Transmission Through a Conducting Screen Perforated Periodically with Apertures

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Abstract—A general solution to the problem of determining first the aperture field distribution and then the transmission and reflection coefficients of an infinite planar conducting sheet perforated periodically with apertures has been formulated. The excitation is considered to be a plane wave incident at any arbitrary angle. The aperture dimensions and array element spacings were assumed to be comparable with the wavelength of the incident electromagnetic field. The solution given can include the effect of a dielectric slab used to support the thin conducting sheet.

The solution is obtained by matching the tangential field components at the surface of the screen. The resulting integral equation is solved by the method of moments which reduces the integral equation to a system of linear algebraic equations that can be solved with the use of a digital computer. Accurate results for both the magnitude and phase of the aperture field distribution and the transmission coefficients for the propagating modes are determined explicitly for a specific example of slots arranged in an equilateral triangular lattice. The balance of power flow between the reflected and the transmitted waves has been checked with satisfactory results. The solution can be applied to the problem of scattering from a conducting screen with periodic apertures and to the complementary problem of scattering from a set of conducting plates by the use of Babinet's principle.

Introduction

MEASUREMENTS [1] have demonstrated that a thin conducting metal screen perforated with slots has bandpass bandstop characteristics when illuminated by an incident electromagnetic wave of variable frequency. This property makes these structures useful for many applications such as microwave filters, bandpass radomes, artificial dielectric and antenna reflectors, or ground planes.

The problem of scattering by a conducting thin screen perforated periodically with apertures has been investigated by Kieburtz and Ishimaru [2] with the help of a variational approach. The success or failure of a variational solution depends on the ability to chose an appropriate trial function for the aperture field distribution. A method of solution is established directly for the aperture field distribution; hence the problems of reflection, transmission, and the aperture susceptance can be treated in a straightforward manner. The method is quite general and applies to rectangular or circular apertures distributed periodically along any two coordinates. All aperture dimensions and the array element spacings are assumed to be comparable with the wavelength of the electromagnetic field. The formulation includes the presence of a dielectric slab which supports the thin conducting screen. The complementary problem of arrays of conducting plates can also be handled by the use of Babinet's principle.

The basic procedure is to expand the unknown electric field distribution near the conducting screen in a set of Floquet mode functions and relate the unknown magnetic fields on the two sides of the screen to the corresponding modal admittances in these two regions. By appropriately matching the tangential field components at the screen, an integral equation is obtained for the unknown electric field in each aperture. To simplify the computations, the unknown aperture field distribution is then expanded into a new set of functions that are orthogonal over the aperture itself. Hence this expansion should yield a faster convergence than the Floquet mode expansion. By the method of moments [3], the integral equation is reduced to a set of linear algebraic equations which can be solved with the use of a digital computer.

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Modal Formulations

Consider an electromagnetic wave to be incident on a thin conducting screen perforated with apertures distributed periodically along any two skewed coordinates S_1 and S_2 at an angle α as shown in Fig. 1. θ is the angle between the propagation vector $\bar{k}$ and the normal to the plane of the screen, and ϕ is the angle between the x axis and the projection of $\bar{k}$ on the x–y plane. It is assumed that the thickness of the conducting screen is negligible compared to a wavelength and that the apertures are all identical. The screen is backed with a dielectric sheet of thickness h.

The electromagnetic fields near the screen must satisfy the periodicity requirements imposed by Floquet's theorem; thus the scalar mode potential with the exp $(j\omega t)$ time dependence omitted can be written as [4]

$$\psi_{pq} = \exp\left(-j(u_{pq}x + v_{pq}y + \gamma_{pq}z)\right) \tag{1}$$

where

$$u_{pq} = k \sin\theta \cos\phi + \frac{2\pi p}{d_x} \tag{2}$$

$$v_{pq} = k \sin\theta \sin\phi + \frac{2\pi q}{d_y} - \frac{2\pi p}{d_x \tan\alpha}, \tag{3}$$

