Determination of Correct Values for Propagation Constant, Intrinsic Impedance and Refraction Index of Metamaterials

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***Abstract*- In this paper the correct values of propagation constant, intrinsic impedance and index of refraction of metamaterials are determined. These characteristic parameters are functions of complex constitutive parameters m and e, of which the real and imaginary parts may be positive or negative for DPS, DNG, ENG and MNG. The correct sign of square roots of the appropriate formulas are determined. It is shown that the real part of refraction index in ENG and MNG media may be both positive or negative, which is obtained by the sign of the imaginary part of propagation constant. The characteristic impedance is directly obtained by its formula.**

I. INTRODUCTION

Metamaterials are artificially realizable materials, of which the real part of at least one of the parameters m or e is negative. For common matter called double positive (DPS) material, the real parts of both m and e are positive. Epsilon negative (ENG) materials have only the real part of e negative. For mu negative (MNG) materials, only the real part of m is negative. For double negative (DNG) materials, the real parts of both m and e are negative [1]. The theoretical study of DNG materials were fist investigated by Veselago [2]. The method of fabrication of MNG, ENG and DNG metamaterial media were developed in [3], [4], and [5]-[6], respectively. The propagation of waves in metamaterial media is investigated by analytical and numerical methods in the literature [7], which requires the computation of correct values of propagation constant, intrinsic impedance and refraction index. In lossy materials the constitutive parameters m and e are complex. However, determination of the characteristic parameters of metamaterials requires computation of expressions $\pm\sqrt{me}$ and $\pm\sqrt{m/e}$, which may be positive or negative [8]. We intend to investigate the correct sign of these expressions, which lead to the evaluation of propagation constant, intrinsic impedance and refraction index of homogeneous, isotropic and linear metamaterials in the macroscopic view point.

II. SELECTION OF CORRECT SIGN OF SQUARE ROOTS

Consider a linear x polarized plane wave traveling along the z-direction

$$\begin{cases} \vec{E} = \hat{x}E_0 e^{-g_z z + jwt} \\ \vec{H} = \hat{y}H_0 e^{-g_z z + jwt} \end{cases} \tag{1}$$

The propagation constant, index of refraction and intrinsic impedance of the media are:

$$\begin{cases} g_z = a + jb = jw\sqrt{me} = jk_0 n, \\ \quad n = n' + jn'' = \sqrt{m_r e_r}, \end{cases} \tag{2}$$

$$Z_0 = E_0 / H_0 = g_z/(jwe) = (jwm)/g_z = \sqrt{m/e} = h_0\sqrt{m_r/e_r} \tag{3}$$

where $k_0 = w/c$, $c = 1/\sqrt{m_0 e_0}$, $h_0 = \sqrt{m_0/e_0}$, $e_r = e/e_0$ and $m_r = m/m_0$.

The average value of power is

$$\vec{S} = \frac{1}{2}\mathrm{Re}(\vec{E}\times\vec{H}^*) = \hat{z}\frac{1}{2}\left|\frac{E_0}{Z_0}\right|^2 \mathrm{Re}(Z_0)e^{-2(az)} \tag{4}$$

The propagation constant, intrinsic impedance and refraction index involve square roots, which may have two opposite signs. However, the correct sign should be selected, according to the physical principles. Since the time dependence is chosen as $\exp(+jwt)$ and the wave propagates in the positive z-direction, the sign selected for $g_Z = a + jb$ should be considered with the principle of conservation of power and attenuation of wave in the positive z-direction. Consequently, the sign of a should be positive [9]-[10], and we have from (2)

$$\begin{cases} a = -k_0 n'' \\ b = k_0 n' \end{cases} \tag{5}$$

Therefore, the correct sign for the square root amounts to the selection of $a > 0$ or equivalently $n'' < 0$.

Since $\vec{E} \perp \vec{H} \perp \vec{S}$ for a plane wave in a linear homogeneous and isotropic media and $\vec{S}$ is in the $\hat{z}$-direction, according to (4) the real part of Z_0 should be positive $(\mathrm{Re}(Z_0) > 0)$.

Consequently, this criterion may be used to select the correct sign for $\sqrt{m_r / e_r}$. This is obvious, because positive resistance indicates consumption of energy in an active medium. The criteria for selection of correct sign of square root is summarized below

$$a > 0\,,\ \mathrm{Re}(Z_0) > 0\,,\ n'' < 0 \tag{6}$$

III. SELECTION OF THE CORRECT SIGN FOR THE REAL AND IMAGINARY PARTS OF m AND e

It is proved that for the establishment of principle of causality, m and e should be dispersive [2], [11]. Since metamaterial media are realized by metallic rods or split ring resonators, they are necessarily lossy [3]-[4]. As for as the signs of real and imaginary parts of m and e are concerned, there are four cases for each of $e_r = \pm e_r' \pm j e_r''$ and $m_r = \pm m_r' \pm j m_r''$, which may lead to 16 cases for the wave propagation. Now, with the selection of time dependence as $\exp(+jwt)$, the imaginary parts of m and e should be negative, so that the wave attenuates in its direction of propagation in common and metamaterial media[12]-[14]. This condition leads to the fact that the dissipated power in the Poynting's theorem should be positive. For plane waves,

