L \text{ or } R_r)$

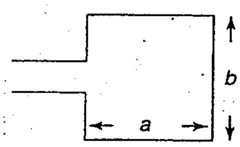
5-28. From equation (5-24)

$$R_r = \eta\left(\frac{\pi}{6}\right)(k^2a^2)^2 = \eta\cdot\frac{2\pi}{3}\cdot\left(\frac{kS}{\lambda}\right)^2 = 120\pi\cdot\frac{2\pi}{3}\cdot\left(\frac{2\pi S}{\lambda^2}\right)^2$$

$$= 120\cdot\frac{2}{3}\cdot4\cdot\pi^4\cdot\left(\frac{S}{\lambda^2}\right)^2 = 31170.909\cdot\frac{S^2}{\lambda^4} \simeq 31{,}171\frac{S^2}{\lambda^4}$$

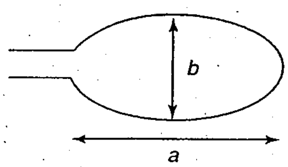
a. Area

$S = ab,\qquad R_r = 31170.909\cdot\frac{a^2b^2}{\lambda^4} \simeq 31{,}171\frac{a^2b^2}{\lambda^4}$



b. Area

$S = \pi\frac{a}{2}\frac{b}{2},\qquad R_r = 31170.909\cdot\frac{\pi^2a^2b^2}{16\lambda^4} \simeq 31{,}171\frac{\pi^2a^2b^2}{16\lambda^4}$



5-29. $f = 100$ MHz $\Rightarrow \lambda = c/f = 3 \times 10^8/10^8 = 3$ meters

$$C = 2\pi a \Rightarrow a = \frac{C}{2\pi} = \frac{\lambda/20}{2\pi} = \frac{\lambda}{40\pi} = \frac{3}{40\pi} = 0.0239 \text{ m} = 0.00796\lambda$$

(a)
$$R_r = 20\pi^2 \left(\frac{C}{\lambda}\right)^4 = 20\pi^2 \left(\frac{1}{20}\right)^4 = \frac{20\pi^2}{16} \times 10^{-4} = 1.2337 \times 10^{-3} \text{ ohms}$$

$$R_L = \frac{a}{b}\sqrt{\frac{\omega\mu_0}{2\sigma}} = \frac{\lambda/40\pi}{\lambda/400\pi}\sqrt{\frac{2\pi \times 10^8(4\pi \times 10^{-7})}{2(5.7 \times 10^7)}} = 0.00838$$

$$R_{in} = R_r + R_L = 0.0012337 + 0.00838 = 0.0096137$$

(b)
$$L_a = \mu_0 a\left[\ln\left(\frac{8a}{b}\right) - 2\right] = 4\pi \times 10^{-7}(0.0239)\left[\ln\left(\frac{8\dfrac{\lambda}{40\pi}}{\lambda/400}\right) - 2\right]$$

$$= 0.3 \times 10^{-7}\left[\ln\left(\frac{80}{\pi}\right) - 2\right] = 0.3 \times 10^{-7}[3.2373 - 2]$$

$$= 37.12 \times 10^{-9} \text{ henries}$$

$$X_a = \omega L_a = 2\pi f L_a = 2\pi(10^8)(37.12 \times 10^{-9}) = 23.323 \text{ ohms}$$

$$L_i = \frac{a}{\omega b}\sqrt{\frac{\omega\mu_0}{2\sigma}} = \frac{\dfrac{\lambda}{40\pi}}{2\pi(10^8)\left(\dfrac{\lambda}{400}\right)}\sqrt{\frac{2\pi(10^8)(4\pi \times 10^{-7})}{2(5.7 \times 10^7)}} = 0.1333 \times 10^{-10}$$

$$X_i = 2\pi f L_i = 2\pi(10^8)(0.1333 \times 10^{-10}) = 0.83771 \times 10^{-2} = 0.0084 \text{ ohms}$$

$$X_t = X_a + X_i = 23.323 + 0.0084 = 23.3314 \text{ ohms } \textbf{(inductive)}$$

(c) Capacitance

$$X_c = \frac{1}{2\pi f C} = 23.3314$$

$$C = \frac{1}{23.3314(2\pi \times 10^8)} = 6.82 \times 10^{-11} = 68.2 \times 10^{-12} \text{ farads}$$

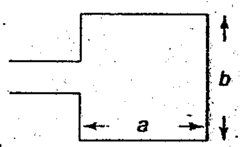
5-30. From the solution of Problem 5-28, the radiation resistance of a loop is

$$R_r = 31{,}171\frac{(\text{Area})^2}{\lambda^4} = 31{,}171\frac{(S)^2}{\lambda^4}$$

Thus for rectangular and elliptical loops:

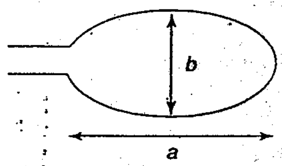
a. Area

$S =$ $\qquad R_r \simeq 31{,}171\dfrac{a^2b^2}{\lambda^4}$



b. Area

$$S = \pi\left(\frac{a}{2}\right)\left(\frac{b}{\lambda 2}\right), \qquad R_r \simeq 31{,}171\frac{\pi^2 a^2 b^2}{16\lambda^4}$$



5-31. In Far-Field ($kr \gg 1$) region

$$\underline{E}_a \doteq E_\phi \hat{a}_\phi = -j\eta\frac{kI_{\text{in}}}{4\pi r}\cdot l_e \cdot e^{-jkr} \qquad (\rightarrow l_e\text{: effective length})$$

$$E_\phi \simeq \eta\frac{k^2 a^2 I_0 e^{-jkr}}{4n}\cdot \sin\theta = \eta\frac{\pi S I_0 e^{-jkr}}{\lambda^2 r}\sin\theta$$

$$= -j\eta\frac{kI_0(jk\cdot s\cdot \sin\theta)}{4\pi r}e^{-jkr} \qquad \left(\leftarrow \frac{\pi}{\lambda^2} = \frac{k^2}{4\pi}\right)$$

$$\therefore\ l_e = jk\cdot S\cdot \sin\theta\ \hat{a}_\phi$$

5-32. $C = 2\pi a = 1.4\lambda \Rightarrow a = \dfrac{1.4\lambda}{2\pi} = 0.2228\lambda$

$$\Omega = 2\ln\left(2\pi\frac{a}{b}\right) = 2\ln\left(2\pi\frac{0.2228}{0.01555}\right) = 9.0$$

a. From Figure 5.13

$$Z_{\text{in}} = R_{\text{in}} + jX_{\text{in}} = 320 - j40$$

b. $$|\Gamma| = \left|\frac{Z_{\text{in}} - Z_c}{Z_{\text{in}} + Z_c}\right| = \left|\frac{320 - j40 - 300}{320 - j40 + 300}\right| = 0.0718$$

$$\text{VSWR} = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.0718}{1-0.0718} = 1.155$$

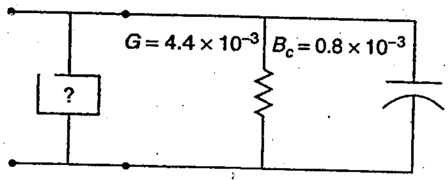
c. $Y_{\text{in}} = \dfrac{1}{Z_{\text{in}}} = \dfrac{1}{320 - j40} = (3.0769 + j0.3846)\times 10^{-3} = G_c + jB_c$

To resonate the circuit, the unknown element must have an inductive admittance of

$$Y_{\text{unknown}} = -j0.3846\times 10^{-3} = -j\frac{1}{\omega L} \Rightarrow L = \frac{1}{0.3846\times 10^{-3}(2\pi f)}$$

$$= \frac{1}{0.3846\times 10^{-3}(2\pi \times 10^8)}$$

$$L = \frac{10^{-5}}{0.769\pi} = 4.138\times 10^{-6}\ \text{h}$$



Therefore the unknown element across the terminals of the loop must be an inductor of $L = 1.989 \times 10^{-6}$ henries

5-33. a. From Figure 5.13 (a, b)

$$Z_{\text{in}} = 90 - j110$$

b. Inductor;

$$X_L = +110 = \omega L = 2\pi f L$$
$$L = \frac{110}{2\pi f} = \frac{110}{2\pi \cdot (10^9)} = \frac{110}{2\pi} \times 10^{-9}$$

c. $Z_{\text{in}} = 90$

$$|\Gamma| = \left|\frac{Z_{\text{in}} - Z_c}{Z_{\text{in}} + Z_c}\right| = \frac{90 - 78}{90 + 78} = \frac{12}{168} = 0.0714$$
$$\text{VSWR} = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.0714}{1 - 0.0714} = \frac{1.0714}{0.9285} = 1.1538$$

5-34. a. From Figure 5.13(b) $Z_{\text{in}} = R_{\text{in}} + jX_{\text{in}} = R_{\text{in}} \Rightarrow X_{\text{in}} = 0$ when

$$\Omega = 2\ln\left(2\pi\frac{a}{b}\right) = \begin{cases} 12 \Rightarrow 2\pi(a/b) = e^6 = 403.429 \Rightarrow \frac{a}{b} = 64.21 \\ 11 \Rightarrow 2\pi(a/b) = e^{5.5} = 244.692 \Rightarrow \frac{a}{b} = 38.94 \\ 10 \Rightarrow 2\pi(a/b) = e^5 = 148.413 \Rightarrow \frac{a}{b} = 23.62 \\ 9 \Rightarrow 2\pi(a/b) = e^{4.5} = 90.017 \Rightarrow \frac{a}{b} = 14.33 \end{cases}$$

b. These occur when the smallest circumference of the loop is ¡from Figure 5.13(b)¿

$\Omega = 12 \Rightarrow C = 2\pi a \simeq 1.08\lambda \Rightarrow a = 0.1719\lambda \Rightarrow b = 0.1719\lambda/64.21 = 2.68 \times 10^{-3}\lambda$

$\Omega = 11 \Rightarrow C = 2\pi a \simeq 1.10\lambda \Rightarrow a = 0.175\lambda \Rightarrow b = 0.175\lambda/38.94 = 4.496 \times 10^{-3}\lambda$

$\Omega = 10 \Rightarrow C = 2\pi a \simeq 1.14\lambda \Rightarrow a = 0.1814\lambda \Rightarrow b = 0.1814\lambda/23.62 = 7.68 \times 10^{-3}\lambda$

$\Omega = 9 \Rightarrow C = 2\pi a \simeq 1.28\lambda \Rightarrow a = 0.2037\lambda \Rightarrow b = 0.2037\lambda/14.33 = 14.216 \times 10^{-3}\lambda$