$$\text{for } p, q = 0, \pm 1, \pm 2, \cdots, \pm \infty$$

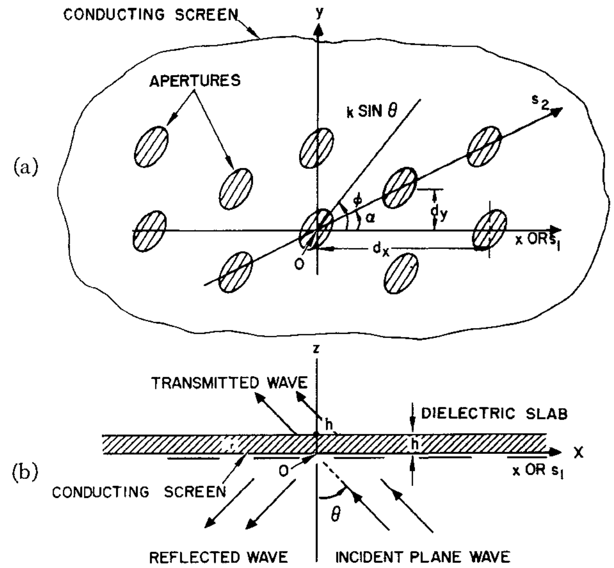

Fig. 1. Geometry of thin conducting screen perforated periodically with apertures.

$$\gamma_{pq} = \sqrt{k^2 - t_{pq}^2}, \qquad \text{for } k^2 > t_{pq}^2 \tag{4a}$$

$$= -j\sqrt{k^2 - t_{pq}^2}, \qquad \text{for } k^2 < t_{pq}^2 \tag{4b}$$

with

$$t_{pq}^2 = u_{pq}^2 + v_{pq}^2.$$

The modal propagation constant γ_{pq} is positive real for the propagating modes and is negative imaginary for evanescent modes. For the case of an isosceles triangular array when the element spacings and the angle of incidence satisfy

$$\left(\frac{\lambda}{d_x}\right)^2 + \left(\frac{\lambda}{2d_y}\right)^2 \geq (1 + \sin\theta)^2$$

and

$$2\left(\frac{\lambda}{d_x}\right), \quad \left(\frac{\lambda}{d_y}\right) \geq (1 + \sin\theta).$$

The distant scattered field only consists of the TE and TM modes with $p=0$, $q=0$; otherwise the higher modes may also exist. The vector orthonormal mode functions for the transverse electric field can be derived from the scalar potentials (1). The resultant TE and TM vector mode functions, transverse with respect to the z axis, are

$$\bar{\Phi}_{pq}^{\mathrm{TE}} = \frac{1}{\sqrt{d_x d_y}}\left(\frac{v_{pq}}{t_{pq}}\hat{x} - \frac{u_{pq}}{t_{pq}}\hat{y}\right)\psi_{pq}, \quad \text{for TE modes} \tag{5}$$

$$\bar{\Phi}_{pq}^{\mathrm{TM}} = \frac{1}{\sqrt{d_x d_y}}\left(\frac{u_{pq}}{t_{pq}}\hat{x} + \frac{v_{pq}}{t_{pq}}\hat{y}\right)\psi_{pq}, \quad \text{for TM modes.} \tag{6}$$

The transverse electric and magnetic fields are related by the modal admittances

$$\xi_{pq}^{\mathrm{TE}} = \frac{\gamma_{pq}}{k}\sqrt{\frac{\epsilon_0}{\mu_0}}, \qquad \text{TE modes} \tag{7}$$

$$\xi_{pq}^{\mathrm{TM}} = \frac{k}{\gamma_{pq}}\sqrt{\frac{\epsilon_0}{\mu_0}}, \qquad \text{TM modes.} \tag{8}$$

The modal admittance for TE and TM waves looking into the dielectric region from $z=+0$ plane can be obtained by the transmission line formula which is

$$Y_{pq} = \xi_{pq}^d \frac{\xi_{pq} + j\xi_{pq}^d \tan(\gamma_{pq}^d h)}{\xi_{pq}^d + j\xi_{pq}\tan(\gamma_{pq}^d h)}. \tag{9}$$

The function ξ_{pq}^d designates the modal admittance in the dielectric region and has the same form as in (7) and (8) for the corresponding TE and TM waves, except that the dielectric constant ϵ_0 should be replaced by $\epsilon_0\epsilon_r$. For convenience of notation, a third subscript r, equal to 1 or 2, will be used to designate the TE and TM modes, respectively.