$$P_d = \frac{w}{4}\int_V (m'' \,|\vec{H}|^2 + e'' \,|\vec{E}|^2)\, dV \tag{7}$$

Consequently, for DPS, DNG, ENG and MNG at least we should have $e_r = \pm e_r' - j e_r''$ and $m_r = \pm m_r' - j m_r''$.

IV. CORRECTION OF CHARACTERISTIC PARAMETERS AS COMPUTED BY THEIR FORMULAS

Table I summarizes the correct values of characteristic parameters of lossy metamaterials derived from their values as computed by the corresponding formulas in (2) and (3) according to the criteria (6). The computed value of refraction index in polar form is $n_c = |n_c| \exp(jj_{nc})$. The imposition of conditions $a > 0$ or $n'' < 0$ leads to changing the sign of n_c in DNG medium and changing its sign only in the interval $45 < j_{nc} \le 90$ for the ENG and MNG media.

In the special case of lossless metamaterial media, Table II gives the correct values of characteristic parameters. The sign of correct values of g_Z, Z_0 and n are chosen according to criteria in (6). As it is shown in Table II, in lossless ENG and MNG metamaterial media, the real part of Z_0 is zero and therefore, the real part of $\vec{S}$ is also zero as indicated by (4). Consequently, the wave can not propagate and will be evanescent.

V. EXAMPLES

Two dispertion models namely Drud model or Lorentz (Resonance) model may be considered for e_r and m_r [2]-[6]. Another model similar to the Lorentz model is introduced for

TABLE I
CORRECT VALUES OF CHARACTERISTIC PARAMETERS Z_0 , g_Z AND n IN LOSSY MEDIA. COMPUTED VALUES ARE $Z_c = \sqrt{m/e}$, $g_c = jw\sqrt{me}$ AND $n_c = \sqrt{m_r e_r} = |n_c| \exp(jj_{nc})$.

	Z_0	g_Z	n
DPS	Z_c	g_c	n_c
DNG	Z_c	$-g_c$	$-n_c$
ENG	Z_c	$\begin{cases} -g_c & 45 < j_{nc} \le 90 \\ g_c & -90 \le j_{nc} < -45 \end{cases}$	$\begin{cases} -n_c & 45 < j_{nc} \le 90 \\ n_c & -90 \le j_{nc} < -45 \end{cases}$
MNG	Z_c	$\begin{cases} -g_c & 45 < j_{nc} \le 90 \\ g_c & -90 \le j_{nc} < -45 \end{cases}$	$\begin{cases} -n_c & 45 < j_{nc} \le 90 \\ n_c & -90 \le j_{nc} < -45 \end{cases}$

TABLE II
CORRECT VALUES OF CHARACTERISTIC PARAMETERS IN LOSSLESS MEDIA. $r = \sqrt{|m'/e'|}$ AND $s = \sqrt{|m_r' e_r'|}$.

	Correct value & Computer calculation	Z_0	g_Z	n
DPS	Correct value	$+r$	$+jws$	$+s$
	Computer calculation	$+r$	$+jws$	$+s$
DNG	Correct value	$+r$	$-jws$	$-s$
	Computer calculation	$+r$	$+jws$	$+s$
ENG	Correct value	$+jr$	$+ws$	$-js$
	Computer calculation	$+jr$	$-ws$	$+js$
MNG	Correct value	$-jr$	$+ws$	$-js$
	Computer calculation	$+jr$	$-ws$	$+js$

m_r [4]-[5]. Assuming time dependence as $\exp(+jwt)$, the Drude model for e_r and the Lorentz model for m_r are as follows:

$$e_r = 1 - \frac{w_{ep}^2}{w(w - j\Gamma_e)} \tag{8}$$

$$m_r = 1 - \frac{w_{mp}^2 - w_{m0}^2}{w^2 - w_{m0}^2 - j\Gamma_m w} \tag{9}$$

where w_{m0} is the angular frequency of magnetic resonance, w_{ep} and w_{mp} are the electric and magnetic plasma angular frequencies, respectively and Γ_e and Γ_m are the electric and magnetic damping factors. Four cases of DPS, DNG, ENG and MNG may be considered by the appropriate selection of parameters in the above dispersion models for e_r and m_r .