5-35. $I(\phi) = I_0 \cos\phi$

a. $$\bar{A}(\bar{r}) = \frac{\mu I_0}{4\pi} a \int_0^{2\pi} \hat{a}_\phi \cos\phi' \frac{e^{-jkr}}{R}\, d\phi' \simeq \frac{\mu I_0}{4\pi} a \frac{e^{-jkr}}{r} \int_0^{2\pi} \hat{a}_\phi \cos\phi' \cdot e^{jk\hat{a}r\cdot\bar{r}'}\, d\phi'$$
$$= \frac{\mu I_0}{4\pi} a \frac{e^{-jkr}}{r} \left\{ -\hat{a}_x \int_0^{2\pi} \cos\phi' \sin\phi' e^{jka\sin\theta\cos(\phi-\phi')}\, d\phi' + \hat{a}_y \int_0^{2\pi} \cos^2\phi' \cdot e^{jka\sin\theta\cos(\phi-\phi')}\, d\phi' \right\}$$
$$= \frac{\mu I_0 a}{8\pi} \frac{e^{-jkr}}{r} \left\{ -\hat{a}_x \int_0^{2\pi} \sin(2\phi') e^{jka\sin\theta\cos(\phi-\phi')}\, d\phi' + \hat{a}_y \int_0^{2\pi} (\cos(2\phi')+1) e^{jka\sin\theta\cos(\phi-\phi')}\, d\phi' \right\}$$
$$= \frac{\mu I_0 a}{4} \frac{e^{-jkr}}{r} \{ \hat{a}_x J_2(ka\sin\theta)\sin 2\phi - \hat{a}_y J_2(ka\sin\theta)\cos 2\phi + \hat{a}_y J_0(ka\sin\theta) \}$$
$$= \frac{\mu I_0 a}{2} \frac{e^{-jkr}}{r} \left\{ -\hat{a}_\phi J_2(ka\sin\theta)\cos\phi + \hat{a}_y \tfrac{1}{2}[J_2(ka\sin\theta) + J_0(ka\sin\theta)] \right\}$$
$$= \frac{\mu I_0 a}{2} \frac{e^{-jkr}}{r} \left\{ -\hat{a}_\phi J_2(ka\sin\theta)\cos\phi + \hat{a}_y \frac{J_1(ka\sin\theta)}{ka\sin\theta} \right\}$$
$$A_\phi \simeq \frac{-\mu I_0 a}{2} \frac{e^{-jkr}}{r} \left\{ J_2(ka\sin\theta) - \frac{J_1(ka\sin\theta)}{ka\sin\theta} \right\} \cos\phi$$
$$= \frac{\mu I_0 a}{2} \frac{e^{-jkr}}{r} J_1'(ka\sin\theta)\cos\phi$$

$$A_\theta \simeq \frac{\mu I_0 a}{2} \frac{e^{-jkr}}{r} \frac{J_1(ka\sin\theta)}{ka\sin\theta} \cos\theta\sin\phi$$
$$E_\phi \simeq \frac{j\eta ka}{2} I_0 \frac{e^{-jkr}}{r} J_1'(ka\sin\theta)\cos\phi$$
$$E_\theta \simeq \frac{j\eta ka}{2} I_0 \frac{e^{-jkr}}{r} \frac{J_1(ka\sin\theta)}{ka\sin\theta} \cos\theta\sin\phi$$

b. $\theta = 0, \phi = \pi/2$

$$E_\phi = 0$$
$$E_\theta = \frac{j\eta ka}{4} I_0 \frac{e^{-jkr}}{r}$$
$$W_{av} \simeq \frac{|\bar{E}|^2}{2\eta} = \frac{\eta}{32} I_0 \frac{(ka)^2}{r^2}$$
$$U\left(\theta = 0, \phi = \frac{\pi}{2}\right) = \frac{\eta}{32} I_0 (ka)^2$$

CHAPTER 6

6-1. (a) $E_t = E_1 + E_2 + E_3 = 2E_0 \frac{e^{-jkr}}{r} + E_0 \frac{e^{-jkr_1}}{r_1} + E_0 \frac{e^{-jkr_2}}{r_2}$

where the center element is placed at the origin. For far-field observations

$$\left.\begin{array}{l} r_1 \simeq r - d\cos\theta \\ r_2 \simeq r + d\cos\theta \end{array}\right\} \text{ for phase variations}$$

$$r_1 \simeq r_2 \simeq r \quad \text{for amplitude variations}$$

and

$$\begin{aligned} E_t &= E_0 \frac{e^{-jkr}}{r}\{2 + e^{jkd\cos\theta} + e^{-jkd\cos\theta}\} \\ &\simeq E_0 \frac{e^{-jkr}}{r}\left\{2\left[1 + \frac{1}{2}(e^{jkd\cos\theta} + e^{-jkd\cos\theta})\right]\right\} \\ &= E_0 \frac{e^{-jkr}}{r}\{2[1 + \cos(kd\cos\theta)]\} \end{aligned}$$

Computer Program Directivity

$$\begin{aligned} U &= \cos^4\left(\frac{\pi}{4}\cos\theta\right) \\ \text{At.} d &= \lambda/4 \\ P_{\text{rad}} &= 8.7119 \\ D_0 &= 1.44244 \\ &= 1.5910 \text{ dB} \\ kd &= \frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} = \frac{\pi}{2} \\ \text{AF}(\theta) &= 4\cos^2\left(\frac{\pi}{4}\cos\theta\right) \end{aligned}$$

Thus the array factor is equal to

$$\text{AF}(\theta) = 2[1 + \cos(kd\cos\theta)] = 4\cos^2\left(\frac{kd}{2}\cos\theta\right)$$

which in normalized form can also be written as

$$\text{AF}(\theta)_n = 1 + \cos(kd\cos\theta) = 2\cos^2\left(\frac{kd}{2}\cos\theta\right)$$

(b) The nulls of the pattern can be found using either of the above forms for the array factor. For example

one form

$$AF(\theta) = 1 + \cos(kd\cos\theta_n) = 0$$

$$\cos(kd\cos\theta_n) = -1$$

$$kd\cos\theta_n = \cos^{-1}(-1) = n\pi, n = \pm 1, \pm 3, \ldots$$

$$\theta_n = \cos^{-1}(n\lambda/2d), n = \pm 1, \pm 3, \pm 5, \ldots$$

the other form

$$2\cos^2\left(\frac{kd}{2}\cos\theta_n\right) = 0$$

$$\frac{kd}{2}\cos\theta_n = \cos^{-1}(0) = \frac{n\pi}{2},$$

$$n = \pm 1, \pm 3, \ldots$$

$$\theta_n = \cos^{-1}(n\lambda(2d)), n = \pm 1, \pm 3, \ldots$$

which are of identical form. Therefore both forms yield the same results. Thus for $d = \lambda/4$

$$\theta_n = \cos^{-1}\left(\frac{n\lambda}{2d}\right)_{d=\lambda/4} = \cos^{-1}(2n), n = \pm 1, \pm 3, \ldots \Rightarrow \text{No nulls exist.}$$

(c) Similarly the maxima of the pattern can be found using either of the two forms for the array factor. For example

one form

$$AF(\theta) = 1 + \cos(kd\cos\theta_m) = 2$$

$$\cos(kd\cos\theta_m) = 1$$

$$kd\cos\theta_m = \cos^{-1}(1) = 2m\pi,$$

$$m = 0, \pm 1, \ldots,$$

$$\theta_m = \cos^{-1}\left(\frac{m\lambda}{d}\right), m = 0, \pm 1, \pm 2, \ldots,$$

other form

$$AF(\theta) = 2\cos^2\left(\frac{kd}{2}\cos\theta_m\right) = 2$$

$$\cos\left(\frac{kd}{2}\cos\theta_m\right) = \pm 1$$

$$\frac{kd}{2}\cos\theta_m = \cos^{-1}(\pm 1) = m\pi,$$

$$m = 0, \pm 1, \ldots$$

$$\theta_m = \cos^{-1}\left(\frac{m\lambda}{d}\right), m = 0, \pm 1, \pm 2, \ldots$$

which are of identical form. Therefore both yield the same results. Thus for $d = \lambda/4$.

$$\theta_m = \cos^{-1}(4m), \quad m = 0, \pm 1, \pm 2, \rightarrow \begin{cases} m = 0: & \theta_0 = \cos^{-1}(0) = 90^\circ \\ m = \pm 1: & \theta_1 = \cos^{-1}(4) \Rightarrow \text{Does not exist.} \end{cases}$$

The same is true for other values of m (i.e, $m = \pm 2, \pm 3, \ldots$).
Therefore the only maximum occurs at $\theta = 90°$.

(d) Computer Program ***Directivity***:
When $d = \lambda/4$

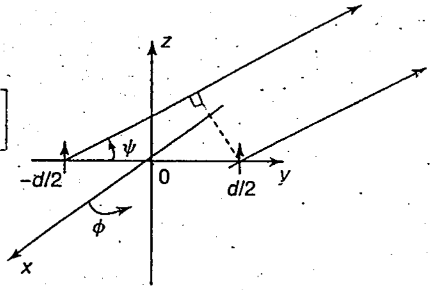
$$AF(\theta) = 4\cos^2\left(\frac{kd}{2}\cos\theta\right) = 4\cos^2\left(\frac{\pi}{4}\cos\theta\right)$$

$$U_n = \cos^4\left(\frac{\pi}{4}\cos\theta\right)$$

$$D_a = 1.4384 = 1.5787 \text{ dB}$$

6-2. one dipole $E_\theta = j\eta \dfrac{kI_0 l e^{-jkr}}{4\pi r} \cdot \sin\theta$

Array Factor:



$$(\mathrm{AF})_2 = E_0 \left[e^{j\frac{\pi}{2}} \cdot e^{-j\frac{\lambda}{8}\cdot\frac{2\pi}{\lambda}\cos\psi} + e^{+j\frac{\lambda}{8}\cdot\frac{2\pi}{\lambda}\cos\psi} \right]$$

$$= E_0 e^{j\frac{\pi}{4}} \cdot \left[e^{-j\frac{\pi}{4}(\cos\psi - 1)} + e^{j\frac{\pi}{4}(\cos\psi - 1)} \right]$$

$$= E_0 e^{j\frac{\pi}{4}} \cdot 2 \cdot \cos\left(\frac{\pi}{4}(\cos\psi - 1)\right)$$

$$\doteqdot E_0 e^{j\frac{\pi}{4}} \cdot 2\cos\left(\frac{\pi}{4}(\sin\theta\sin\phi - 1)\right)$$

$(\hat{a}_y \cdot \hat{a}_r = \sin\theta \cdot \sin\phi = \cos\psi) = \sin\theta$

At y, z plane, $\phi = 90°$

a. (1) $|E_\theta(\theta)|_{\phi=0°} \propto \left|\sin\theta \cdot \cos\left(\frac{\pi}{4}\right)\right|$, ($x$-$z$ plane)

$0 < \theta < \pi \quad -\pi < \theta < \pi$

$\downarrow \quad\quad \downarrow$

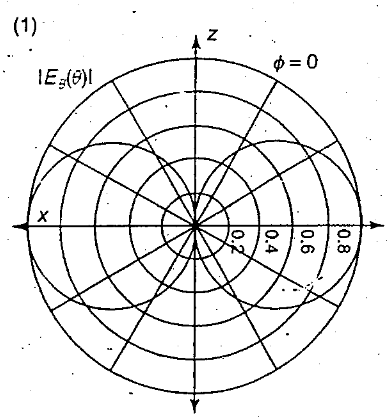
(2) $|E_\theta(\theta)|_{\phi=90°} \propto \left|\sin\theta \cdot \cos\left(\frac{\pi}{4}(\sin\theta - 1)\right)\right|$, ($y$-$z$ plane)

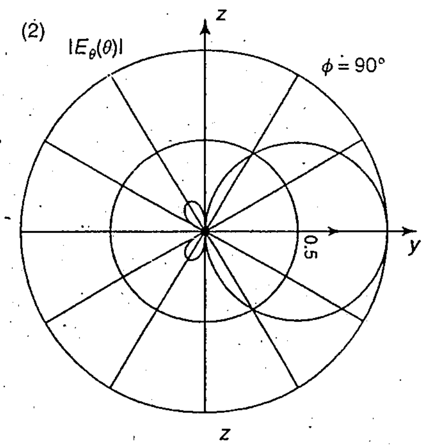
(3) $|E_\theta(\phi)|_{\theta=90°} \propto \left|\cos\left(\frac{\pi}{4}(\sin\phi - 1)\right)\right|$, ($x, y \cdot$ plane)

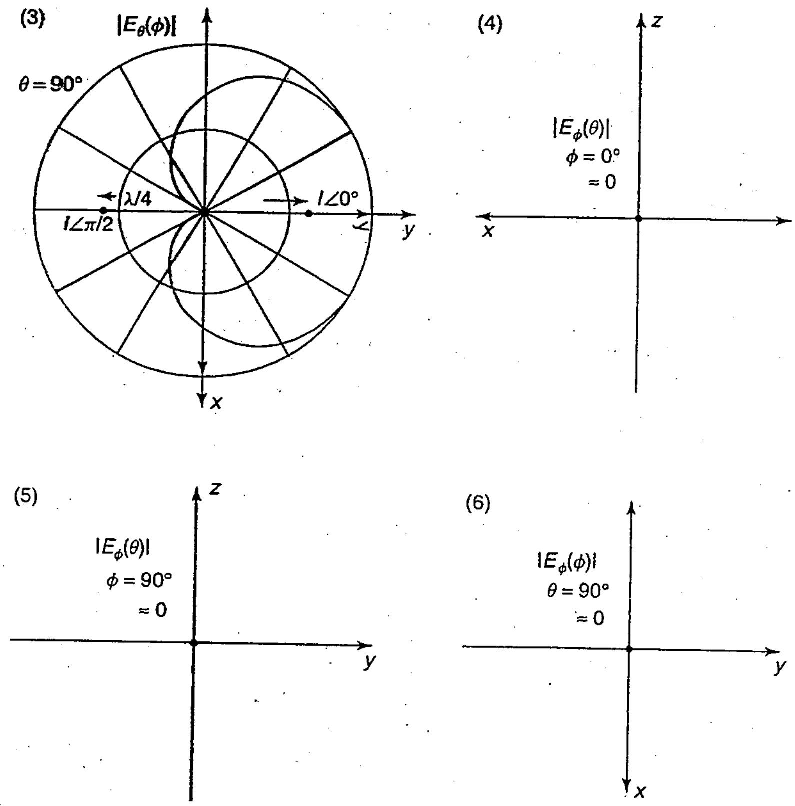
(4) $|E_\phi(\theta)|_{\phi=0°} \propto 0$