It is known that a homogenous electromagnetic wave can always be decomposed into a combination of two plane waves with either the E or the H field perpendicular to the plane of incidence that correspond to the TE and TM Floquet modes with $p=0$, $q=0$. A plane wave with unit electric field intensity, incident in the ϕ plane at an angle θ to the array normal, has the reflection coefficients R_{pqr} and the transmission coefficients B_{pqr} for the corresponding modes $\bar{\Phi}_{pqr}$ in both regions. Matching the tangential field components at the $z=0$ plane yields

$$\begin{aligned}\bar{E}_t &= \sum_{r=1}^{2} A_{00r}\bar{\Phi}_{00r} + \sum_p\sum_q\sum_{r=1}^{2} R_{pqr}\bar{\Phi}_{pqr} \\ &= \sum_p\sum_q\sum_{r=1}^{2} B_{pqr}\bar{\Phi}_{pqr}, \quad \text{in each aperture}\end{aligned} \tag{10a}$$

$$\bar{E}_t = 0, \qquad \text{over the conductor in a unit cell} \tag{10b}$$

$$\begin{aligned}-z \times \bar{H}_t &= \sum_{r=1}^{2} A_{00r}\xi_{00r}\bar{\Phi}_{00r} - \sum_p\sum_q\sum_{r=1}^{2} R_{pqr}\xi_{pqr}\bar{\Phi}_{pqr} \\ &= \sum_p\sum_q\sum_{r=1}^{2} B_{pqr}Y_{pqr}\bar{\Phi}_{pqr},\end{aligned} \tag{11}$$

over each aperture.

The reflection coefficients R_{pqr} and the transmission coefficients B_{pqr} can be obtained from (10a) as

$$R_{pqr} = B_{pqr} = \iint_{\text{aperture}} \bar{E}_t \cdot \bar{\Phi}_{pqr}^* \, da \tag{12a}$$

for all p, q, and r with the exception of the incident terms for which p, q are both equal to zero. In this case

$$A_{00r} + R_{00r} = B_{00r}, \qquad \text{for } r = 1 \text{ or } 2. \tag{12b}$$

Substituting (12a) into (11), we obtain an integral equation which must be satisfied over the aperture only:

$$2 \sum_{r=1}^{2} A_{00r}\xi_{00r}\overline{\Phi}_{00r} = \sum_p \sum_q \sum_{r=1}^{2} (\xi_{pqr} + Y_{pqr})\overline{\Phi}_{pqr} \iint_{\text{aperture}} \overline{E}_t \cdot \overline{\Phi}_{pqr}{}^* \, da. \tag{13}$$

To solve the above integral equation for the unknown aperture electric field distribution, the electric field in the aperture will be expanded in another set of complete orthonormal functions $\overline{\Psi}_{mnl}$ that span the space of the aperture itself. The new set of orthonormal functions is chosen to satisfy the aperture boundary condition and thus provides a faster convergence to the aperture field than the Floquet mode expression of (10a)

$$\overline{E}_t = \sum_m \sum_n \sum_{l=1}^{2} F_{mnl}\overline{\Psi}_{mnl}. \tag{14}$$

Multiplying both sides of (13) by the complex conjugate of $\overline{\Psi}_{mnl}$ and integrating over the aperture, we obtain

$$2 \sum_{r=1}^{2} A_{00r}\xi_{00r}C_{00r}{}^{*MNL} = \sum_p \sum_q \sum_{r=1}^{2} (\xi_{pqr} + Y_{pqr})C_{pqr}{}^{*MNL} \iint_{\text{aperture}} \overline{E}_t \cdot \overline{\Phi}_{pqr}{}^* \, da \tag{15}$$

where

$$C_{pqr}{}^{MNL} = \iint_{\text{aperture}} \overline{\Psi}_{MNL} \cdot \overline{\Phi}_{pqr}{}^* \, da \tag{16}$$

and the asterisk designates the complex conjugate. The integral equation (15) generates a system of linear algebraic equations with the mode coefficients F_{mnl} as unknowns, which can be written in the matrix form

$$[Y_{MNL}{}^{mnl}][F_{mnl}] = 2[I_{mnl}] \tag{17}$$

where $[Y_{MNL}{}^{mnl}]$ is a square admittance matrix, the row index is designated by the subscript M, N, L, and the column index by the superscript m, n, l. The matrix elements are given by

