# CHAPTER 9

**9.1.** In Fig. 9.4, let $B = 0.2\cos 120\pi t$ T, and assume that the conductor joining the two ends of the resistor is perfect. It may be assumed that the magnetic field produced by $I(t)$ is negligible. Find:

a) $V_{ab}(t)$: Since $B$ is constant over the loop area, the flux is $\Phi = \pi(0.15)^2 B = 1.41 \times 10^{-2}\cos 120\pi t$ Wb. Now, $emf = V_{ba}(t) = -d\Phi/dt = (120\pi)(1.41 \times 10^{-2})\sin 120\pi t$. Then $V_{ab}(t) = -V_{ba}(t) = \underline{-5.33\sin 120\pi t \text{ V}}$.

b) $I(t) = V_{ba}(t)/R = 5.33\sin(120\pi t)/250 = \underline{21.3\sin(120\pi t) \text{ mA}}$

**9.2.** In the example described by Fig. 9.1, replace the constant magnetic flux density by the time-varying quantity $\mathbf{B} = B_0 \sin\omega t\ \mathbf{a}_z$. Assume that $\mathbf{v}$ is constant and that the displacement $y$ of the bar is zero at $t = 0$. Find the emf at any time, $t$.

The magnetic flux through the loop area is

$$\Phi_m = \int_s \mathbf{B}\cdot d\mathbf{S} = \int_0^{vt}\int_0^d B_0 \sin\omega t\ (\mathbf{a}_z \cdot \mathbf{a}_z)\, dx\, dy = B_0\, v\, t\, d \sin\omega t$$

Then the emf is

$$emf = \oint \mathbf{E}\cdot d\mathbf{L} = -\frac{d\Phi_m}{dt} = \underline{-B_0\, d\, v\ [\sin\omega t\ +\ \omega t\cos\omega t]}\ \text{V}$$

**9.3.** Given $\mathbf{H} = 300\,\mathbf{a}_z\ \cos(3\times 10^8 t - y)$ A/m in free space, find the emf developed in the general $\mathbf{a}_\phi$ direction about the closed path having corners at

a) (0,0,0), (1,0,0), (1,1,0), and (0,1,0): The magnetic flux will be:

$$\Phi = \int_0^1\int_0^1 300\mu_0 \cos(3\times 10^8 t - y)\, dx\, dy = 300\mu_0 \sin(3\times 10^8 t - y)|_0^1$$
$$= 300\mu_0 \left[\sin(3\times 10^8 t - 1) - \sin(3\times 10^8 t)\right]\ \text{Wb}$$

Then

$$\text{emf} = -\frac{d\Phi}{dt} = -300(3\times 10^8)(4\pi\times 10^{-7})\left[\cos(3\times 10^8 t - 1) - \cos(3\times 10^8 t)\right]$$
$$= \underline{-1.13\times 10^5\left[\cos(3\times 10^8 t - 1) - \cos(3\times 10^8 t)\right]\ \text{V}}$$

b) corners at (0,0,0), (2$\pi$,0,0), (2$\pi$,2$\pi$,0), (0,2$\pi$,0): In this case, the flux is

$$\Phi = 2\pi\times 300\mu_0 \sin(3\times 10^8 t - y)|_0^{2\pi} = 0$$

The emf is therefore $\underline{0}$.

**9.4.** A rectangular loop of wire containing a high-resistance voltmeter has corners initially at $(a/2, b/2, 0)$, $(-a/2, b/2, 0)$, $(-a/2, -b/2, 0)$, and $(a/2, -b/2, 0)$. The loop begins to rotate about the $x$ axis at constant angular velocity $\omega$, with the first-named corner moving in the $\mathbf{a}_z$ direction at $t = 0$. Assume a uniform magnetic flux density $\mathbf{B} = B_0\mathbf{a}_z$. Determine the induced emf in the rotating loop and specify the direction of the current.

The magnetic flux though the loop is found (as usual) through

$$\Phi_m = \int_s \mathbf{B} \cdot d\mathbf{S}, \text{ where } \mathbf{S} = \mathbf{n}\, da$$

Because the loop is rotating, the direction of the normal, $\mathbf{n}$, changing, and is in this case given by

$$\mathbf{n} = \cos\omega t\, \mathbf{a}_z - \sin\omega t\, \mathbf{a}_y$$

Therefore,

$$\Phi_m = \int_{-b/2}^{b/2}\int_{-a/2}^{a/2} B_0\mathbf{a}_z \cdot (\cos\omega t\, \mathbf{a}_z - \sin\omega t\, \mathbf{a}_y)\; dx\, dy = abB_0 \cos\omega t$$

The integral is taken over the entire loop area (regardless of its immediate orientation). The important result is that the component of $\mathbf{B}$ that is normal to the loop area is varying sinusoidally, and so it is fine to think of the $\mathbf{B}$ field itself rotating about the $x$ axis in the opposite direction while the loop is stationary. Now the emf is

$$emf = \oint \mathbf{E} \cdot d\mathbf{L} = -\frac{d\Phi_m}{dt} = \underline{ab\,\omega B_0 \sin\omega t} \text{ V}$$

The direction of the current is the same as the direction of $\mathbf{E}$ in the emf expression. It is easiest to picture this by considering the $\mathbf{B}$ field rotating and the loop fixed. By convention, $d\mathbf{L}$ will be counter-clockwise when looking down on the loop from the upper half-space (in the opposite direction of the normal vector to the plane). The current will be counter-clockwise whenever the emf is positive, and will be clockwise whenever the emf is negative.