(5) $|E_\phi(\theta)|_{\phi=90°} \propto 0$

(6) $|E_\phi(\theta)|_{\theta=90°} \propto 0$







6-3. Method I

a. Derive the array factor;

$$\mathrm{AF} = -e^{jkd\cos\theta} - j + e^{-jkd\cos\theta} = -2j\sin(kd\cos\theta) - j$$
$$\mathrm{AF} = 2\sin(kd\cos\theta) + 1$$
$$\mathrm{AF} = 2\sin(\pi\cos\theta) + 1$$

b. $2\sin(\pi\cos\theta) = -1$

$$kd\cos\theta = \sin^{-1}\left(-\tfrac{1}{2}\right) = -\frac{\pi}{6}, \frac{-5\pi}{6}, \frac{-13\pi}{6}, \ldots, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}$$

$$\theta_n = \cos^{-1}\left(\frac{x}{\pi}\right)$$

$$-\frac{\pi}{6} \rightarrow \quad \theta_1 = 99.59°$$

$$-\frac{5\pi}{6} \rightarrow \quad \theta_2 = 146.44°$$

Method II

Uniform array with $\beta = -\pi/2$

a. $$\mathrm{AF} = \frac{\sin\frac{N\psi}{2}}{N\sin\frac{\psi}{2}} = \frac{\sin\frac{3}{2}\left[\pi\cos\theta - \frac{\pi}{2}\right]}{3\sin\frac{1}{2}\left[\pi\cos\theta - \frac{\pi}{2}\right]}$$

b. $$\theta_n = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2n}{N}\pi\right)\right] \quad n = 1, 2$$

$$= \cos^{-1}\left[\frac{1}{\pi}\left(\frac{\pi}{2} \pm \frac{2\pi}{3}n\right)\right] \quad n \neq 3, 6, 9$$

$n = 1;\quad \cos^{-1}\left[-\frac{1}{6}\right] = 99.59°$

$n = 2;\quad \cos^{-1}\left[-\frac{5}{6}\right] = 146.44°$

6-4. a. $$\mathrm{AF} = 1 + e^{j(kd\cos\theta + \pi/2)} + e^{-j(kd\cos\theta + \pi/2)}$$

$$= 1 + 2\cos(kd\cos\theta + \pi/2)$$

$$\therefore\ \mathrm{AF} = 1 - 2\sin(kd\cos\theta)$$

b. To find the nulls,

$$\mathrm{AF} = 1 - 2\sin(kd\cos\theta) = 0$$

$$2\sin(kd\cos\theta) = 1, \quad \pi\cos\theta = \sin^{-1}\left(\tfrac{1}{2}\right) = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \ldots$$

$$\cos\theta = \frac{1}{6}, \frac{5}{6}, \frac{13}{6}, \ldots$$

$$\theta_{\text{null}} = 80.4°, 33.6°$$

6-5. (a) $$\mathrm{E} = \frac{e^{-jkr_2}}{r_2} + \frac{e^{-jkr_1}}{r_1} - \frac{e^{-jkr_3}}{r_3} - \frac{e^{-jkr_4}}{r_4}$$

$$= \frac{e^{-jkr}}{r}\left[e^{+jk\frac{3d}{2}\cos\theta} + e^{jk\frac{d}{2}\cos\theta} - e^{-jk\frac{d}{2}\cos\theta} - e^{-jk\frac{3d}{2}\cos\theta}\right]$$

$$r_1 = r - \frac{d}{2}\cos\theta, r_2 = r - \frac{3d}{2}\cos\theta, r_3 = r + \frac{d}{2}\cos\theta, r_4 = r + \frac{3d}{2}\cos\theta$$

$$\mathrm{AF} = 2j\left[\sin\left(\frac{3kd}{2}\cos\theta\right) + \sin\left(\frac{kd}{2}\cos\theta\right)\right]$$

(b) let $x = kd\cos\theta, y = \frac{kd}{2}\cos\theta \Rightarrow \mathrm{AF} = 4j\left[\sin(kd\cos\theta)\cos\left(\frac{kd}{2}\cos\theta\right)\right]$

$$\mathrm{AF}(d = \lambda/2) = 4j\left[\sin(\pi\cos\theta)\cos\left(\frac{\pi}{2}\cos\theta\right)\right]$$

$$\therefore\ \theta_n = 0°, 90°, 180°$$

6-6. $\mathbf{E}_{\text{total}} = \dfrac{e^{-jkr}}{r}[2 + e^{jkd\cos\psi} + e^{-jkd\cos\psi}]$

$$= \frac{e^{-jkr}}{r}[2 + 2\cos(kd\cos\psi)]$$

$\cos\psi = \hat{a}_y \cdot \hat{a}_r = \sin\theta\sin\phi$

So,

$$\text{AF} = 2 + 2\cos(kd\sin\theta\sin\phi)$$

or

$$\text{AF} = 2[1 + \cos(kd\sin\theta\sin\phi)]$$

6-7. $d = \lambda/10$, broadside, uniform $\Rightarrow N = ?$

(a). $\text{HPBW} = 60° = \pi/3$ rad

$$\Theta_h \simeq 2\left[\frac{\pi}{2} - \cos^{-1}\left(\frac{1.391\lambda}{\pi N d}\right)\right]$$

$$\frac{\pi}{3} \simeq 2\left[\frac{\pi}{2} - \cos^{-1}\left(\frac{1.391\lambda}{\pi N d}\right)\right]$$

$$\frac{\pi}{6} = \frac{\pi}{2} - \cos^{-1}\left(\frac{1.391\lambda}{\pi N d}\right) \Rightarrow \cos^{-1}\left(\frac{1.391\lambda}{\pi N d}\right) = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3} = 60°$$

$$\frac{1.391\lambda}{\pi N d} = \cos(60°) = \frac{1}{2}$$

$$N = \frac{1.391(2)\lambda}{\pi d} = \frac{1.391(2)(10)}{\pi} = 8.855 \simeq 9$$

$$\boxed{N \simeq 9}$$

(b). $\text{FMBW} = 60° = \pi/3$ rad

$$\frac{\pi}{3} = 2\left[\frac{\pi}{2} - \cos^{-1}\left(\frac{\lambda}{Nd}\right)\right]$$

$$\frac{\pi}{6} = \frac{\pi}{2} - \cos^{-1}\left(\frac{\lambda}{Nd}\right) \Rightarrow \cos^{-1}\left(\frac{\lambda}{Nd}\right) = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3} = 60°$$

$$\frac{\lambda}{Nd} = \cos(60°) = \tfrac{1}{2}$$

$$N = \frac{2\lambda}{d} = \frac{2\lambda(10)}{\lambda} = 20$$

$$\boxed{N = 20}$$

6-8. $N = 3, \quad d = \lambda/2$

(a) $\theta_n = \cos^{-1}\left(\pm\frac{n}{N}\frac{\lambda}{d}\right) = \cos^{-1}\left(\pm\frac{n}{3}\frac{2}{1}\right) = \cos^{-1}\left(\pm n\frac{2}{3}\right), n = 1, 2, 3, \ldots$
$n \neq N = 3$

$$\underline{n=1}: \quad \theta_1 = \cos^{-1}\left(\pm\frac{2}{3}\right) = \begin{cases} 48.19^\circ \\ 131.81^\circ \end{cases}$$

$$\underline{n=2}: \quad \theta_2 = \cos^{-1}\left(\pm\frac{4}{3}\right) = \text{does not exist}$$

(b) $\theta_m = \cos^{-1}\left(\pm\frac{m\lambda}{d}\right) = \cos^{-1}(\pm 2m), m = 0, 1, 2, \ldots$

$$\underline{m=0}: \quad \theta_0 = \cos^{-1}(0) = 90^\circ$$
$$\underline{m=1}: \quad \theta_1 = \cos^{-1}(\pm 2) = \text{does not exist}$$

(c) $$\Theta_h \simeq 2\left[90^\circ - \cos^{-1}\left(\frac{1.391\lambda}{\pi N d}\right)\right] = 2\left[90^\circ - \cos^{-1}\left(\frac{1.391(2)}{3\pi}\right)\right]$$
$$= 2[90^\circ - \cos^{-1}(0.295)]$$
$$= 2(90^\circ - 72.83^\circ) = 2(17.17^\circ) = 34.34^\circ$$

or

$$\Theta_h = \left[\cos^{-1}\left(\cos\theta_0 - \frac{2.782}{Nkd}\right) - \cos^{-1}\left(\cos\theta_0 + \frac{2.782}{Nkd}\right)\right]_{\substack{\theta_0=90^\circ \\ N=3, d=\lambda/2}}$$
$$= \left[\cos^{-1}\left(-\frac{2\times 2.782}{6\pi}\right) - \cos^{-1}\left(\frac{2\times 2.782}{6\pi}\right)\right]$$
$$= [\cos^{-1}(-0.295) - \cos^{-1}(0.295)]$$
$$= 107.168 - 72.832 = 34.34^\circ$$

(d) $D_0 = 2N\left(\frac{d}{\lambda}\right) = 2(3)\left(\frac{1}{2}\right) = 3 = 4.7712 \text{ dB}$

(e)
$$(\text{AF})_n = \frac{1}{N}\frac{\sin\left[\frac{N}{2}(kd\cos\theta + \beta)\right]}{\sin\left[\frac{1}{2}(kd\cos\theta + \beta)\right]}\Bigg|_{\substack{N=3 \\ \beta=0 \\ d=\lambda/2}} = \frac{\sin\left(\frac{3\pi}{2}\cos\theta\right)}{3\sin\left(\frac{\pi}{2}\cos\theta\right)}$$

$$(\text{AF})_n(\theta=0^\circ) = \frac{1}{3}(\text{AF})_n(\theta=90^\circ) \Rightarrow \frac{(\text{AF})_{\theta=0^\circ}}{(\text{AF})_{\theta=90^\circ}} = \frac{1}{3} = 20\log_{10}\left(\frac{1}{3}\right) = -9.54 \text{ dB}$$

or approximately:

$$(\text{AF})_n = \frac{\sin(\frac{3\pi}{2}\cos\theta)}{3\sin(\frac{\pi}{2}\cos\theta)} \simeq \frac{\sin(\frac{3\pi}{2}\cos\theta)}{\frac{3\pi}{2}\cos\theta}$$

$$\text{AF}_n(\theta = 0^\circ) = \frac{2}{3\pi}(\text{AF})_n(\theta = 0^\circ) \Rightarrow \frac{(\text{AF})_n(\theta = 0^\circ)}{(\text{AF})_n(\theta = 90^\circ)} = \frac{2}{3\pi} = 0.2122 = -13.46 \text{ dB}$$