$$Y_{MNL}{}^{mnl} = \sum_p \sum_q \sum_{r=1}^{2} (\xi_{pqr} + Y_{pqr})C_{pqr}{}^{*MNL}C_{pqr}{}^{mnl} \tag{18}$$

$$I_{mnl} = \sum_{r=1}^{2} A_{00r}\xi_{00r}C_{00r}{}^{*mnl}. \tag{19}$$

Equation (17) is the expression for an admittance matrix for a multiterminal network where $[Y_{MNL}{}^{mnl}]$ is the admittance matrix, F_{mnl} is the terminal voltage, I_{mnl} is the equivalent current source, and $C_{pqr}{}^{mnl}$ is the coupling coefficient between two different types of modes. The accuracy of the mode coefficient F_{mnl} obtained from (17) depends upon the number of modes used to approximate the aperture electric field in both (10a) and (14). After the field distribution over the aperture is determined, the distant reflected and the transmitted fields can be calculated in a straightforward manner from (12a).

Transmission Through Rectangular Apertures

In this section (12a), (14), and (17) are employed to calculate the reflection from, or transmission through, a perfectly conducting screen perforated with rectangular apertures. An equilateral grid arrangement was used for the layout of the apertures. When the origin of the rectangular coordinates is at the center of an aperture and each rectangular aperture has the dimensions a and b parallel to the x and y axis, the set of orthonormal mode functions can be written as

$$\overline{\Psi}_{mn}{}^{\text{TE}} = g\left[\frac{n\pi}{b} e_{mnx}\hat{x} - \frac{m\pi}{a} e_{mny}\hat{y}\right] \tag{20}$$

$$\overline{\Psi}_{mn}{}^{\text{TM}} = g\left[\frac{m\pi}{a} e_{mnx}\hat{x} + \frac{n\pi}{b} e_{mny}\hat{y}\right] \tag{21}$$

and

$$g = \left(\frac{\epsilon_m\epsilon_n}{ab}\right)^{1/2}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{-1/2} \tag{22}$$

$$e_{mnx} = \cos\left(\frac{m\pi}{a}x - \frac{m\pi}{2}\right)\sin\left(\frac{n\pi}{b}y - \frac{n\pi}{2}\right) \tag{23}$$

$$e_{mny} = \sin\left(\frac{m\pi}{a}x - \frac{m\pi}{2}\right)\cos\left(\frac{n\pi}{b}y - \frac{n\pi}{2}\right) \tag{24}$$

where ϵ_m is the Neumann factor, $\epsilon_m = 1$ for $m = 0$, and $\epsilon_m = 2$ for $m \geq 1$.

A computation was made that considered the lowest possible 10 modes in (14) for rectangular apertures and about 200 Floquet modes whose transverse components of wave numbers t_{pq} were less than 11 times the free-space wave number $k = 2\pi/\lambda$ in (10a). For the case of narrow slots with a rectangular grid arrangement, the numerical results are in excellent agreement with the values measured by Ott [1]. They also agree with the corresponding solutions given by Babinet's principle for the complementary problem of scattering by conducting plates [5].

Since only one propagating beam is permitted for the case considered here, the distant reflected and transmitted waves are confined to the $\overline{\Phi}_{001}$ and $\overline{\Phi}_{002}$ modes only. Shown in Fig. 2 is the variation of the transmission coefficient as a function of frequency when the screen is illuminated by a normally incident plane wave with

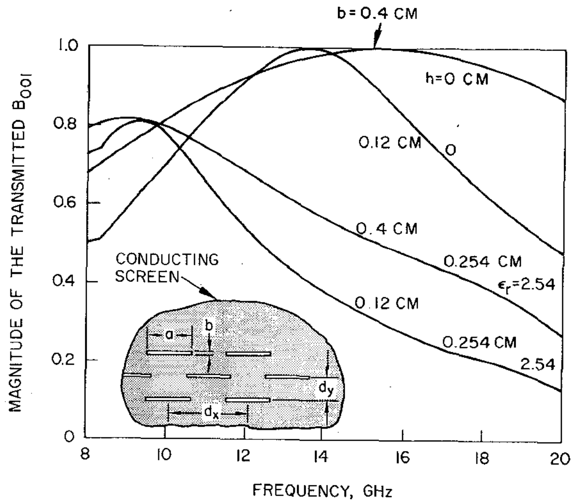

Fig. 2. Transmission coefficient of plane wave with E field parallel to the y axis at normal angle of incidence ($d_x = 2.0$ cm, $d_y = 0.577$ cm, $\alpha = 30°$, $a = 1.2$ cm).