As an example of a lossy medium, the variation of e_r and m_r as a function of frequency are drawn in Fig. 1(a) with the Drude and Lorentz models, respectively, with parameters $\Gamma_e = \Gamma_m = 10^{10}\, rad/\sec$, $f_{ep} = 15GHz$, $f_{mp} = 18GHz$, and $f_{m0} = 12GHz$. In Figs. 1(b) to 1(d), the corrected values of propagation constant g_z, characteristic impedance Z_0 and refraction index n using Table I are drawn.

Metamaterials having negative real parts for the refraction index are called Negative Refraction Index (NRI) [15]-[16].

According to Fig. 1(d), DNG medium is NRI. However, ENG and MNG media could be both NRI or non-NRI. According to Fig. 1(b) and (5), an appropriate criteria for designating a medium as NRI, could be the sign of b . Media with $b < 0$ are NRI.

As an example of lossless media, the real part of e_r and m_r are drawn in Fig. 1(a) versus frequency by the Drude and Lorentz models, respectively. In Figs. 2(b) to 2(d), the values of g_z, Z_0 and n are drawn according to the corrections in Table II. The appropriate parameters are chosen as in the lossy example.

VI. Conclusion

Metamaterials should be necessarily lossy and dispersive to be causal and physically realizable. The imaginary parts of e and m should be negative, so that the conservation of energy could hold. By the assumption of time dependence as $\exp(+jwt)$ and $\exp(-g_z z + jwt) = \exp(-az)\exp(j(wt - bz))$ the real part of g_z namely a , should be chosen positive and the imaginary part of the refraction index namely n'' should be negative. Then the characteristic impedance may be computed by the appropriate relations in (3). Consequently, the characteristic parameters of metamaterials as computed by their appropriate formulas should be checked for their sign. The criteria of selection of sign of the square roots in lossy media are $a > 0$, $n'' < 0$ and $\mathrm{Re}(Z_0) > 0$. ENG and MNG media may or may not be NRI, where the sign of b could be used as a criterion.