6-9. Placing one element at the origin and the other at d distance above it, the array factor is equal to

$$\text{AF}(\theta) = 1 + e^{j(kd\cos\theta+\beta)}$$

$$= 2e^{j\frac{1}{2}(kd\cos\theta+\beta)}\left[\frac{e^{-j\frac{1}{2}(kd\cos\theta+\beta)} + e^{+j\frac{1}{2}(kd\cos\theta+\beta)}}{2}\right]$$

$$\text{AF}(\theta) = 2e^{j\frac{1}{2}(kd\cos\theta+\beta)}\cos\left[\tfrac{1}{2}(kd\cos\theta+\beta)\right]$$

which in normalized form can be written as

$$(\text{AF})_n = \cos\left[\tfrac{1}{2}(kd\cos\theta+\beta)\right]$$

a. $\beta = -kd = -\frac{2\pi}{\lambda}\left(\frac{\lambda}{4}\right) = -\frac{\pi}{2}$.

b. For $d = \lambda/4$, $\quad (\text{AF})_n = \cos\left[\frac{\pi}{4}(\cos\theta - 1)\right]$

c. $(\text{AF})_{n|\max} = 1 = \cos\left[\frac{\pi}{4}(\cos\theta_m - 1)\right] \Rightarrow \theta_m = 0^\circ$

$$(\text{AF})_n = 0.707 = \cos\left[\frac{\pi}{4}(\cos\theta_n - 1)\right] \Rightarrow \frac{\pi}{4}(\cos\theta_n - 1) = \cos^{-1}(0.707)$$

$$= \begin{cases} +\dfrac{\pi}{4} \text{ For } +\pi/4 & \Rightarrow \cos\theta_h - 1 = 1 \Rightarrow \cos\theta_h = 2 \Rightarrow \theta_h = \cos^{-1}(2) \\ & \Rightarrow \text{Does not exist} \\ -\dfrac{\pi}{4} \text{ For } -\pi/4 & \Rightarrow \cos\theta_h - 1 = -1 \\ & \Rightarrow \cos\theta_h = 0 = \theta_h = \cos^{-1}(0) = 90^\circ = \dfrac{\pi}{2} \text{ radians} \end{cases}$$

Therefore $\theta_{1r} = \theta_{2r} = 2\left(\frac{\pi}{2} - 0\right) = \pi$

and $D_0 \simeq \frac{4\pi}{\theta_{1r}\theta_{2r}} = \frac{4\pi}{(\pi)^2} = \frac{4}{\pi} = 1.273 = 1.049$ dB

Computer Program $\left(U = \cos^2\left[\frac{\pi}{4}(\cos\theta - 1)\right]\right)$

$$D_0 = 1.9945 = 2.9984 \text{ dB}$$

6-10. a. $\beta = +kd = +\frac{\pi}{2}$

b. $(\text{AF})_n = \cos\left[\frac{\pi}{4}(\cos\theta + 1)\right]$

$(\text{AF})_{n|\max} = 1 = \cos\left[\frac{\pi}{4}(\cos\theta_m + 1)\right] \Rightarrow \theta_m = 180° = \pi$ radians

$(\text{AF})_n = 0.707 = \cos\left(\frac{\pi}{4}(\cos\theta_h + 1)\right) \Rightarrow \theta_h = 90° = \frac{\pi}{2}$ radians

$\Theta_{1r} = \Theta_{2r} = 2\left(\pi - \frac{\pi}{2}\right) = \pi$

and

$$D_0 \simeq \frac{4\pi}{\pi^2} = \frac{4}{\pi} = 1.273 = 1.049 \text{ dB}$$

Computer Program result.

$$U = \cos^2\left[\frac{\pi}{4}(\cos\theta + 1)\right]$$

$$D_0 = 1.9945 = 2.9984 \text{ dB}$$

6-11. a. $\beta = -kd = -\frac{2\pi}{\lambda}\left(\frac{\lambda}{2}\right) = -\pi = -180°$

b. $\theta_n = \cos^{-1}\left[1 - \frac{n\lambda}{Nd}\right] = \cos^{-1}\left(1 - \frac{n\lambda}{4\lambda/2}\right) = \cos^{-1}\left(1 - \frac{n}{2}\right),$

$n = 1, 2, 3, \ldots, \; n \neq 4, 8, \ldots$

$n = 1$: $\theta_1 = \cos^{-1}(1/2) = 60°$

$n = 2$: $\theta_2 = \cos^{-1}(0) = 90°$

$n = 3$: $\theta_3 = \cos^{-1}(-1/2) = 120°$

c. $\theta_m = \cos^{-1}(1 - m\lambda/d) = \cos^{-1}(1 - m\lambda/\lambda/2)$

$= \cos^{-1}(1 - 2m), \; m = 0, 1, 2, \ldots$

$m = 0$: $\theta_0 = \cos^{-1}(1) = 0°$

$m = 1$: $\theta_1 = \cos^{-1}(-1) = 180°$

d. $\Theta_0 = 2\cos^{-1}\left(1 - \frac{\lambda}{Nd}\right) = 2\cos^{-1}\left(1 - \frac{\lambda}{4\lambda/2}\right) = 2\cos^{-1}\left(1 - \frac{1}{2}\right)$

$= 2\cos^{-1}\left(\frac{1}{2}\right) = 2(60°)$

$\Theta_0 = 120°$

e. $D_0 = 2N\left(\frac{d}{\lambda}\right) = 2(4)\left(\frac{\lambda/2}{\lambda}\right) = 4 = 6.0211$ dB because end-fire in both directions

D_0 (computer programs) $= 8.2085 = 9.1427$ dB (using approximate AF)

D_0 (computer program) $= 4 = 6.021$ dB (using exact AF)

6-12. a. $D_0 = 4N\left(\frac{d}{\lambda}\right)$

$$20 = 10\log_{10} D_0 \text{ (dimensionless)} \Rightarrow D_0 \text{ (dimensionless)} = 10^2 = 100$$

$$100 = 4N\left(\frac{\lambda}{4\lambda}\right) = N \Rightarrow N = 100$$

b. $L = 99\left(\frac{\lambda}{4}\right) = \frac{99}{4}\lambda = 24.175\lambda$

c.
$$\Theta_{3dB} = \Theta_h = 2\cos^{-1}\left(1 - \frac{1.391\lambda}{Nd\pi}\right)\Big|_{n=100} = 2\cos^{-1}\left(1 - \frac{1.391\lambda}{\pi\left(\frac{\lambda}{4}\right)100}\right)$$

$$= 2\cos^{-1}\left(1 - \frac{1.391(4)}{100\pi}\right) = 2\cos^{-1}(1 - 0.01771) = 2\cos^{-1}(0.98228)$$

$$\Theta_h = 2(10.799^\circ) = 21.598^\circ \simeq 21.6^\circ$$

d. Sidelobe (dB) $\simeq -13.5$ dB

e. $\beta = \pm kd = \pm\frac{2\pi}{\lambda}\left(\frac{\lambda}{4}\right) = \pm\frac{\pi}{2} = \pm 90^\circ$

6-13. $N = 8$, Ordinary End-Fire Array

(a) $0 < d < \lambda/2$

(b) $0 < d < \lambda/2$

(c) $d = \lambda/2$

(d) $d = n\lambda, \quad n = 1, 2, 3, \ldots$

6-14. $N = 6$, $\theta_0 = 0^\circ$ and 180° simultaneously

(a) $d = \lambda/2$

(b) $\beta = \pm kd = \pm\frac{2\pi}{\lambda}\left(\frac{\lambda}{2}\right) = \pm\pi = \pm 180^\circ$

(c) $D_0 = \left(\frac{1}{2}\right)4N\left(\frac{d}{\lambda}\right) = \frac{1}{2}(4)(6)\left(\frac{\lambda}{2\lambda}\right) = 6 = 7.78$ dB

(d)
$$(\mathrm{AF})_n = \frac{\sin\left[\frac{N}{2}(kd\cos\theta + \beta)\right]}{N\sin\left[\frac{1}{2}(kd\cos\theta + \beta)\right]}\Bigg|_{\substack{N=6\\ d=\lambda/2\\ \beta=-kd=-\pi}}$$

$$= \frac{\sin\left[\frac{Nkd}{2}(\cos\theta - 1)\right]}{N\sin\left[\frac{kd}{2}(\cos\theta - 1)\right]} = \frac{\sin[3\pi(\cos\theta - 1)]}{3\sin\left[\frac{\pi}{2}(\cos\theta - 1)\right]}$$

$$(\text{AF})_n(\theta = 90°) = \frac{\sin(-3\pi)}{3\sin(\pi/2)} = \frac{0}{3} = 0$$

$$(\text{AF})_n(\theta = 0°) = \frac{\sin(0)}{3\sin(0)} = 1$$

$$\frac{(\text{AF})(\theta_0 = 90°)}{(\text{AF})(\theta_0 = 0°)} = \frac{0}{1} = 0 = -\infty \text{ dB}$$

6-15. a. Choose different phase excitation. That is

$$\beta = \pm\left(kd + \frac{2.94}{N}\right) \simeq \pm\left(kd + \frac{\pi}{N}\right)$$

$$\beta = \pm\left(\frac{2\pi}{\lambda}\frac{\lambda}{4} + \frac{2.94}{100}\right) = \pm\left(\frac{\pi}{2} + 0.0294\right)$$

$$= \pm(1.570796 + 0.0294) = \pm(1.6) = \pm 91.684°$$

b. Directivity increase by 1.789 factor = 2.526 dB

c. The HPBW will decrease because sidelobe level will increase.

$$\Theta_h = 2\cos^{-1}\left(1 - \frac{0.1398\lambda}{Nd}\right) = 2\cos^{-1}\left(1 - \frac{0.1398\lambda}{100\lambda/4}\right)$$

$$= 2\cos^{-1}\left(1 - \frac{0.1398\,(4)}{100}\right)$$

$$= 2\cos^{-1}(1 - 0.005592) = 2\cos^{-1}(0.9944) = 2(6.066°) = 12.13°$$

decreased by 9.47°

d. sidelobe level will increase. It will be higher than −13.5 dB

6-16. a. $d = \left(\frac{N-1}{N}\right)\frac{\lambda}{4} = 0.225\lambda$

b. $\beta = kd + \frac{2.94}{10} = 2\pi(0.225) + 0.294 = 1.7077$ rad

c. $\theta_n = \cos^{-1}\left(1 + (1-2n)\frac{\lambda}{2dN}\right)$

$$\theta_n = \cos^{-1}\left(1 + (1-2n)\frac{1}{4.5}\right)$$

$$\theta_1 = \cos^{-1}(0.777) = 38.9°, \theta_2 = \cos^{-1}(0.333) = 70.53°,$$

$$\theta_3 = \cos^{-1}(-0.111) = 96.38°, \theta_4 = \cos^{-1}(-0.555) = 123.7°.$$

d. First null Beamwidth

$$\theta_n = 2\cos^{-1}\left(1 - \frac{\lambda}{2dN}\right) = 2\cos^{-1}\left(1 - \frac{1}{2(0.225)\cdot 10}\right) = 77.88°$$

e. $D_0 = 1.789\left[4N \cdot \left(\frac{d}{\lambda}\right)\right] = 1.789[4 \cdot 10 \cdot (0.225)] = 16.101 = 12.068$ dB

6-17. $N = 6$

(a) $d < \lambda/2$

(b) Choose $d = 3\lambda/8$

$$D_0 = 4N\left(\frac{d}{\lambda}\right) = 4(6)\left(\frac{3}{8}\right) = 9 = 9.54 \text{ dB}(\mathbf{d = 3\lambda/8})$$

$$D_0(d = 5\lambda/24) = 4(6)\left(\frac{5}{24}\right) = 5 = 6.99 \text{ dB}(\,d = \mathbf{5\lambda/24})$$

Hansen-Woodyard End-Fire ($d = 5\lambda/24$):

(c) $D_0 = 1.805\left[4N\left(\frac{d}{\lambda}\right)\right] = 1.805\left[4(6)\left(\frac{d}{\lambda}\right)\right] = 1.805\left[24\left(\frac{d}{\lambda}\right)\right]$

$$= 1.805\left[24\left(\frac{5}{24}\right)\right]$$

$$= 1.805(5) = 9.025 = 9.5545 \text{ dB}$$

(d) $d = \left(\frac{N-1}{N}\right)\frac{\lambda}{4} = \left(\frac{6-1}{6}\right)\left(\frac{\lambda}{4}\right) = \frac{5\lambda}{24}$