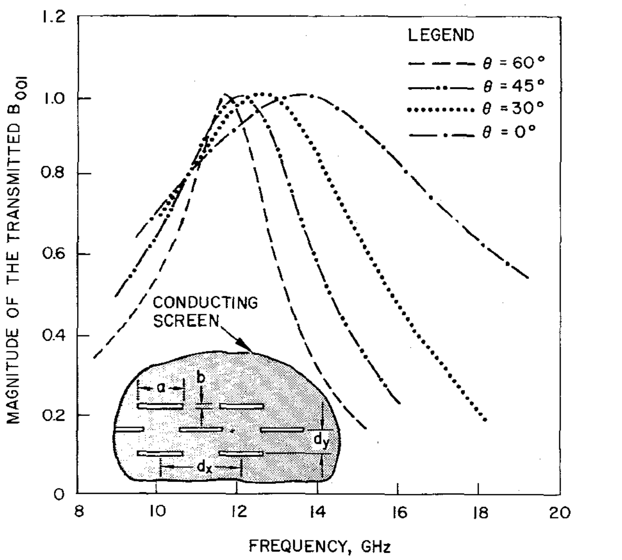

Fig. 3. Transmission coefficient of incident plane wave with E field perpendicular to the x–z plane ($d_x = 2.0$ cm, $d_y = 0.577$ cm, $\alpha = 30°$, $a = 1.2$ cm, $b = 0.12$ cm).

E field perpendicular to the x axis. The bandwidth increases with slot width. The resonance frequency, where the transmission coefficient is purely real, also increases with the slot width instead of decreasing. For the case where the screen is loaded with a dielectric slab, the resonance frequency decreases as expected.

The transmission coefficient of a plane wave with E field perpendicular to the x–z plane, incident at different angles of θ is shown in Fig. 3. The resonant frequency decreases with the increase in the angle of incidence. The graph in Fig. 4 shows the transmission coefficient of a plane wave with H field perpendicular to the y–z plane, incident at different angles of θ. It is seen that the resonant frequency is insensitive to the angle of incidence in this plane and the bandwidth increases as the angle of incidence increases. This phenomenon is different from the case shown in Fig. 3 in which the bandwidth decreased instead of increasing. The dips in the higher frequency region are due to the forced resonances that occur just prior to the onset of grating lobes. Beyond these frequencies where the dips occur both reflected and transmitted waves may propagate as multiple beams in the free space.

In the previous cases the slot arrays are all illuminated by a plane wave incident in either x–z or y–z planes. Because of the symmetry in the geometry the reflected and transmitted waves both have the same polarization as that of the incident wave. In Table I the three most significant mode coefficients F_{mnl} in (14) for the cases of beam scan in the x–z and y–z planes are given. The other mode coefficients are not listed because they are much smaller than these for the cases considered here. Nevertheless, it is seen that the aperture field distribution is dominated by the TE_{10} mode. This phenomenon does not hold for the case when a plane wave is incident in a ϕ plane that does not have geometrical symmetry or for

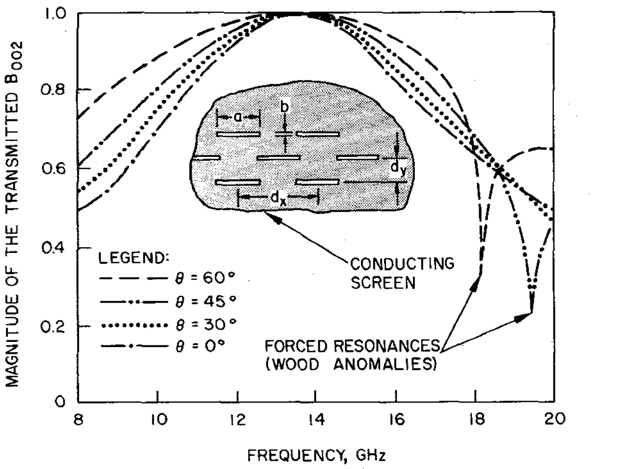

Fig. 4. Transmission coefficient of incident plane wave with H field perpendicular to y–z plane ($d_x = 2.0$ cm, $d_y = 0.577$ cm, $\alpha = 30°$, $a = 1.2$ cm, $b = 0.12$ cm).

the case when the electrical length of the slots are far from the resonant frequency of the TE_{10} mode. In general, 10 waveguide modes appear to be sufficient to give a good approximation to the aperture field distribution if the slot length is less than one wavelength. Illustrated in Table II is the type of convergence for the transmission coefficient B_{001} that one can expect when various numbers of Floquet modes in (10a) are considered in the computation.