**9.5.** The location of the sliding bar in Fig. 9.5 is given by $x = 5t + 2t^3$, and the separation of the two rails is 20 cm. Let $\mathbf{B} = 0.8x^2\mathbf{a}_z$ T. Find the voltmeter reading at:

a) $t = 0.4$ s: The flux through the loop will be

$$\Phi = \int_0^{0.2}\int_0^{x} 0.8(x')^2\, dx'\, dy = \frac{0.16}{3}x^3 = \frac{0.16}{3}(5t + 2t^3)^3 \text{ Wb}$$

Then

$$\text{emf} = -\frac{d\Phi}{dt} = \frac{0.16}{3}(3)(5t+2t^3)^2(5+6t^2) = -(0.16)[5(.4)+2(.4)^3]^2[5+6(.4)^2] = \underline{-4.32 \text{ V}}$$

b) $x = 0.6$ m: Have $0.6 = 5t + 2t^3$, from which we find $t = 0.1193$. Thus

$$\text{emf} = -(0.16)[5(.1193) + 2(.1193)^3]^2[5 + 6(.1193)^2] = \underline{-.293 \text{ V}}$$

**9.6.** Let the wire loop of Problem 9.4 be stationary in its $t = 0$ position and find the induced emf that results from a magnetic flux density given by $\mathbf{B}(y, t) = B_0 \cos(\omega t - \beta y)\, \mathbf{a}_z$, where $\omega$ and $\beta$ are constants.

We begin by finding the net magnetic flux through the loop:

$$\Phi_m = \int_s \mathbf{B} \cdot d\mathbf{S} = \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} B_0 \cos(\omega t - \beta y)\, \mathbf{a}_z \cdot \mathbf{a}_z \, dx\, dy$$
$$= \frac{B_0 a}{\beta} \left[ \sin(\omega t + \beta b/2) - \sin(\omega t - \beta b/2) \right]$$

Now the emf is

$$emf = \oint \mathbf{E} \cdot d\mathbf{L} = -\frac{d\Phi_m}{dt} = -\frac{B_0 a \omega}{\beta} \left[ \cos(\omega t + \beta b/2) - \cos(\omega t - \beta b/2) \right]$$

Using the trig identity, $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$, we may write the above result as

$$emf = \underline{+2 B_0 \, a \frac{\omega}{\beta} \sin(\omega t) \sin(\beta b/2) \text{ V}}$$

**9.7.** The rails in Fig. 9.7 each have a resistance of 2.2 $\Omega$/m. The bar moves to the right at a constant speed of 9 m/s in a uniform magnetic field of 0.8 T. Find $I(t)$, $0 < t < 1$ s, if the bar is at $x = 2$ m at $t = 0$ and

a) a 0.3 $\Omega$ resistor is present across the left end with the right end open-circuited: The flux in the left-hand closed loop is

$$\Phi_l = B \times \text{area} = (0.8)(0.2)(2 + 9t)$$

Then, $\text{emf}_l = -d\Phi_l/dt = -(0.16)(9) = -1.44$ V. With the bar in motion, the loop resistance is increasing with time, and is given by $R_l(t) = 0.3 + 2[2.2(2 + 9t)]$. The current is now

$$I_l(t) = \frac{\text{emf}_l}{R_l(t)} = \underline{\frac{-1.44}{9.1 + 39.6t} \text{ A}}$$

Note that the sign of the current indicates that it is flowing in the direction opposite that shown in the figure.

b) Repeat part *a*, but with a resistor of 0.3 $\Omega$ across each end: In this case, there will be a contribution to the current from the right loop, which is now closed. The flux in the right loop, whose area decreases with time, is

$$\Phi_r = (0.8)(0.2)[(16 - 2) - 9t]$$

and $\text{emf}_r = -d\Phi_r/dt = (0.16)(9) = 1.44$ V. The resistance of the right loop is $R_r(t) = 0.3 + 2[2.2(14 - 9t)]$, and so the contribution to the current from the right loop will be

$$I_r(t) = \frac{-1.44}{61.9 - 39.6t} \text{ A}$$

The minus sign has been inserted because again the current must flow in the opposite direction as that indicated in the figure, with the flux decreasing with time. The total current is found by adding the part *a* result, or

$$I_T(t) = \underline{-1.44 \left[ \frac{1}{61.9 - 39.6t} + \frac{1}{9.1 + 39.6t} \right] \text{ A}}$$