6-18. $N = 10,\; d = \lambda/4$

(a) Broadside (Table 6.1 and 6.2) $\Rightarrow \beta = 0$

$$\text{HPBW} = 2\left[90^\circ - \cos^{-1}\left(\frac{1.394 \times 4}{10\pi}\right)\right] = 2(90^\circ - 79.80^\circ) = 20.4^\circ$$

$$\text{FNBW} = 2\left[90^\circ - \cos^{-1}\left(\frac{4}{10}\right)\right] = 2(90^\circ - 66.42^\circ) = 47.16^\circ$$

$$\text{FSLBW} = 2\left[90^\circ - \cos^{-1}\left(\frac{6}{10}\right)\right] = 2(90^\circ - 53.13^\circ) = 73.74^\circ$$

From (6-17a) $\Rightarrow$ Relative sidelobe maximum $= -13.46$ dB

From Table 6.7 $\Rightarrow D_0 = 2N\left(\frac{d}{\lambda}\right) = 2 \cdot 10 \cdot \frac{1}{4} = 5 = 6.99$ dB

Using the computer program of Chapter 2

$$D_0 = 5.21 \Rightarrow 7.17 \text{ dB}$$

(b) Ordinary End-Fire (Tables 6.3 and 6.4) $\Rightarrow \beta = \pm kd = \pm\pi/2 = \pm 90^\circ$

$$\text{HPBW} = 2\cos^{-1}\left[1 - \frac{1.391(4)}{10\pi}\right] = 2(34.62^\circ) = 69.25^\circ$$

$$\text{FNBW} = 2\cos^{-1}\left[1 - \frac{4}{10}\right] = 2\cos^{-1}(0.6) = 2(53.13^\circ) = 106.26^\circ$$

$$\text{FSLBW} = 2\cos^{-1}\left[1 - \frac{3(4)}{20}\right] = 2(66.42) = 132.84^\circ$$

From (6-17a) $\Rightarrow$ Relative side lobe maximum $= -13.46$ dB

From Table 6.7 $\Rightarrow D_0 = 4 \cdot N\left(\frac{d}{\lambda}\right) = 4(10)\frac{1}{4} = 10 = 10$ dB

Using the computer program of Chapter 2 $\Rightarrow D_0 = 10.05 = 10.02$ dB

(c) Hansen-Woodyard End-Fire (Tables 6.5 and 6.6)

$\beta = \pm\left(kd + \frac{\pi}{10}\right) = \pm(90° + 18°) = \pm 108°$

$$\text{HPBW} = 2\cos^{-1}\left[1 - \frac{1.398(4)}{10}\right] = 2(19.25) = 38.5°$$

$$\text{FNBW} = 2\cos^{-1}\left[1 - \frac{4}{2(10)}\right] = 2(36.87) = 73.74°$$

$$\text{FSLBW} = 2\cos^{-1}\left[1 - \frac{4}{10}\right] = 2(53.13) = 106.26°$$

From Figure 6.9 $\Rightarrow$ Relative side lobe maximum $\simeq -9$ dB

From Table 6.8 $\Rightarrow D_0 = 1.789\left[4N\left(\frac{d}{\lambda}\right)\right] = 1.789(4)(10)\left(\frac{1}{4}\right)$
$= 17.89 = 12.5$ dB

Using the computer program of Chapter 2 $\Rightarrow D_0 = 18.02 = 12.56$ dB

6-19.
$$(\text{AF})_n = \frac{\sin\left[\frac{N}{2}(kd\cos\theta + \beta)\right]}{N\sin\left[\frac{1}{2}(kd\cos\theta + \beta)\right]} = \frac{\sin\left[5\left(\frac{\pi}{2}\cos\theta + \beta\right)\right]}{10 \cdot \sin\left[\frac{1}{2}\left(\frac{\pi}{2}\cos\theta + \beta\right)\right]}$$

$$\theta_0 = 45° \Rightarrow \beta = -kd\cos\theta_0 = -\frac{2\pi}{\lambda}\left(\frac{\lambda}{4}\right)\cos 45° = -1.1107 \text{ radians}$$

a. Using (6-22)

$$\Theta_h = \text{HPBW} = \cos^{-1}\left[\cos 45° - 0.443\frac{\lambda}{(\text{Ltd})}\right] - \cos^{-1}\left[\cos 45° + 0.443\frac{\lambda}{(\text{Ltd})}\right]$$

$$= \cos^{-1}\left[0.707 - 0.443\frac{1}{2.25 + 0.25}\right] - \cos^{-1}\left[0.707 + 0.443\frac{1}{2.25 + 0.25}\right]$$

$$\Theta_h = \cos^{-1}(0.5299) - \cos^{-1}(0.8843) = 58° - 27.83° = 30.2°$$

b. $D_0 = \frac{U_{\max}}{U_0}, U_{\max} = 1$

$$U_0 = \tfrac{1}{2}\int_0^{\pi}\left[\frac{\sin\left[5 \cdot \frac{\pi}{2}(\cos\theta - 0.707)\right]}{5 \cdot \frac{\pi}{2}(\cos\theta - 0.707)}\right]^2 \sin\theta\, d\theta$$

Let $z = 5 \cdot \frac{\pi}{2}(\cos\theta - 0.707)$, $dz = -\frac{5\pi}{2}\sin\theta\, d\theta$

$$U_0 = -\frac{1}{5\pi}\int_{2.3012}^{-13.4067}\left[\frac{\sin z}{z}\right]^2 dz, \quad (Nkd \to \text{large})$$

$$U_0 \simeq -\frac{1}{5\pi}\int_{\infty}^{-\infty}\left(\frac{\sin z}{z}\right)^2 dz = \frac{\pi}{5\pi} = \frac{1}{5}$$

$$D_0 = \frac{U_{max}}{U_0} \simeq \frac{1}{(1/5)} = 5 = D_0|_{\theta_0=90^\circ}$$

since for $\theta_0 = 90^\circ$ $\quad D_0 \simeq 2N\left(\frac{d}{\lambda}\right)$

and for $\theta_0 = 0^\circ$ $\quad D_0 \simeq 4N\left(\frac{d}{\lambda}\right)$

we might expect for $\theta_0 = 45^\circ$ the value of D_0 to be somewhere between $D_0|_{\theta_0=0^\circ}$ and $D_0|_{\theta_0=90^\circ}$

A possibility is $D_0|_{\theta_0=45^\circ} \simeq 3N\left(\frac{d}{\lambda}\right) = 3\cdot(10)\cdot\left(\frac{1}{4}\right)$, $D_0 \simeq 7.5$

Using computer program:

$D_0 = 5.321$

6-20. $$(\text{AF})_n = \frac{\sin\left[\frac{N}{2}(kd\cos\theta + \beta)\right]}{N\sin\left[\frac{1}{2}(kd\cos\theta+\beta)\right]}$$

a. For $\beta = 0 \Rightarrow (\text{AF})_n = \dfrac{\sin\left(\frac{N}{2}kd\cos\theta\right)}{N\sin\left(\frac{1}{2}kd\cos\theta\right)}$

In order for the array not to have any minor lobes, we can assume that its first null occurs at $\theta = 0^\circ$ or 180°. Thus

$$(\text{AF})_n = \frac{\sin\left(\frac{N}{2}kd\right)}{N\sin\left(\frac{1}{2}kd\right)} = 0 \Rightarrow \frac{N}{2}kd = \pi \Rightarrow d = \frac{2\pi}{kN} = \frac{\lambda}{N}$$

This assures that there are no minor lobes formed.

b. For $\beta = kd$ the maximum occurs at $\theta = 180^\circ$ and the array factor can be written as $(\text{AF})_n = \dfrac{\sin\left[\frac{N}{2}kd(\cos\theta+1)\right]}{N\sin\left[\frac{1}{2}kd(\cos\theta+1)\right]}$

In order for the array not to have any minor lobes, we can assume that the first null is formed at $\theta = 0^\circ$.

Thus $\left.\frac{N}{2}kd(\cos\theta+1)\right|_{\theta=0} = Nkd = \pi \Rightarrow d = \frac{\pi}{Nk} = \frac{\lambda}{2N}$

6-21. $kd = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}$

a. $\beta = 0$ radians

b. $\beta = -\pi/2$

c. $\beta = +\pi/2$

d. $\beta = -1.36 = -\frac{\sqrt{3}}{4}\pi = -0.433\ \pi$

e. $\beta = -\left(\frac{\pi}{2} + 0.147\right)$ or $-\left(\frac{\pi}{2} + 0.157\right) = -\frac{11}{20}\pi = -1.72$

f. $\beta = +\left(\frac{\pi}{2} + 0.147\right)$ or $+\left(\frac{\pi}{2} + 0.157\right) = \frac{11}{20}\pi = 1.72$

6-22. $N = 19,\ d = \lambda/4$

a. $\beta = -kd\cos\theta_0\big|_{\substack{\theta_0=30^\circ \\ d=\lambda/4}} = -\frac{2\pi}{\lambda}\left(\frac{\lambda}{4}\right)\cos(30^\circ) = -\frac{\pi}{2}\frac{\sqrt{3}}{2} = -\frac{\pi\sqrt{3}}{4} = -1.3603$

$\beta = -\frac{\pi\sqrt{3}}{4} = -1.3603\ (\text{rad}) = -77.942^\circ$

b. $\theta_h = \cos^{-1}\left[\cos\theta_0 - 0.443\frac{\lambda}{L+d}\right]_{\theta_0=30^\circ} - \cos^{-1}\left[\cos\theta_0 + 0.443\frac{\lambda}{L+d}\right]_{\theta_0=30^\circ}$

$= \cos^{-1}\left[0.866 - \frac{0.443}{5}\right] - \cos^{-1}\left[0.866 + \frac{0.443}{5}\right]$

$= \cos^{-1}(0.7774) - \cos^{-1}(0.9546) = 38.9769^\circ - 17.3309^\circ = 21.6459^\circ$

$\theta_h = 21.6459^\circ$

c. $\boxed{-13.5\ \text{dB}}$

Computer Result.

HPBW $\Rightarrow$ 23 degree

D_0(Directivity) = 10.103 dB

6-23. $D_0 \simeq 2N\ (d/\lambda)$

a. $d = \frac{\lambda}{4}, D_0 = 2.10 \cdot \frac{1}{4} = 5 = 6.99$ dB

Computer Program: $D_0 = 7.132$ dB

b. $d = \frac{\lambda}{2}, D_0 = 2.10 \cdot \frac{1}{2} = 10 = 10$ dB

Computer Program: $D_0 = 10.00$ dB

c. $d = \frac{3\lambda}{4}, D_0 = 2.10 \cdot (0.75) = 15 = 11.76$ dB

Computer Program: $D_0 = 11.624$ dB

d. $d = \lambda, D_0 = 2.10 \cdot (1) = 20 = 13.0$ dB

Computer Program: $D_0 = 10.011$ dB

6-24. The recommended element spacing is

$d = \frac{1}{1+\cos\theta}$, where θ is the scan angle in degrees

a. $\theta_0 = 30°$

$$d = \frac{1}{1+\cos 30°} = 0.5359 \text{ wavelength}$$

b. $\theta_0 = 45°$

$$d = \frac{1}{1+\cos 45°} = \frac{1}{1+0.7071} = 0.58578 \text{ wavelength}$$

c. $\theta_0 = 60°$

$$d = \frac{1}{1+\cos 60°} = \frac{1}{1+0.5} = 0.6667 \text{ wavelength}$$

Although a narrow element pattern can sometimes accommodate larger spacing, using this rule will ensure that the array factor has only one maximum in the visible region.