The graph in Fig. 5 shows the transmitted TE- and TM-mode coefficients for an incident TE plane wave (E field perpendicular to the plane of incidence). When the plane of incidence is the plane with geometrical symmetry, the reflected or transmitted wave has no TM-mode component. In general, as the angle of incidence θ increases, the magnitudes of both the TE and TM transmission coefficients decrease.

TABLE I

Three Most Significant Mode Coefficients of the Aperture Electric Field Distribution*

Angle of Incidence (degrees)	F_{101} (TE_{10} mode)	F_{201} (TE_{20} mode)	F_{301} (TE_{30} mode)
Incident Plane Wave with E Field Perpendicular to x–z Plane			
$\theta=1$, $\phi=0$	$3.103-j0.0074$	$0.0000+j0.0071$	$0.1255-j0.0003$
$\theta=31$, $\phi=0$	$2.835-j1.315$	$0.0856+j0.1846$	$0.1462-j0.0678$
$\theta=61$, $\phi=0$	$0.6335-j1.514$	$0.1553+j0.0650$	$0.0441-j0.1054$
Incident Plane Wave with H Field Perpendicular to y–z Plane			
Angle of Incidence (degrees)	F_{101} (TE_{10} mode)	F_{301} (TE_{30} mode)	F_{112} (TM_{11} mode)
$\theta=1$, $\phi=90$	$3.102-j0.0060$	$0.1254-j0.0002$	$0.0000+j0.0072$
$\theta=31$, $\phi=90$	$3.084-j0.0402$	$0.1221-j0.0016$	$0.0053+j0.4139$
$\theta=61$, $\phi=90$	$3.063-j0.0102$	$0.1273-j0.0004$	$0.0019+j0.5624$

* $d_x=2.0$ cm, $d_y=0.577$ cm, $\alpha=30°$, $a=1.2$ cm, $b=0.12$ cm, and $f=13.6$ GHz.

TABLE II

Convergence of the Transmission Coefficient B_{001} with an Incident Plane Wave with E Field Perpendicular to the x–z Plane Versus the Total Number of Modes Considered in (10)

Total Number of Modes	$\theta=1°$, $\phi=0°$	$\theta=31°$, $\phi=0°$	$\theta=61°$, $\phi=0°$
650	$0.999911\angle -0.76$	$0.909437\angle -24.57$	$0.397952\angle -66.55$
355	$0.999875\angle -0.90$	$0.906842\angle -24.93$	$0.392846\angle -66.87$
219	$0.999995\angle -0.13$	$0.907121\angle -24.89$	$0.386110\angle -67.29$
115	$0.999161\angle -2.35$	$0.901169\angle -25.68$	$0.389377\angle -67.08$
45	$0.998332\angle -3.31$	$0.927109\angle -22.01$	$0.456679\angle -62.87$

The transmitted TM- and TE-mode coefficients due to an incident TM plane wave (H field perpendicular to the plane of incidence) are given in Fig. 6. It is seen that when the plane of incidence is the y–z plane ($\phi=90°$), the wave is totally transmitted. In Fig. 6(b) it is interesting to see that the magnitude of the transmitted TE coefficient B_{001} may become greater than unity. This is possible because the Poynting vectors of a TE wave and a TM wave that have the same magnitude of mode coefficients are different by a $\cos^2\theta$ factor. Since only one propagating beam is permitted for all cases given herein,