**9.8.** A perfectly-conducting filament is formed into a circular ring of radius $a$. At one point a resistance $R$ is inserted into the circuit, and at another a battery of voltage $V_0$ is inserted. Assume that the loop current itself produces negligible magnetic field.

a) Apply Faraday's law, Eq. (4), evaluating each side of the equation carefully and independently to show the equality: With no **B** field present, and no time variation, the right-hand side of Faraday's law is zero, and so therefore

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

This is just a statement of Kirchoff's voltage law around the loop, stating that the battery voltage is equal and opposite to the resistor voltage.

b) Repeat part $a$, assuming the battery removed, the ring closed again, and a linearly-increasing **B** field applied in a direction normal to the loop surface: The situation now becomes the same as that shown in Fig. 9.4, except the loop radius is now $a$, and the resistor value is not specified. Consider the loop as in the $x$-$y$ plane with the positive $z$ axis directed out of the page. The $\mathbf{a}_\phi$ direction is thus counter-clockwise around the loop. The **B** field (out of the page as shown) can be written as $\mathbf{B}(t) = B_0 t\, \mathbf{a}_z$. With the normal to the loop specified as $\mathbf{a}_z$, the direction of $d\mathbf{L}$ is, by the right hand convention, $\mathbf{a}_\phi$. Since the wire is perfectly-conducting, the only voltage appears across the resistor, and is given as $V_R$. Faraday's law becomes

$$\oint \mathbf{E} \cdot d\mathbf{L} = V_R = -\frac{d\Phi_m}{dt} = -\frac{d}{dt}\int_s B_0 t\, \mathbf{a}_z \cdot \mathbf{a}_z \, da = -\pi a^2 B_0$$

This indicates that the resistor voltage, $V_R = \pi a^2 B_0$, has polarity such that the positive terminal is at point $a$ in the figure, while the negative terminal is at point $b$. Current flows in the clockwise direction, and is given in magnitude by $I = \pi a^2 B_0 / R$.

**9.9.** A square filamentary loop of wire is 25 cm on a side and has a resistance of 125 Ω per meter length. The loop lies in the $z = 0$ plane with its corners at $(0,0,0)$, $(0.25,0,0)$, $(0.25,0.25,0)$, and $(0,0.25,0)$ at $t = 0$. The loop is moving with velocity $v_y = 50$ m/s in the field $B_z = 8\cos(1.5 \times 10^8 t - 0.5x)$ $\mu$T. Develop a function of time which expresses the ohmic power being delivered to the loop: First, since the field does not vary with $y$, the loop motion in the $y$ direction does not produce any time-varying flux, and so this motion is immaterial. We can evaluate the flux at the original loop position to obtain:

$$\Phi(t) = \int_0^{.25}\int_0^{.25} 8 \times 10^{-6} \cos(1.5 \times 10^8 t - 0.5x)\, dx\, dy$$
$$= -(4 \times 10^{-6}) \left[\sin(1.5 \times 10^8 t - 0.13) - \sin(1.5 \times 10^8 t)\right] \text{ Wb}$$

Now, $emf = V(t) = -d\Phi/dt = 6.0 \times 10^2 \left[\cos(1.5 \times 10^8 t - 0.13) - \cos(1.5 \times 10^8 t)\right]$, The total loop resistance is $R = 125(0.25 + 0.25 + 0.25 + 0.25) = 125\,\Omega$. Then the ohmic power is

$$P(t) = \frac{V^2(t)}{R} = \underline{2.9 \times 10^3 \left[\cos(1.5 \times 10^8 t - 0.13) - \cos(1.5 \times 10^8 t)\right]^2 \text{ Watts}}$$

**9.10** a) Show that the ratio of the amplitudes of the conduction current density and the displacement current density is $\sigma/\omega\epsilon$ for the applied field $E = E_m \cos\omega t$. Assume $\mu = \mu_0$. First, $D = \epsilon E = \epsilon E_m \cos\omega t$. Then the displacement current density is $\partial D/\partial t = -\omega\epsilon E_m \sin\omega t$. Second, $J_c = \sigma E = \sigma E_m \cos\omega t$. Using these results we find $|J_c|/|J_d| = \sigma/\omega\epsilon$.

b) What is the amplitude ratio if the applied field is $E = E_m e^{-t/\tau}$, where $\tau$ is real? As before, find $D = \epsilon E = \epsilon E_m e^{-t/\tau}$, and so $J_d = \partial D/\partial t = -(\epsilon/\tau)E_m e^{-t/\tau}$. Also, $J_c = \sigma E_m e^{-t/\tau}$. Finally, $|J_c|/|J_d| = \underline{\sigma\tau/\epsilon}$.