6-25. Since the excitation coefficient of each element is identical, $\beta = 0$. Thus
$\text{AF} = e^{j\psi_0} + e^{j\psi_x} + e^{j\psi_y} + e^{j\psi_z}$

where

$$\begin{aligned}
\psi_0 &= 0 && \text{; For element at origin} \\
\psi_x &= kd\cos\gamma_x = kd\,\hat{a}_x \cdot \hat{a}_r = kd\sin\theta \cdot \cos\phi && \text{; For element along } x\text{-axis} \\
\psi_y &= kd\cos\gamma_y = kd\,\hat{a}_y \cdot \hat{a}_r = kd\sin\theta\cos\phi && \text{; For element along } y\text{-axis} \\
\psi_z &= kd\cos\gamma_z = kd\,\hat{a}_z \cdot \hat{a}_r = kd\cos\theta && \text{; For element along } z\text{-axis}
\end{aligned}$$

6-26. $(\text{AF})_n = \cos\theta; 0° \le \theta \le 90°, 0° \le \phi \le 360°$

(a) Replace $\cos\theta$ by $\sin\theta\sin\phi$

$$(\text{AF})_n \simeq \sin\theta\sin\phi; 0° \le \theta \le 180°, 0° \le \phi \le 180°$$

(b) 1. xy-plane$(\theta = 90°) \Rightarrow (\text{AF})_n = \sin\phi$

$\sin\phi_h = 0.707 \Rightarrow \phi h = 45° \Rightarrow \Phi_h = 2(90° - 45°) = 90°$

2. yz-plane $(\phi = 90°) \Rightarrow (\text{AF})_n \simeq \sin\theta$

$$\sin\theta_h = 0.707 \Rightarrow \theta_h = 45° \Rightarrow \Theta_h = 2(90° - 45°) = 90°$$

(c)
$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}}, U_{\max} = \sin^2\theta \sin^2\phi|_{\max} = 1$$

$$P_{\text{rad}} = \int_0^{\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi = \int_0^{\pi}\sin^2\phi\, d\phi \int_0^{\pi}\sin^3\theta\, d\theta$$
$$= \frac{\pi}{2}\left(\frac{4}{3}\right) = \frac{4\pi}{6} \text{(see below)}$$

$$\int_0^{\pi}\sin^2\phi\, d\phi = \tfrac{1}{2}\int_0^{\pi}[1-\cos(2\phi)]\, d\phi = \tfrac{1}{2}\left[\phi - \tfrac{1}{2}\sin^2\phi\right]_0^{\pi} = \frac{\pi}{2},$$

$$\int_0^{\pi}\sin^3\theta\, d\theta = \left(-\cos\theta + \frac{1}{3}\cos^3\theta\right)_0^{\pi} = \frac{4}{3}$$

$$D_0 = \frac{4\pi(1)}{4\pi/6} = 6 = 7.782 \text{ dB}$$

6-27. $(\text{AF})_n \simeq \cos^2\theta;\quad 0^\circ \le \theta \le 90^\circ,\quad 0^\circ \le \phi \le 360^\circ$

(a) Replace $\cos\theta$ by $\sin\theta\sin\phi$

$$(\text{AF})_n \simeq \sin^2\theta\sin^2\phi; 0^\circ \le \theta \le 180^\circ, 0^\circ \le \phi \le 180^\circ$$

(b) 1. xy-plane $(\theta = 90^\circ) \Rightarrow (\text{AF})_n = \sin^2\phi$

$$\sin^2\phi_h = 0.707 \Rightarrow \phi_h = \sin^{-1}(0.841) = 57.228^\circ$$
$$\Rightarrow \Phi_h = 2(90^\circ - 57.228^\circ) = 65.544^\circ$$

2. yz-plane $(\phi = 90^\circ) \Rightarrow (\text{AF})_n = \sin^2\theta$

$$\sin^2\theta_h = 0.707 \Rightarrow \theta_h = \sin^{-1}(0.841) = 57.228^\circ$$
$$\Rightarrow \Theta_h = 2(90^\circ - 57.228^\circ) = 65.544^\circ$$

(c) $D_0 = \dfrac{4\pi U_{\max}}{P_{\text{rad}}}, U_{\max} = \sin^4\theta\sin^4\phi|_{\max} = 1$

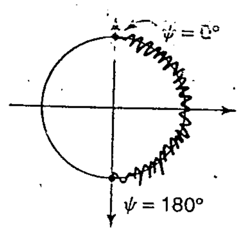
$$P_{\text{rad}} = \int_0^{\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi = \int_0^{\pi}\sin^4\phi\, d\phi\int_0^{\pi}\sin^5\theta\, d\theta$$

$$\int_0^{\pi}\sin^4\phi\, d\phi = \frac{1}{4}\int_0^{\pi}\left[\frac{3}{2} - 2\cos(2\phi) + \tfrac{1}{2}\cos(4\phi)\right] = \frac{3\pi}{8}, \int_0^{\pi}\sin^5\theta\, d\theta = \frac{16}{15}$$

$$D_0 = \frac{4\pi(1)}{\frac{3\pi}{8}(16/15)} = \frac{4(15)(8)}{3(16)} = \frac{5(32)}{16} = 10 = 10 \text{ dB}$$

6-28. $\beta = 0°,\ d = \lambda/4$

a. $\psi = kd\cos\theta + \beta = \dfrac{2\pi}{\lambda}\dfrac{\lambda}{4}\cos\theta + 0 = \dfrac{\pi}{2}\cos\theta$



b. $\theta = 0° \Rightarrow \psi_1 = \dfrac{\pi}{2}\cos\theta° = \dfrac{\pi}{2}\cos 0° = \dfrac{\pi}{2} \Rightarrow z_1 = j$

$\theta = 45° \Rightarrow \psi_2 = \dfrac{\pi}{2}\cos 45° = \dfrac{\pi}{2}\dfrac{1}{\sqrt{2}} \Rightarrow z_2 = 0.444 + j0.896$

$$\text{AF} = (z - j)(z - 0.444 - j0.896) = (-0.896 + j0.444) + z(-0.444 - j1.896) + z^2$$

3 elements needed

c. $a_1 = -0.896 + j0.444$

$a_2 = -0.444 - j1.896$

$a_3 = 1$

6-29. $\beta = \pi/4,\ d = \lambda/4$

a. $\psi = kd\cos\theta + \beta = \dfrac{2\pi}{\lambda}\cdot\dfrac{\lambda}{4}\cos\theta + \dfrac{\pi}{4} = \dfrac{\pi}{2}\cos\theta + \dfrac{\pi}{4}$

Visible region: $\theta = 0° \Rightarrow \psi = \dfrac{\pi}{2} + \dfrac{\pi}{4} = \dfrac{3\pi}{4}$

$\theta = 180° \Rightarrow \psi = -\dfrac{\pi}{2} + \dfrac{\pi}{4} = -\dfrac{\pi}{4}$

b. $\text{AF} = (z - z_1)(z - z_2)(z - z_3)$

$= a_1 + a_2 z + a_3 z^2 + a_4 z^3 \quad \rightarrow$ 4 elements required

c. $\psi_{10°} = kd\cos\theta + \beta = 90\cdot\cos(10°) + 45° = 133.633°$

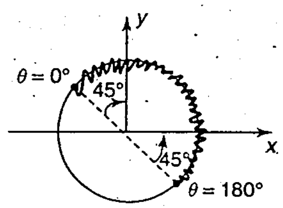
$= -0.690 + j0.724 = z_1$

$\psi_{70°} = 90°\cos(70°) + 45° = 75.782° = 0.2456 + j0.9694 = z_2$

$\psi_{110°} = 90°\cos(110°) + 45° = 14.218° = 0.9694 + j0.2456 = z_3$

So

$$\begin{aligned}
\text{AF} &= (z + 0.690 - j0.724)(z - 0.2456 - j0.9694)(z - 0.9694 - j0.2456) \\
&= [z^2 + (0.4444 - j1.6934)z + (-0.8713 - j0.491)](z - 0.9694 - j0.2456) \\
\text{AF} &= z^3 + z^2(-0.5250 - j1.9390) + z(-1.718 + j1.041) + (0.724 + j0.690)
\end{aligned}$$



So

$$\begin{aligned}
a_1 &= +0.724 + j0.690 = 1.000138\angle 43.62^\circ \\
a_2 &= -1.718 + j1.041 = 2.00878\angle 148.786^\circ \\
a_3 &= -0.5250 - j1.9390 = 2.0088\angle -105.1500^\circ \\
a_4 &= 1
\end{aligned}$$

6-30. a. $\psi = kd\cos\theta + \beta = 72^\circ \cos\theta$

b. $\theta = 0^\circ$: $\psi = \psi_1 = 0.4\pi \Rightarrow z_1 = 0.31 + j0.95$

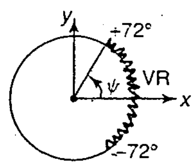
$\theta = 50^\circ$: $\psi = \psi_2 = 0.257\pi \Rightarrow z_2 = 0.69 + j0.723$

$\theta = 100^\circ$: $\psi = \psi_3 = -0.218 \Rightarrow z_3 = 0.976 - j0.216$

$\text{AF} = (z - 0.31 - j0.95)(z - 0.69 - j0.723)(z - 0.976 + j0.216)$

$\text{AF} = z^3 + z^2(-1.98 - j1.46) + z(0.865 + j2.298) + (0.272 - j0.96)$

4 elements required



c. $a_1 = 0.272 - j0.962 = 1\angle -74.22^\circ$

$a_2 = 0.865 + j2.298 = 2.46\angle 69.37^\circ$

$a_3 = -1.98 - j1.46 = 2.46\angle 216.4^\circ$

$a_4 = 1\angle 0^\circ = 1 + j0$

6-31. a. The excitation coefficients for a 3-element array are 1, 2, 1.

Placing one element at the origin, one above it, and the other below it, the problem is identical to that of Problem 6-1. Thus the array factors are identical and equal to

b. $(\text{AF})_n = 1 + \cos(kd\cos\theta) = 2\cos^2\left(\frac{kd}{2}\cos\theta\right)$

c. The nulls of the pattern can be found using either of the above forms, as it was demonstrated in Problem 6-1. Using either one.

$$d = \lambda \Rightarrow \theta_n = \cos^{-1}(n\lambda/2d) = \cos^{-1}(n/2), n = \pm1, \pm3, \pm5, \ldots$$

$n = \pm1$: $\theta_1 = \cos^{-1}(\pm1/2) = \cos^{-1}(\pm0.5) = 60°, 120°$

$n = \pm3$: $\theta_3 = \cos^{-1}(\pm3/2) = \cos^{-1}(\pm1.5) =$ Does not exist.

$n = \pm5$: $\theta_5 = \cos^{-1}(\pm5/2) = \cos^{-1}(2.5) =$ Does not exist.

The same holds for $\|n\| \geq n$.

d. The maxima of the pattern can also be found either of the forms. Using the results of Problem 6-1.

$$d = \lambda \Rightarrow \theta_m = \cos^{-1}(m\lambda/d) = \cos^{-1}(m), m = 0, \pm1, \pm2, \pm3, \ldots$$

$m = 0$: $\theta_0 = \cos^{-1}(0) = 90°$

$m = \pm1$: $\theta_1 = \cos^{-1}(\pm1) = 0°, 180°$

$m = \pm2$: $\theta_2 = \cos^{-1}(\pm2) =$ Does not exist. The same holds for $n \geq 3$.

6-32. For a three-element binomial array the array factor is that given in Problem 6-1 and 6-31. Thus in normalized form it can be written as

$$(\text{AF})_n = \cos^2\left(\frac{kd}{2}\cos\theta\right)$$

whose maximum occurs at $\theta = 90°$. In order not to have a side lobe, the argument of the outer cosine function at $\theta = 0°$ or $180°$ must be equal or less than $\pi/2$. Thus

$$\left|\frac{kd}{2}\cos\theta\right|_{\substack{\theta=0° \\ \theta=180°}} \leq \frac{\pi}{2} \Rightarrow d \leq \frac{\pi}{k} = \frac{\lambda}{2}$$

6-33. The excitation coefficients of a 4-element binomial array are 1, 3, 3, 1 or

a. $\left.\begin{array}{l} a_1 = 3 \\ a_2 = 1 \end{array}\right\} N = 2M = 4 \Rightarrow M = 2$

b. $(\text{AF})_4 = \sum_{n=1}^{M=2} an \cos[(2n-1)u], u = \frac{\pi d}{\lambda}\cos\theta$, using (6-61a) and (6-61c).