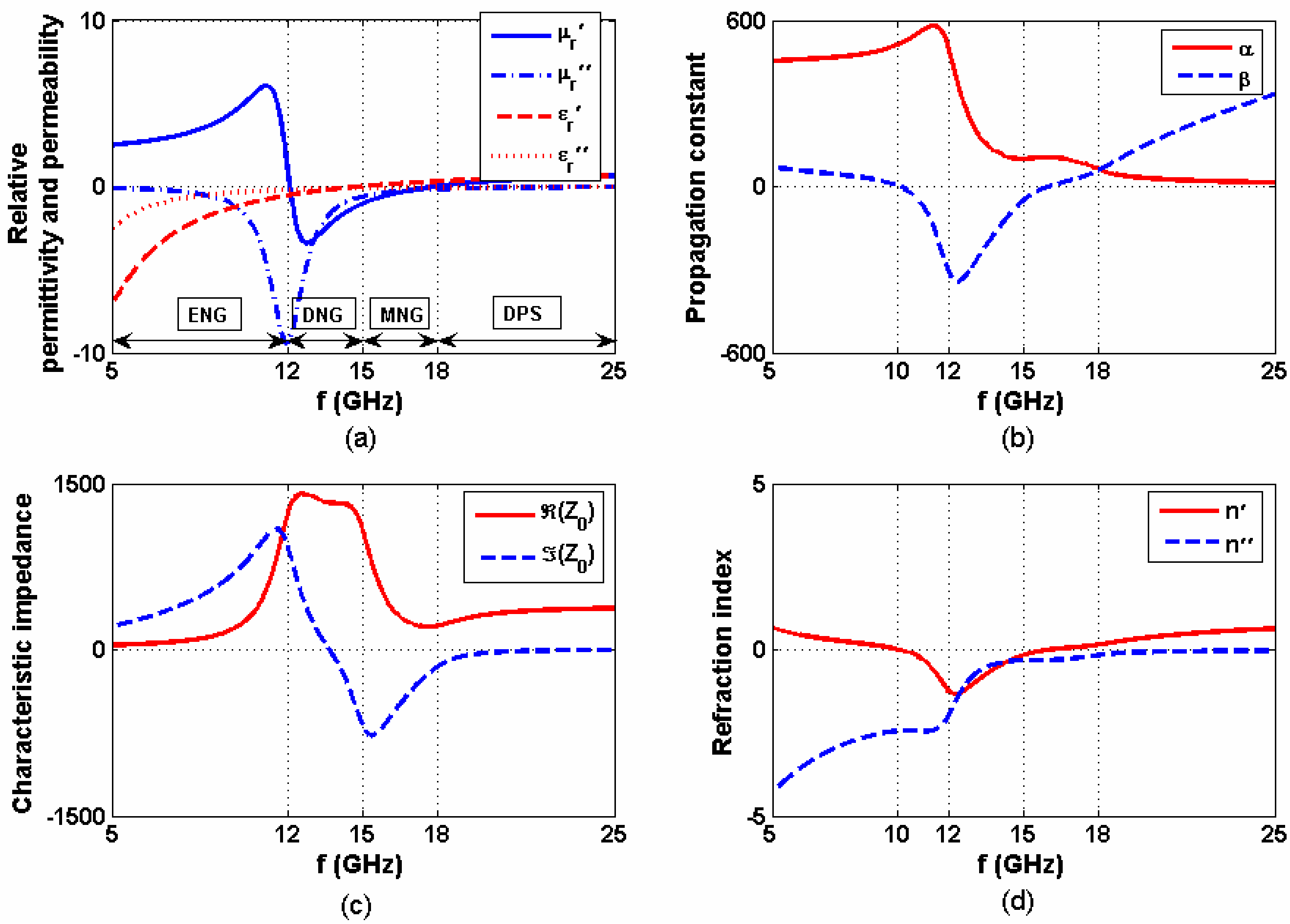

Figure 1. Corrected characteristic parameters in lossy DPS, DNG, ENG and MNG media. (a) e_r with Drude model and m_r with Lorentz model; (b) propagation constant g_z ; (c) characteristic impedance Z_0 ; (d) refraction index n .

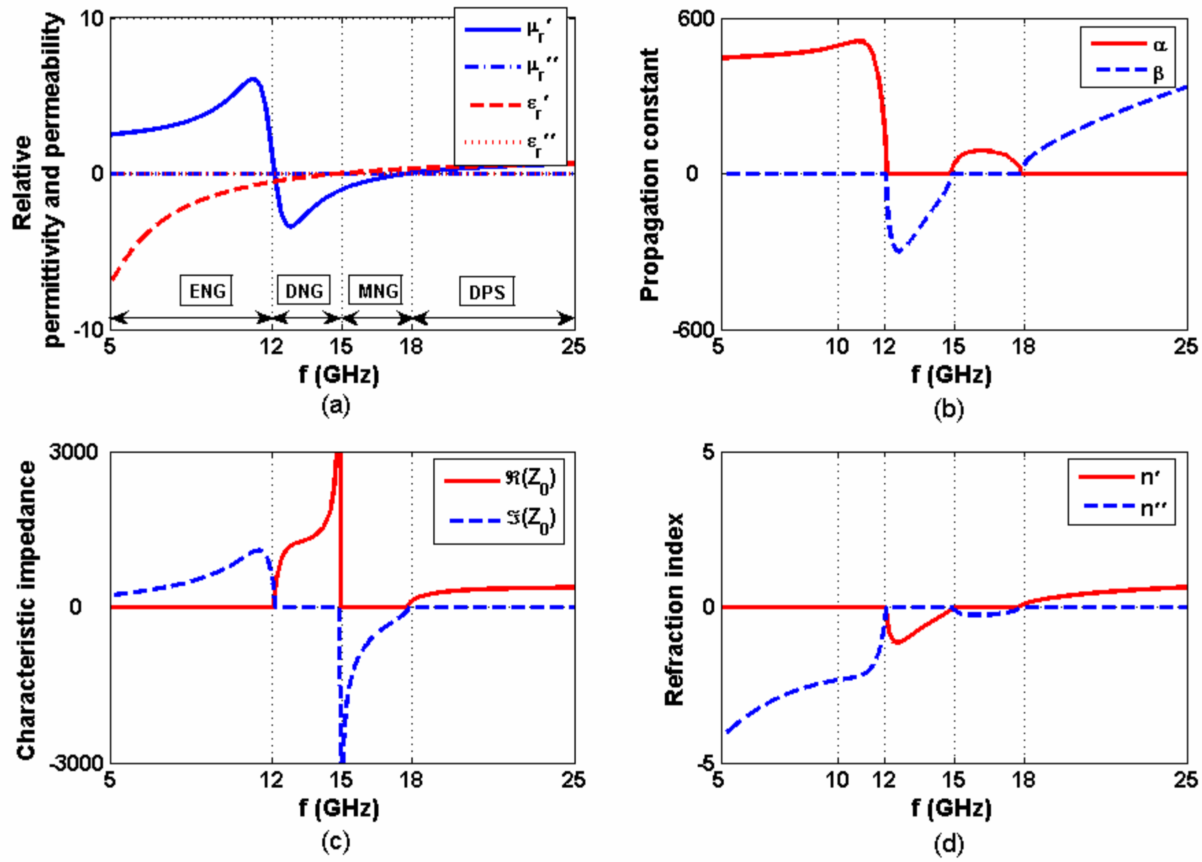

Figure 2. Corrected characteristic parameters in lossless DPS, DNG, ENG and MNG media. (a) e_r with Drude model and m_r with Lorentz model; (b) propagation constant g_z; (c) characteristic impedance Z_0; (d) refraction index n.

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