**9.11.** Let the internal dimension of a coaxial capacitor be $a = 1.2$ cm, $b = 4$ cm, and $l = 40$ cm. The homogeneous material inside the capacitor has the parameters $\epsilon = 10^{-11}$ F/m, $\mu = 10^{-5}$ H/m, and $\sigma = 10^{-5}$ S/m. If the electric field intensity is $\mathbf{E} = (10^6/\rho)\cos(10^5 t)\mathbf{a}_\rho$ V/m, find:

a) **J**: Use

$$\mathbf{J} = \sigma\mathbf{E} = \underline{\left(\frac{10}{\rho}\right)\cos(10^5 t)\mathbf{a}_\rho \text{ A/m}^2}$$

b) the total conduction current, $I_c$, through the capacitor: Have

$$I_c = \int\int \mathbf{J}\cdot d\mathbf{S} = 2\pi\rho l J = 20\pi l \cos(10^5 t) = \underline{8\pi\cos(10^5 t) \text{ A}}$$

c) the total displacement current, $I_d$, through the capacitor: First find

$$\mathbf{J}_d = \frac{\partial\mathbf{D}}{\partial t} = \frac{\partial}{\partial t}(\epsilon\mathbf{E}) = -\frac{(10^5)(10^{-11})(10^6)}{\rho}\sin(10^5 t)\mathbf{a}_\rho = -\frac{1}{\rho}\sin(10^5 t) \text{ A/m}$$

Now

$$I_d = 2\pi\rho l J_d = -2\pi l \sin(10^5 t) = \underline{-0.8\pi\sin(10^5 t) \text{ A}}$$

d) the ratio of the amplitude of $I_d$ to that of $I_c$, the quality factor of the capacitor: This will be

$$\frac{|I_d|}{|I_c|} = \frac{0.8}{8} = \underline{0.1}$$

**9.12.** Find the displacement current density associated with the magnetic field (assume zero conduction current):

$$\mathbf{H} = A_1 \sin(4x)\cos(\omega t - \beta z)\,\mathbf{a}_x + A_2 \cos(4x)\sin(\omega t - \beta z)\,\mathbf{a}_z$$

The displacement current density is given by

$$\frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{H} = \underline{(4A_2 + \beta A_1)\sin(4x)\cos(\omega t - \beta z)\,\mathbf{a}_y}\ \text{A/m}^2$$

**9.13.** Consider the region defined by $|x|$, $|y|$, and $|z| < 1$. Let $\epsilon_r = 5$, $\mu_r = 4$, and $\sigma = 0$. If $\mathbf{J}_d = 20\cos(1.5 \times 10^8 t - bx)\mathbf{a}_y\ \mu\text{A/m}^2$;

a) find **D** and **E**: Since $\mathbf{J}_d = \partial\mathbf{D}/\partial t$, we write

$$\mathbf{D} = \int \mathbf{J}_d dt + C = \frac{20 \times 10^{-6}}{1.5 \times 10^8}\sin(1.5 \times 10^8 - bx)\mathbf{a}_y$$
$$= \underline{1.33 \times 10^{-13}\sin(1.5 \times 10^8 t - bx)\mathbf{a}_y\ \text{C/m}^2}$$

where the integration constant is set to zero (assuming no dc fields are present). Then

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = \frac{1.33 \times 10^{-13}}{(5 \times 8.85 \times 10^{-12})}\sin(1.5 \times 10^8 t - bx)\mathbf{a}_y$$
$$= \underline{3.0 \times 10^{-3}\sin(1.5 \times 10^8 t - bx)\mathbf{a}_y\ \text{V/m}}$$

b) use the point form of Faraday's law and an integration with respect to time to find **B** and **H**: In this case,

$$\nabla \times \mathbf{E} = \frac{\partial E_y}{\partial x}\mathbf{a}_z = -b(3.0 \times 10^{-3})\cos(1.5 \times 10^8 t - bx)\mathbf{a}_z = -\frac{\partial \mathbf{B}}{\partial t}$$

Solve for **B** by integrating over time:

$$\mathbf{B} = \frac{b(3.0 \times 10^{-3})}{1.5 \times 10^8}\sin(1.5 \times 10^8 t - bx)\mathbf{a}_z = \underline{(2.0)b \times 10^{-11}\sin(1.5 \times 10^8 t - bx)\mathbf{a}_z\ \text{T}}$$

Now

$$\mathbf{H} = \frac{\mathbf{B}}{\mu} = \frac{(2.0)b \times 10^{-11}}{4 \times 4\pi \times 10^{-7}}\sin(1.5 \times 10^8 t - bx)\mathbf{a}_z$$
$$= \underline{(4.0 \times 10^{-6})b\sin(1.5 \times 10^8 t - bx)\mathbf{a}_z\ \text{A/m}}$$

c) use $\nabla \times \mathbf{H} = \mathbf{J}_d + \mathbf{J}$ to find $\mathbf{J}_d$: Since $\sigma = 0$, there is no conduction current, so in this case

$$\nabla \times \mathbf{H} = -\frac{\partial H_z}{\partial x}\mathbf{a}_y = \underline{4.0 \times 10^{-6} b^2 \cos(1.5 \times 10^8 t - bx)\mathbf{a}_y\ \text{A/m}^2} = \mathbf{J}_d$$

d) What is the numerical value of $b$? We set the given expression for $\mathbf{J}_d$ equal to the result of part $c$ to obtain:

$$20 \times 10^{-6} = 4.0 \times 10^{-6} b^2 \quad \Rightarrow \quad b = \underline{\sqrt{5.0}\ \text{m}^{-1}}$$