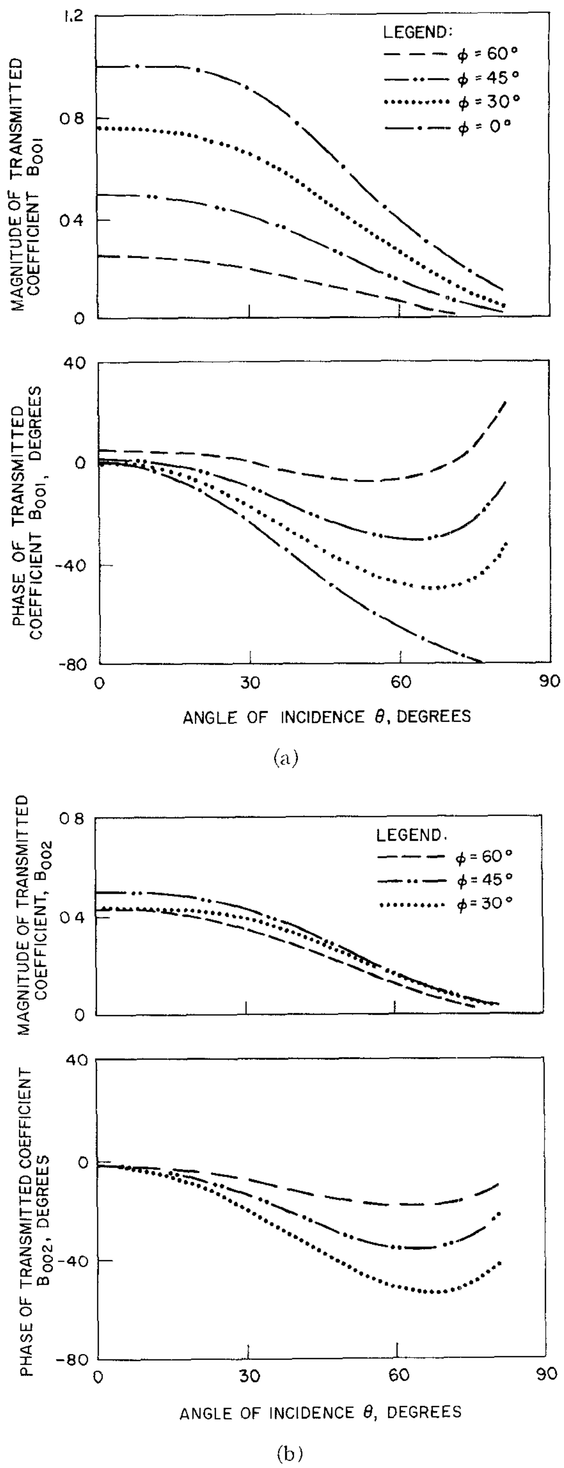

Fig. 5. Transmitted TE- and TM-mode coefficients due to incident TE plane wave ($d_x=2.0$ cm, $d_y=0.577$ cm, $\alpha=30°$, $a=1.2$ cm, $b=0.12$ cm, $f=13.6$ GHz. (a) Transmitted TE wave. (b) Transmitted TM wave.

the power flow of the reflected and transmitted waves must satisfy the law of energy conservation. Thus

$$\frac{\mathrm{Re}(Y_{001})}{\xi_{001}}\,|B_{001}|^2+\frac{\xi_{002}+\mathrm{Re}(Y_{002})}{\xi_{001}}\,|B_{002}|^2+|R_{001}|^2=1 \tag{25a}$$