Thus

$$(\mathrm{AF})_4 = a_1 \cos(u) + a_2 \cos(3u) = 3\cos\left(\frac{\pi d}{\lambda}\cos\theta\right) + \cos\left(\frac{3\pi d}{\lambda}\cos\theta\right)$$

which can also be written, using (6-66) for $m = 3$, as

$$\begin{aligned}(\mathrm{AF})_4 &= 3\cos\left(\frac{\pi d}{\lambda}\cos\theta\right) + 4\cos^3\left(\frac{\pi d}{\lambda}\cos\theta\right) - 3\cos\left(\frac{\pi d}{\lambda}\cos\theta\right)\\ &= 4\cos^3\left(\frac{\pi d}{\lambda}\cos\theta\right)\\ (\mathrm{AF})_4 &= 4\cos^3\left(\frac{\pi d}{\lambda}\cos\theta\right)\end{aligned}$$

c. The nulls occur when

$$\begin{aligned}(\mathrm{AF})_4 &= 4\cos^3\left(\frac{\pi d}{\lambda}\cos\theta_n\right) = 0 \Rightarrow \frac{\pi d}{\lambda}\cos\theta_n = \cos^{-1}(0)\\ &= \pm\frac{(2n+1)\pi}{2}, n = 0, 1, 2, \ldots\end{aligned}$$

or

$$\theta_n = \cos^{-1}\left[\pm\frac{(2n+1)\lambda}{2d}\right] \underset{=}{d = 3\lambda/4} \cos^{-1}\left[\pm\frac{(2n+1)2}{3}\right], n = 0, 1, 2, \ldots$$

$n = 0$: $\theta_0 = \cos^{-1}\left(\pm\frac{2}{3}\right) = 48.19°, 131.81°$

$n = 1$: $\theta_1 = \cos^{-1}(\pm 2)$ = Does not exist. The same holds for $n \geq 2$.

6-34. a. Using Pascal's triangle ($2M + 1 = 5 \Rightarrow M = 2$), $kd = 5\pi/4$

$$\Rightarrow d = \frac{\frac{5\pi}{4}}{2\pi/\lambda} = \frac{5}{8}\lambda$$

$2a_1 = 6 \Rightarrow a_1 = 3; a_2 = 4, a_3 = 1$

b. $$\begin{aligned}(\mathrm{AF})_5 &= \sum_{n=1}^{M+1} a_n \cos[2(n-1)u] = \sum_{n=1}^{3} a_n \cos[2(n-1)u]\\ &= a_1 + a_2\cos(2u) + a_3\cos(4u)\\ &= 3 + 4\cos(2u) + \cos(4u) \underset{=}{x = 2u}\ 3 + 4\cos(x) + \cos(2x)\end{aligned}$$

$$\cos^2(x) = \frac{1 + \cos(2x)}{2} \Rightarrow \cos(2x) = 2\cos^2(x) - 1$$

$$(\text{AF})_5 = 3 + \cos(x) + 2\cos^2 x - 1 = 2 + 4\cos(x) + 2\cos^2(x)$$
$$= 2[1 + 2\cos x + \cos^2 x] = 2[1 + \cos x]^2$$
$$= 2[1 + \cos(2u)]^2 = 2(2)^2[\cos^2 u]^2 = 8\cos^4 u,\ u = \frac{\pi d}{\lambda}\cos\theta\Big|_{d=\frac{5\lambda}{8}}$$
$$= \frac{5\pi}{8}\cos\theta$$

$$(\text{AF})_5 = 8\cos^4(u), u = \frac{5\pi}{8}\cos\theta$$

c. $[(\text{AF})_5]_n = \cos^4(u), u = (5\pi/8)\cos\theta$

d. $(\text{AF})_n = \cos^4(u) = 0 \Rightarrow u = \frac{5\pi}{8}\cos\theta_n = \cos^{-1}(0) = \frac{n\pi}{2}$,

$n = \pm 1, \pm 2, \pm 3, \ldots$

$$\theta_n = \cos^{-1}\left[\frac{n\pi}{2}\left(\frac{8}{5\pi}\right)\right] = \cos^{-1}\left(n\frac{4}{5}\right), n = \pm 1, \pm 3, \pm 5, \ldots$$

$n = 1$: $\theta_1 = \cos^{-1}(4/5) = 36.87°$ **$n = -1$:** $\theta_{-1} = \cos^{-1}(-4/5) = 143.13°$

$n = 3$: $\theta_3 = \cos^{-1}(12/5) =$ does not exist **$n = -3$:** $\theta_{-3} = \cos^{-1}(-12/5) =$ does not exist

$n = 5$: $\theta_5 = \cos^{-1}(4) =$ does not exist **$n = -5$:** $\theta_{-5} = \cos^{-1}(-4) =$ does not exist

Nulls @ $\theta = 36.87°, 143.13°$

6-35. The excitation coefficients for a 4-element binomial array are 1, 3, 3, 1 or

a. $a_1 = 3$, $a_2 = 1$

b. Since the elements are placed along the x-axis

$$\cos\gamma = \hat{a}_x \cdot \hat{a}_r = \hat{a}_x \cdot (\hat{a}_x \sin\theta\cos\phi + \hat{a}_y \sin\theta\sin\phi + \hat{a}_z\cos\theta)$$
$$= \sin\theta\cos\phi$$

The array factor for this array is similar to that of Problem 6-33. The

$$(\text{AF})_4 = 3\cos\left(\frac{\pi d}{\lambda}\sin\theta\cos\phi\right) + \cos\left(\frac{3\pi d}{\lambda}\sin\theta\cos\phi\right)$$
$$= 4\cos^3\left(\frac{\pi d}{\lambda}\sin\theta\cos\phi\right)$$

c. The total field is obtained using the pattern multiplication rule of (6-5) by multiplying the field of a single $\lambda/2$ dipole, as given by (4-84), with the array factor above. Thus

$$E_\theta(\text{total}) = E_\theta(\text{single}) \times (\text{AF})$$
$$= j\eta\frac{I_0 e^{-jkr}}{2\pi r}\cdot\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\left[4\cos^3\left(\frac{\pi d}{\lambda}\sin\theta\cos\phi\right)\right]$$

6-36. The answers to this problem are identical to those of Problem 6-35, except that cos γ is equal to

$$\cos\gamma = \hat{a}_y \cdot \hat{a}_r = \sin\theta \sin\phi$$

Therefore $\sin\theta\cos\phi$ in Problem 6-35 must be replaced by $\sin\theta\sin\phi$

6-37. a. From (6-63), $a_1 = 10$, $a_2 = 5$, $a_3 = 1$, $\leftarrow$ Verified with computer program

b. Since the array is broadside, the progressive phase shift between the elements as required by (6-18a) is zero ($\beta = 0$.)

c. $(AF)_6 = 2\sum_{n=1}^{3} \text{an} \cos[(2n-1)u], u = \frac{\pi d}{\lambda}\cos\theta = \frac{\pi}{2}\cos\theta.$

Computer Program $\Rightarrow D_0 = 6.089$ dB At $d = \lambda/2$

d. $$\underline{E} = \hat{a}_\theta j\eta \frac{I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{\pi}{4}\cos\theta\right) - \cos\left(\frac{\pi}{4}\right)}{\sin\theta}\right]\left\{10\cos\left(\frac{\pi}{2}\cos\theta\right) + 5\cos\left(\frac{3\pi}{2}\cos\theta\right) + \cos\left(\frac{5\pi}{2}\cos\theta\right)\right\}$$

6-38. a. From (6-63), $a_1 = 10$, $a_2 = 15$, $a_3 = 6$, $a_4 = 1 \leftarrow$ Verified with computer program ($D_0 = 6.467$ dB) at $d = \frac{\lambda}{2}$

b. Same answer like (b) in Problem 6.37.

c. $AF = \sum_{n=1}^{4} a_n \cdot \cos[2(n-1)u] = 10 + 15\cos 2u + 6\cos 4u + \cos 6u$

$$\left(\leftarrow u = \frac{\pi d}{\lambda}\cos\theta = \frac{\pi}{2}\cos\theta\right)$$

d. Field of E_θ at origin: From (4-62a)

$$E_\theta \simeq j\frac{\eta I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{\pi}{4}\cos\theta\right) - \cos\left(\frac{\pi}{4}\right)}{\sin\theta}\right] \leftarrow \text{one dipole of } \frac{\lambda}{4} \text{ length.}$$

Array:

$$E_\theta \simeq j\eta\frac{I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{\pi}{4}\cos\theta\right) - 0.707}{\sin\theta}\right][10 + 15\cos 2u + 6\cos 4u + \cos 6u]$$

$$\left(\leftarrow u = \frac{\pi d}{\lambda}\cos\theta\right)$$

6-39. The excitation coefficients for a 5-element binomial array are 1, 4, 6, 4, 1 or $a_1 = 3$, $a_2 = 4$, and $a_3 = 1$. Thus the array factor can be written using (6-61b) and (6-61c) as

a. $(\mathrm{AF})_5 = \sum_{n=1}^{3} a_n \cos[2(n-1)u] = a_1 + a_2\cos(2u) + a_3\cos(4u)$

Using (6-69) for $m = 2$ and $m = 4$, the array factor can also be written as $(\mathrm{AF})_5 = a_1 + a_2[2\cos^2 u - 1] + a_3[8\cos^4 u - 8\cos^2 u + 1]$

$$(\mathrm{AF})_5 = 3 + 4(2\cos^2 u - 1) + (8\cos^4 u - 8\cos^2 u + 1) = 8\cos^4 u$$
$$= 8\cos^4\left(\frac{\pi d}{\lambda}\cos\theta\right)$$

b. Using the computer program of Chapter 2.

$$D_0 = 3.668 = 5.64 \text{ dB}$$

c. The nulls of the pattern are obtained from

$$(\mathrm{AF})_5 = 8\cos^4\left(\frac{\pi d}{\lambda}\cos\theta_n\right)\bigg|_{d=\lambda} = 8\cos^4(\pi\cos\theta_n) = 0$$
$$\pi\cos\theta_n = \cos^{-1}(0) = \pm\left(\frac{2n+1}{2}\right)\pi, n = 0,1,2,3,4,\ldots.$$
$$\theta_n = \cos^{-1}\left[\pm\left(\frac{2n+1}{2}\right)\right], n = 0,1,2,\ldots$$
$$n = 0: \theta_0 = \cos^{-1}\left(\pm\tfrac{1}{2}\right) = 60^\circ, 120^\circ$$
$$n = 1: \theta_1 = \cos^{-1}\left(\pm\tfrac{3}{2}\right) = \text{Does not exist. The same holds for } n \geq 2.$$

6-40. $N = 3. = 2M + 1 \Rightarrow M = 1,\ d = \lambda/2$

(a) $1\,②\,1 \Rightarrow 2a_1 = 2 \Rightarrow \boxed{a_1 = 1}, \boxed{a_2 = 1}$

(b) $(\mathrm{AF})_{2M+1} = \sum_{n=1}^{M+1} a_n\cos[2(n-1)u] = a_1 + a_2\cos(2u)$

$$(\mathrm{AF})_3 = \left(\frac{1 + 1\cos(2u)}{2}\right)2 = 2\cos^2(u)$$

$$\boxed{(\mathrm{AF})_3 = 1 + \cos(2u) = 2\cos^2(u), u = \frac{\pi d}{\lambda}\sin\theta\sin\phi = \frac{\pi}{2}\sin\theta\sin\phi}$$

(c) $D_0 = \left.\dfrac{(2N-2)(2N-4)\cdots 2}{(2N-3)(2N-5)\cdots 1}\right|_{N=3} = \dfrac{4(2)}{3(1)} = \dfrac{8}{3} = 2.667$ (6-65a)

$$\boxed{D_0 = \tfrac{8}{3} = 2.667 = 4.26 \text{ dB}}$$

(d) $\boxed{D_0\left(l = \dfrac{\lambda}{2}\right) = 1.643 = 2.156 \text{ dB}}$

(e) $D_0 \leq 1.643(2.667) = \boxed{4.382 = 6.417 \text{ dB}}$

(f) **alternate:**

$$\boxed{D_0 \simeq 1.77\sqrt{N} = 1.77\sqrt{3} = 3.0657 = 4.865 \text{ dB}} \quad \text{(6-65b)}$$