**9.14.** A voltage source, $V_0 \sin\omega t$, is connected between two concentric conducting spheres, $r = a$ and $r = b$, $b > a$, where the region between them is a material for which $\epsilon = \epsilon_r\epsilon_0$, $\mu = \mu_0$, and $\sigma = 0$. Find the total displacement current through the dielectric and compare it with the source current as determined from the capacitance (Sec. 6.3) and circuit analysis methods: First, solving Laplace's equation, we find the voltage between spheres (see Eq. 39, Chapter 6):

$$V(t) = \frac{(1/r) - (1/b)}{(1/a) - (1/b)} V_0 \sin\omega t$$

Then

$$\mathbf{E} = -\nabla V = \frac{V_0 \sin\omega t}{r^2(1/a - 1/b)}\mathbf{a}_r \quad \Rightarrow \quad \mathbf{D} = \frac{\epsilon_r\epsilon_0 V_0 \sin\omega t}{r^2(1/a - 1/b)}\mathbf{a}_r$$

Now

$$\mathbf{J_d} = \frac{\partial \mathbf{D}}{\partial t} = \frac{\epsilon_r\epsilon_0\omega V_0 \cos\omega t}{r^2(1/a - 1/b)}\mathbf{a}_r$$

The displacement current is then

$$I_d = 4\pi r^2 J_d = \frac{4\pi\epsilon_r\epsilon_0\omega V_0 \cos\omega t}{(1/a - 1/b)} = C\frac{dV}{dt}$$

where, from Eq. 6, Chapter 6,

$$C = \frac{4\pi\epsilon_r\epsilon_0}{(1/a - 1/b)}$$

**9.15.** Let $\mu = 3\times10^{-5}$ H/m, $\epsilon = 1.2\times10^{-10}$ F/m, and $\sigma = 0$ everywhere. If $\mathbf{H} = 2\cos(10^{10}t - \beta x)\mathbf{a}_z$ A/m, use Maxwell's equations to obtain expressions for $\mathbf{B}$, $\mathbf{D}$, $\mathbf{E}$, and $\beta$: First, $\mathbf{B} = \mu\mathbf{H} = \underline{6\times10^{-5}\cos(10^{10}t - \beta x)\mathbf{a}_z \text{ T}}$. Next we use

$$\nabla\times\mathbf{H} = -\frac{\partial \mathbf{H}}{\partial x}\mathbf{a}_y = 2\beta\sin(10^{10}t - \beta x)\mathbf{a}_y = \frac{\partial \mathbf{D}}{\partial t}$$

from which

$$\mathbf{D} = \int 2\beta\sin(10^{10}t - \beta x)\,dt + C = \underline{-\frac{2\beta}{10^{10}}\cos(10^{10}t - \beta x)\mathbf{a}_y \text{ C/m}^2}$$

where the integration constant is set to zero, since no dc fields are presumed to exist. Next,

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = -\frac{2\beta}{(1.2\times10^{-10})(10^{10})}\cos(10^{10}t - \beta x)\mathbf{a}_y = \underline{-1.67\beta\cos(10^{10}t - \beta x)\mathbf{a}_y \text{ V/m}}$$

Now

$$\nabla\times\mathbf{E} = \frac{\partial E_y}{\partial x}\mathbf{a}_z = 1.67\beta^2\sin(10^{10}t - \beta x)\mathbf{a}_z = -\frac{\partial \mathbf{B}}{\partial t}$$

So

$$\mathbf{B} = -\int 1.67\beta^2\sin(10^{10}t - \beta x)\mathbf{a}_z dt = (1.67\times10^{-10})\beta^2\cos(10^{10}t - \beta x)\mathbf{a}_z$$

We require this result to be consistent with the expression for $\mathbf{B}$ originally found. So

$$(1.67\times10^{-10})\beta^2 = 6\times10^{-5} \quad \Rightarrow \quad \beta = \underline{\pm600 \text{ rad/m}}$$

**9.16.** Derive the continuity equation from Maxwell's equations: First, take the divergence of both sides of Ampere's circuital law:

$$\underbrace{\nabla \cdot \nabla \times \mathbf{H}}_{0} = \nabla \cdot \mathbf{J} + \frac{\partial}{\partial t}\nabla \cdot \mathbf{D} = \underline{\nabla \cdot \mathbf{J} + \frac{\partial \rho_v}{\partial t} = 0}$$

where we have used $\nabla \cdot \mathbf{D} = \rho_v$, another Maxwell equation.