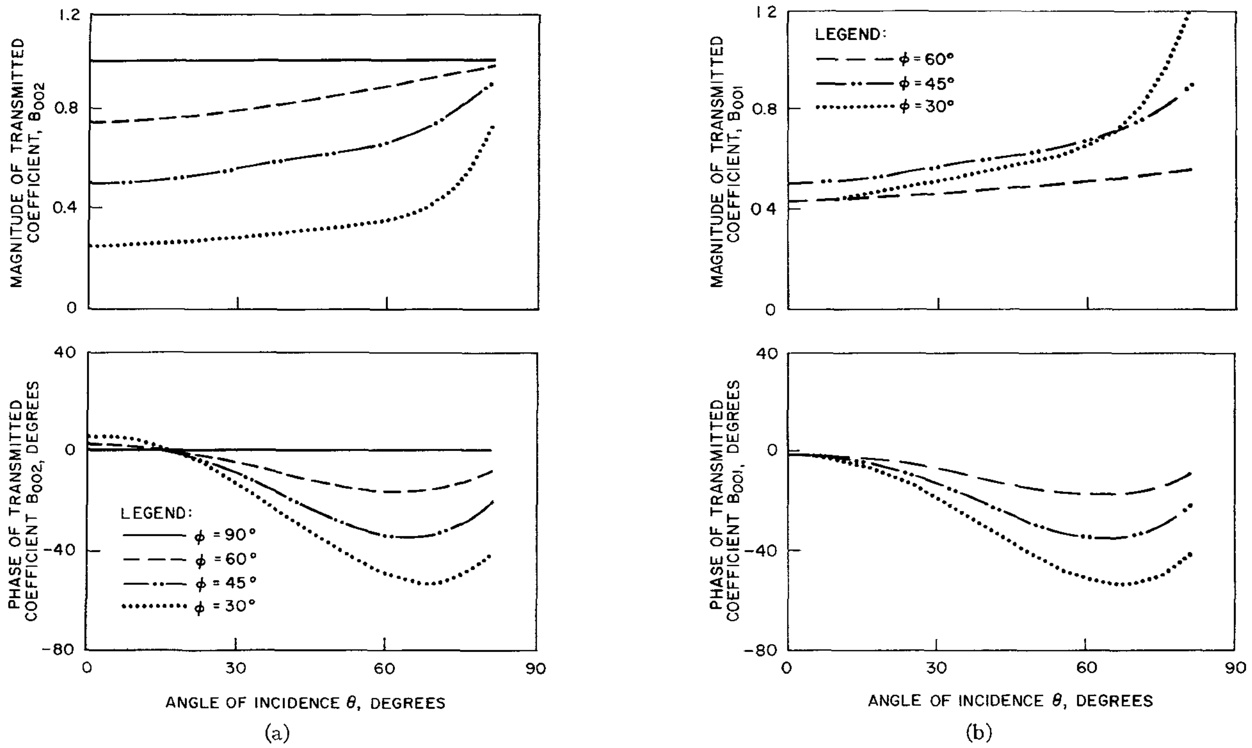

Fig. 6. Transmitted TM- and TE-mode coefficients due to incident TM plane wave ($d_x = 1.0$ cm, $d_y = 0.577$ cm, $\alpha = 30°$, $a = 1.2$ cm, $b = 0.12$ cm, $f = 13.6$ GHz. (a) Transmitted TM wave. (b) Transmitted TE wave.

for an incident TE wave, and

$$\frac{\mathrm{Re}(Y_{002})}{\xi_{002}} \mid B_{002} \mid^2 + \frac{\xi_{001} + \mathrm{Re}(Y_{001})}{\xi_{002}} \mid B_{001} \mid^2 + \mid R_{002} \mid^2 = 1 \qquad \text{(25b)}$$

for an incident TM wave. For example, assume a TM plane wave illuminates the screen from the direction $\theta = 81°$, $\phi = 30°$. From Fig. 6 the distant transmitted field has the components $B_{001} = 1.238\underline{/-41.5°}$, $B_{002} = 0.701\underline{/-41.0°}$, and therefore, the sum of the left-hand side of (25b) is 0.9977 which is very close to unity. All data presented here have been in excellent agreement with the conditions of (25).

From Figs. 5 and 6 it is also seen that at normal incidence the relation between the TE incident wave and the reflected TM wave or vice versa satisfies the reciprocity theorem.

Conclusion

A method of solution has been formulated to evaluate the aperture field distribution and the transmission coefficients of a plane wave incident upon a conducting screen with periodically spaced apertures. The screen may or may not be loaded with a dielectric slab. The aperture field distribution is expanded into Floquet modes to make it possible to distinguish the propagating modes from the evanescent modes conveniently. It is also expanded into a second set of modes that are orthogonal over the aperture and satisfy the aperture boundary condition itself. This step simplifies the computations involved in the evaluation of the aperture field distribution. The accuracy of the solution depends on the numbers of modes used in (10a) and (14) to approximate the aperture field distribution. Accurate numerical results for the aperture field distribution and the transmission coefficients have been obtained for the specific case of rectangular slots arranged in an equilateral triangular grid array. These results satisfy both the law of energy conservation and the reciprocity theorem.

The problem of transmission through a conducting screen with circular apertures can be solved with the methods presented in this paper. Furthermore, this formulation is also applicable to the complementary problems of reflection from or transmission through a set of conducting plates by the use of Babinet's principle.

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