(g) $D_0 \leq 1.643(3.0657) = \boxed{5.037 = 7.02 \text{ dB}}$

6-41. $N = 3$, $d = \lambda/4 \Rightarrow 2a_1 = 2 \Rightarrow a_1 = 1$, $a_2 = 1$,

$$u = \frac{\pi d}{\lambda}\cos\theta = \frac{\pi}{\lambda}\left(\frac{\lambda}{4}\right)\cos\theta = \frac{\pi}{4}\cos\theta$$

$2M + 1 = 3 \Rightarrow M = 1$

$$\text{AF} = \sum_{n=1}^{2} a_n \cos[2(n-1)u] = a_1\cos(0) + a_2\cos(2u) = a_1 + a_2\cos(2-u)$$
$$= 1 + \cos(2-u)$$
$$\text{AF} = 1 + \cos(2u) = 2\cos^2(u) = 2\cos^2\left(\frac{\pi}{4}\cos\theta\right) \Rightarrow (\text{AF})\big|_{\max} = 2$$

(a) $(\text{AF})_{\max} = 2$ when $\theta = 90°$

(b) $(\text{AF}) = 2\cos^2\left(\dfrac{\pi}{4}\cos\theta_h\right)$
$= 2(0.707)$
$\cos^2\left(\dfrac{\pi}{4}\cos\theta_h\right) = 0.707$
$\dfrac{\pi}{4}\cos\theta_h = \cos^{-1}(\sqrt{0.707})$
$= \cos^{-1}(0.84083)$
$= 32.772° = 0.57198$ rads
$\theta_h = \cos^{-1}\left(0.57198\dfrac{4}{\pi}\right)$
$= \cos^{-1}(0.72826)$
$\theta_h = 43.259°$

(or) $(\text{AF}) = \left[1 + \cos\left(\dfrac{\pi}{2}\cos\theta_h\right)\right]$
$= 2(0.707) = 1.414$
$\cos\left(\dfrac{\pi}{2}\cos\theta_h\right) = 0.414$
$\dfrac{\pi}{2}\cos\theta_h = \cos^{-1}(0.414) = 65.5436°$
$= 1.14395$ rads
$\cos\theta_h = 1.14395(2/\pi) = 0.72826$
$\theta_h = \cos^{-1}(0.72826) = 43.259$

$$\theta_h = 43.259°, \quad 180° - 43.259° = 136.741°$$

(c) $\Theta_h = 2(90 - 43.259) = 2(46.741°) = 93.482°$ or $\Theta_h = 136.741 - 43.259 = 93.482$

(d) $D_0(\text{Pozar}) = -172.4 + 191\sqrt{0.818 + 1/93.482} = -172.4 + 173.87267$
$= 1.47267 = 1.681$ dB

$D_0(\text{McDonald}) = 101/[93.482 - 0.0027(93.482)^2] = 1.4452 = 1.599$ dB

6-42. $N = 5$, Binomial $\Rightarrow 2M + 1 = N = 5 \Rightarrow M = 2; d = 3\lambda/4$

(a) $2a_1 = 6 \Rightarrow a_1 = 3,\quad a_2 = 4,\quad a_3 = 1$ (from Pascal's Triangle)
Equation (6-63)

(b)
$$\begin{aligned} \text{AF} &= a_1 + a_2\cos(2u) + a_3\cos(4u) = 3 + 4\cos(2u) + \cos(4u) \\ &= 3 + 4[2\cos^2(u) - 1] + [2\cos^2(2u) - 1] \\ &= 3 + 8\cos^2 u - 4 + 2[2\cos^2(u) - 1]^2 - 1 \\ &= 3 + 8\cos^2 u - 4 + 8\cos^4 u - 8\cos^2 u + 2 - 1 \end{aligned}$$

$$\text{AF} = 8\cos^4 u,\quad u = \frac{\pi}{\lambda} d\cos\theta = \frac{3\pi}{4}\cos\theta \Rightarrow \text{AF} = 8\cos^4[(3\pi/4)\cos\theta]$$

(c) $\text{AF} = 8\cos^4\left(\frac{3\pi}{4}\cos\theta_n\right) = 0 \Rightarrow \frac{3\pi}{4}\cos\theta_n = \cos^{-1}(0) = \pm\frac{n\pi}{2}$,
$n = 1, 3, 5, \ldots$

$$\theta_n = \cos^{-1}\left(\pm\frac{2n}{3}\right),\quad n = 1, 3, 5, \ldots$$

$\underline{n = 1}$: $\theta_1 = \cos^{-1}(\pm 2/3) = \begin{cases} 48.1897° \\ 131.8103° \end{cases}$

$\underline{n = 3}$: $\theta_3 = \cos^{-1}(\pm 2) =$ does not exist

(d) $\text{AF} = 8\cos^4\left(\frac{3\pi}{4}\cos\theta_m\right)\Big|_{\max} = 8 \Rightarrow \cos^4\left(\frac{3\pi}{4}\cos\theta_m\right) = 1$

$$\Rightarrow \frac{3\pi}{4}\cos\theta_m = \cos^{-1}(\pm 1) = \pm m\pi, m = 0, 1, 2, \ldots$$

$\theta_m = \cos^{-1}(\pm 4m/3)$

$\underline{m = 0}$: $\theta_0 = \cos^{-1}(0) = 90°$

$\underline{m = 1}$: $\theta_1 = \cos^{-1}(\pm 4/3) =$ does not exist

6-43. (a) $d = \lambda/2$

(b)
1 4 6 4 1
$2a_1 = 6 \Rightarrow a_1 = 3$
$a_2 = 4$
$a_3 = 1$

$$\begin{aligned} \text{AF} &= 3 + 4\cos(2u) + \cos(4u) = 3 + 4[2\cos^2(u) - 1] + 2\cos^2(2u) - 1 \\ &= 3 + 4[2\cos^2(u) - 1] + 2[2\cos^2(u) - 1]^2 - 1 \end{aligned}$$

$$\text{AF} = 8\cos^2 u = 0 = 8\cos^2\left(\frac{\pi d}{\lambda}\right) \Rightarrow \frac{\pi d}{\lambda} = \frac{\pi}{2} \Rightarrow d = \lambda/2$$

(c) $N = 5$

$$D_0 = \frac{(10-2)(10-4)(10-6)2}{(10-3)(10-5)(10-7)1} = \frac{8(6)(4)2}{7(5)(3)1} = 3.6571 = 10\log_{10}(3.6571)$$
$$= 5.6314 \text{ dB}$$

$$D_0 \simeq 1.77\sqrt{N} = 1.77\sqrt{5} = 3.9578 = 10\log_{10}(3.9578) = 5.9745 \text{ dB}$$

(d) $u = \pi\frac{d}{\lambda}\cos\theta = \pi\frac{\lambda/2}{\lambda}\cos\theta = \frac{\pi}{2}\cos\theta$

$$\text{AF} = 8\cos^2 u = 8\cos^2\left(\frac{\pi}{2}\cos\theta\right)$$

$$\text{HPBW} = \frac{1.06}{\sqrt{N-1}} = \frac{1.06}{\sqrt{5-1}} = \frac{1.06}{2} = 0.53 = 30.37°$$

6-44. Binomial, $d = \lambda/2$

$$\Theta_{3\text{dB}} = 15.18° = 0.26494 \text{ radians}$$

(a) $\text{HPBW} = \frac{1.06}{\sqrt{N-1}} \Rightarrow \sqrt{N-1} = \frac{1.06}{\text{HPBW}} = \frac{1.06}{0.26494} = 4$

$\sqrt{N-1} = 4 \Rightarrow N - 1 = 16 \Rightarrow N = 17$

$N = 17$

(b) $D_0 \simeq 1.77\sqrt{N} = 1.77\sqrt{17} = 1.77(4.123) = 7.298$

$D_0 \simeq 7.298 = 8.632$ dB

(c) No sidelobes formed

Because $d = \lambda/2 \Rightarrow$ sidelobe level $= -\infty$ dB

6-45. Binomial

(a) $d = \lambda/2$

(b) $\text{HPBW}(d = \lambda/2) = \frac{1.06}{\sqrt{N-1}} = \frac{\pi}{10} = 18° \Rightarrow \sqrt{N-1} = 1.06(10)/\pi$

$$\Rightarrow N = 1 + (10.6/\pi)^2$$

$N = 1 + (3.374)^2 = 1 + 11.3844 = 12.3844 \Rightarrow \boxed{N \simeq 12}$ or $\boxed{N \simeq 13}$

(c) For $N = 12$:

$$D_0 = \frac{(2N-2)(2N-4)\cdots\cdots 2}{(2N-3)(2N-5)\cdots 1} = \frac{22\cdot 20\cdot 18\cdot 16\cdot 14\cdot 12\cdot 10\cdot 8\cdot 6\cdot 4\cdot 2}{21\cdot 19\cdot 17\cdot 15\cdot 13\cdot 11\cdot 9\cdot 7\cdot 5\cdot 3\cdot 1}$$
$$= 5.9457 = 7.742 \text{ dB}$$

or

$$D_0 \simeq 1.77\sqrt{N} = 1.77\sqrt{12} = 1.77(3.464) = 6.13 = 7.876 \text{ dB}$$

$$D_0(\text{uniform}) = 2N\left(\frac{d}{\lambda}\right) = 2(12)\left(\tfrac{1}{2}\right) = 12 = 10.792 \text{ dB}$$

D_0 of binomial is smaller, because of larger beamwidth, by

$$10.792 - 7.742 = 3.05 \simeq 3 \text{ dB}$$

6-46. $N = 3 = 2M + 1 \Rightarrow M = 1, d = \lambda/2 \Rightarrow$ Binomial design, $u = \frac{\pi}{\lambda}\, d$

(a) $2a_1 = 2 \Rightarrow a_1 = 1, a_2 = 1$

(b) $$(\text{AF})_3 = \sum_{n=1}^{M+1=2} a_n \cos[2(n-1)u] = a_1 + a_2\cos(2\mu) = 1 + \cos(2u)$$

$$= 2\cos^2(u)$$

$$u = \frac{\pi\, d}{\lambda}\cos\theta = \frac{\pi}{2}\cos\theta$$

$$\text{HPBW} = \frac{1.06}{\sqrt{N-1}} = \frac{1.06}{\sqrt{3-1}} = \frac{1.06}{\sqrt{2}} = 0.74953 = 42.945^\circ$$

(c) $$D_0 = \frac{(2N-2)(2N-4)\cdots 2}{(2N-3)(2N-5)\cdots 1} = \frac{4\cdot 2}{3\cdot 1} = \frac{8}{3} = 2.667 = 4.26 \text{ dB}$$

6-47. $R_0 = 20$ dB $\Rightarrow R_0$ (voltage Ratio) $= 10^1 = 10$

$$z_0 = \tfrac{1}{2}[(10 + \sqrt{10^2 - 1})^{1/2} + (10 - \sqrt{10^2 - 1})^{1/2}] = 2.3452$$

a. The array factor can be written as

$$(\text{AF})_3 = \sum_{n=1}^{2} a_n \cos[2(n-1)u] = a_1 + a_2\cos(2u) = a_1 + a_2[2\cos^2 u - 1]$$

$$(\text{AF})_3 = (a_1 - a_2) + 2a_2\cos^2 u = (a_1 - a_2) + 2a_2\cos^2\left(\frac{\pi\, d}{\lambda}\cos\theta\right)$$

letting $\cos u = z/z_0$, and equating the array factor to the Tschebyscheff polynomial of order 2, we obtain

$$(a_1 - a_2) + 2a_2\left(\frac{z}{z_0}\right)^2 = -1 + 2z^2 \Rightarrow \frac{2a_2}{z_0^2} = 2 \Rightarrow a_2 = z_0^2$$

$$= (2.3452)^2 = 5.5$$

Therefore

$$\left.\begin{matrix} a_1 = 4.5 \\ a_2 = 5.5 \end{matrix}\right\} \text{ or normalized } \left\{\begin{matrix} a_{1n} = 4.5/5.5 = 0.818 \\ a_{2n} = 5.5/5.5 = 1.0 \end{matrix}\right.$$

$((a_1 - a_2) = -1 \Rightarrow a_1 = a_2 - 1 = 4.5)$