**9.17.** The electric field intensity in the region $0 < x < 5$, $0 < y < \pi/12$, $0 < z < 0.06$ m in free space is given by $\mathbf{E} = C\sin(12y)\sin(az)\cos(2\times 10^{10}t)\,\mathbf{a}_x$ V/m. Beginning with the $\nabla \times \mathbf{E}$ relationship, use Maxwell's equations to find a numerical value for $a$, if it is known that $a$ is greater than zero: In this case we find

$$\nabla \times \mathbf{E} = \frac{\partial E_x}{\partial z}\,\mathbf{a}_y - \frac{\partial E_z}{\partial y}\,\mathbf{a}_z$$
$$= C\left[a\sin(12y)\cos(az)\mathbf{a}_y - 12\cos(12y)\sin(az)\mathbf{a}_z\right]\cos(2\times 10^{10}t) = -\frac{\partial \mathbf{B}}{\partial t}$$

Then

$$\mathbf{H} = -\frac{1}{\mu_0}\int \nabla \times \mathbf{E}\; dt + C_1$$
$$= -\frac{C}{\mu_0(2\times 10^{10}}\left[a\sin(12y)\cos(az)\mathbf{a}_y - 12\cos(12y)\sin(az)\mathbf{a}_z\right]\sin(2\times 10^{10}t)\ \text{A/m}$$

where the integration constant, $C_1 = 0$, since there are no initial conditions. Using this result, we now find

$$\nabla \times \mathbf{H} = \left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right]\mathbf{a}_x = -\frac{C(144+a^2)}{\mu_0(2\times 10^{10})}\sin(12y)\sin(az)\sin(2\times 10^{10}t)\,\mathbf{a}_x = \frac{\partial \mathbf{D}}{\partial t}$$

Now

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} = \int \frac{1}{\epsilon_0}\nabla \times \mathbf{H}\; dt + C_2 = \frac{C(144+a^2)}{\mu_0\epsilon_0(2\times 10^{10})^2}\sin(12y)\sin(az)\cos(2\times 10^{10}t)\,\mathbf{a}_x$$

where $C_2 = 0$. This field must be the same as the original field as stated, and so we require that

$$\frac{C(144+a^2)}{\mu_0\epsilon_0(2\times 10^{10})^2} = 1$$

Using $\mu_0\epsilon_0 = (3\times 10^8)^{-2}$, we find

$$a = \left[\frac{(2\times 10^{10})^2}{(3\times 10^8)^2} - 144\right]^{1/2} = \underline{66\ \text{m}^{-1}}$$

**9.18.** The parallel plate transmission line shown in Fig. 9.7 has dimensions $b = 4$ cm and $d = 8$ mm, while the medium between plates is characterized by $\mu_r = 1$, $\epsilon_r = 20$, and $\sigma = 0$. Neglect fields outside the dielectric. Given the field $\mathbf{H} = 5\cos(10^9 t - \beta z)\mathbf{a}_y$ A/m, use Maxwell's equations to help find:

a) $\beta$, if $\beta > 0$: Take

$$\nabla \times \mathbf{H} = -\frac{\partial H_y}{\partial z}\mathbf{a}_x = -5\beta \sin(10^9 t - \beta z)\mathbf{a}_x = 20\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

So

$$\mathbf{E} = \int \frac{-5\beta}{20\epsilon_0}\sin(10^9 t - \beta z)\mathbf{a}_x\, dt = \frac{\beta}{(4\times 10^9)\epsilon_0}\cos(10^9 t - \beta z)\mathbf{a}_x$$

Then

$$\nabla \times \mathbf{E} = \frac{\partial E_x}{\partial z}\mathbf{a}_y = \frac{\beta^2}{(4\times 10^9)\epsilon_0}\sin(10^9 t - \beta z)\mathbf{a}_y = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$$

So that

$$\mathbf{H} = \int \frac{-\beta^2}{(4\times 10^9)\mu_0\epsilon_0}\sin(10^9 t - \beta z)\mathbf{a}_x\, dt = \frac{\beta^2}{(4\times 10^{18})\mu_0\epsilon_0}\cos(10^9 t - \beta z)$$
$$= 5\cos(10^9 t - \beta z)\mathbf{a}_y$$

where the last equality is required to maintain consistency. Therefore

$$\frac{\beta^2}{(4\times 10^{18})\mu_0\epsilon_0} = 5 \quad \Rightarrow \quad \beta = \underline{14.9 \text{ m}^{-1}}$$

b) the displacement current density at $z = 0$: Since $\sigma = 0$, we have

$$\nabla \times \mathbf{H} = \mathbf{J}_d = -5\beta \sin(10^9 t - \beta z) = -74.5\sin(10^9 t - 14.9z)\mathbf{a}_x$$
$$= \underline{-74.5\sin(10^9 t)\mathbf{a}_x \text{ A/m}} \text{ at z} = 0$$

c) the total displacement current crossing the surface $x = 0.5d$, $0 < y < b$, and $0 < z < 0.1$ m in the $\mathbf{a}_x$ direction. We evaluate the flux integral of $\mathbf{J}_d$ over the given cross section:

$$I_d = -74.5b\int_0^{0.1} \sin(10^9 t - 14.9z)\,\mathbf{a}_x \cdot \mathbf{a}_x\, dz = \underline{0.20\left[\cos(10^9 t - 1.49) - \cos(10^9 t)\right] \text{ A}}$$

**9.19.** In the first section of this chapter, Faraday's law was used to show that the field $\mathbf{E} = -\frac{1}{2}kB_0\rho e^{kt}\mathbf{a}_\phi$ results from the changing magnetic field $\mathbf{B} = B_0 e^{kt}\mathbf{a}_z$.

a) Show that these fields do not satisfy Maxwell's other curl equation: Note that $\mathbf{B}$ as stated is constant with position, and so will have zero curl. The electric field, however, varies with time, and so $\nabla\times\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$ would have a zero left-hand side and a non-zero right-hand side. The equation is thus not valid with these fields.

b) If we let $B_0 = 1$ T and $k = 10^6\ s^{-1}$, we are establishing a fairly large magnetic flux density in 1 $\mu$s. Use the $\nabla \times \mathbf{H}$ equation to show that the rate at which $B_z$ should (but does not) change with $\rho$ is only about $5\times 10^{-6}$ T/m in free space at $t = 0$: Assuming that $\mathbf{B}$ varies with $\rho$, we write

$$\nabla \times \mathbf{H} = -\frac{\partial H_z}{\partial \rho}\mathbf{a}_\phi = -\frac{1}{\mu_0}\frac{dB_0}{d\rho}e^{kt} = \epsilon_0\frac{\partial \mathbf{E}}{\partial t} = -\frac{1}{2}\epsilon_0 k^2 B_0 \rho e^{kt}$$

Thus

$$\frac{dB_0}{d\rho} = \frac{1}{2}\mu_0\epsilon_0 k^2 \rho B_0 = \frac{10^{12}(1)\rho}{2(3\times 10^8)^2} = 5.6\times 10^{-6}\rho$$

which is near the stated value if $\rho$ is on the order of 1m.

**9.20.** Given Maxwell's equations in point form, assume that all fields vary as $e^{st}$ and write the equations without explicitly involving time: Write all fields in the general form $\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_0(\mathbf{r})e^{st}$, where $\mathbf{r}$ is a position vector in any coordinate system. Maxwell's equations become:

$$\nabla \times \mathbf{E}_0(\mathbf{r})\, e^{st} = -\frac{\partial}{\partial t}\left(\mathbf{B}_0(\mathbf{r})\, e^{st}\right) = -s\mathbf{B}_0(\mathbf{r})\, e^{st}$$

$$\nabla \times \mathbf{H}_0(\mathbf{r})\, e^{st} = \mathbf{J}_0(\mathbf{r})e^{st} + \frac{\partial}{\partial t}\left(\mathbf{D}_0(\mathbf{r})\, e^{st}\right) = \mathbf{J}_0(\mathbf{r})e^{st} + s\mathbf{D}_0(\mathbf{r})\, e^{st}$$

$$\nabla \cdot \mathbf{D}_0(\mathbf{r})\, e^{st} = \rho_0(\mathbf{r})\, e^{st}$$

$$\nabla \cdot \mathbf{B}_0(\mathbf{r})\, e^{st} = 0$$

In all cases, the $e^{st}$ terms divide out, leaving:

$$\nabla \times \mathbf{E}_0(\mathbf{r}) = -s\mathbf{B}_0(\mathbf{r})$$

$$\nabla \times \mathbf{H}_0(\mathbf{r}) = \mathbf{J}_0(\mathbf{r}) + s\mathbf{D}_0(\mathbf{r})$$

$$\nabla \cdot \mathbf{D}_0(\mathbf{r}) = \rho_0(\mathbf{r})$$

$$\nabla \cdot \mathbf{B}_0(\mathbf{r}) = 0$$

**9.21.** a) Show that under static field conditions, Eq. (55) reduces to Ampere's circuital law. First use the definition of the vector Laplacian:

$$\nabla^2 \mathbf{A} = -\nabla \times \nabla \times \mathbf{A} + \nabla(\nabla \cdot \mathbf{A}) = -\mu \mathbf{J}$$

which is Eq. (55) with the time derivative set to zero. We also note that $\nabla \cdot \mathbf{A} = 0$ in steady state (from Eq. (54)). Now, since $\mathbf{B} = \nabla \times \mathbf{A}$, (55) becomes

$$-\nabla \times \mathbf{B} = -\mu \mathbf{J} \quad \Rightarrow \quad \nabla \times \mathbf{H} = \mathbf{J}$$

b) Show that Eq. (51) becomes Faraday's law when taking the curl: Doing this gives

$$\nabla \times \mathbf{E} = -\nabla \times \nabla V - \frac{\partial}{\partial t} \nabla \times \mathbf{A}$$

The curl of the gradient is identially zero, and $\nabla \times \mathbf{A} = \mathbf{B}$. We are left with

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$$

**9.22.** In a sourceless medium, in which $\mathbf{J} = 0$ and $\rho_v = 0$, assume a rectangular coordinate system in which $\mathbf{E}$ and $\mathbf{H}$ are functions only of $z$ and $t$. The medium has permittivity $\epsilon$ and permeability $\mu$.

a) If $\mathbf{E} = E_x\mathbf{a}_x$ and $\mathbf{H} = H_y\mathbf{a}_y$, begin with Maxwell's equations and determine the second order partial differential equation that $E_x$ must satisfy.
First use

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \Rightarrow \quad \frac{\partial E_x}{\partial z}\,\mathbf{a}_y = -\mu\frac{\partial H_y}{\partial t}\,\mathbf{a}_y$$

in which case, the curl has dictated the direction that $\mathbf{H}$ must lie in. Similarly, use the other Maxwell curl equation to find

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \quad \Rightarrow \quad -\frac{\partial H_y}{\partial z}\,\mathbf{a}_x = \epsilon\frac{\partial E_x}{\partial t}\,\mathbf{a}_x$$

Now, differentiate the first equation with respect to $z$, and the second equation with respect to $t$:

$$\frac{\partial^2 E_x}{\partial z^2} = -\mu\frac{\partial^2 H_y}{\partial t \partial z} \quad \text{and} \quad \frac{\partial^2 H_y}{\partial z \partial t} = -\epsilon\frac{\partial^2 E_x}{\partial t^2}$$

Combining these two, we find

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\epsilon\frac{\partial^2 E_x}{\partial t^2}$$

b) Show that $E_x = E_0\cos(\omega t - \beta z)$ is a solution of that equation for a particular value of $\beta$: Substituting, we find

$$\frac{\partial^2 E_x}{\partial z^2} = -\beta^2 E_0\cos(\omega t - \beta z) \quad \text{and} \quad \mu\epsilon\frac{\partial^2 E_x}{\partial t^2} = -\omega^2\mu\epsilon E_0\cos(\omega t - \beta z)$$

These two will be equal provided the constant multipliers of $\cos(\omega t - \beta z)$ are equal.

c) Find $\beta$ as a function of given parameters. Equating the two constants in part $b$, we find $\underline{\beta = \omega\sqrt{\mu\epsilon}}$.

**9.23.** In region 1, $z < 0$, $\epsilon_1 = 2 \times 10^{-11}$ F/m, $\mu_1 = 2 \times 10^{-6}$ H/m, and $\sigma_1 = 4 \times 10^{-3}$ S/m; in region 2, $z > 0$, $\epsilon_2 = \epsilon_1/2$, $\mu_2 = 2\mu_1$, and $\sigma_2 = \sigma_1/4$. It is known that $\mathbf{E}_1 = (30\mathbf{a}_x + 20\mathbf{a}_y + 10\mathbf{a}_z)\cos(10^9 t)$ V/m at $P_1(0, 0, 0^-)$.

a) Find $\mathbf{E}_{N1}$, $\mathbf{E}_{t1}$, $\mathbf{D}_{N1}$, and $\mathbf{D}_{t1}$: These will be

$$\mathbf{E}_{N1} = \underline{10\cos(10^9 t)\mathbf{a}_z \text{ V/m}} \quad \mathbf{E}_{t1} = \underline{(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) \text{ V/m}}$$

$$\mathbf{D}_{N1} = \epsilon_1 \mathbf{E}_{N1} = (2 \times 10^{-11})(10)\cos(10^9 t)\mathbf{a}_z \text{ C/m}^2 = \underline{200\cos(10^9 t)\mathbf{a}_z \text{ pC/m}^2}$$

$$\mathbf{D}_{t1} = \epsilon_1 \mathbf{E}_{t1} = (2 \times 10^{-11})(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) = \underline{(600\mathbf{a}_x + 400\mathbf{a}_y)\cos(10^9 t) \text{ pC/m}^2}$$

b) Find $\mathbf{J}_{N1}$ and $\mathbf{J}_{t1}$ at $P_1$:

$$\mathbf{J}_{N1} = \sigma_1 \mathbf{E}_{N1} = (4 \times 10^{-3})(10\cos(10^9 t))\mathbf{a}_z = \underline{40\cos(10^9 t)\mathbf{a}_z \text{ mA/m}^2}$$

$$\mathbf{J}_{t1} = \sigma_1 \mathbf{E}_{t1} = (4 \times 10^{-3})(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) = \underline{(120\mathbf{a}_x + 80\mathbf{a}_y)\cos(10^9 t) \text{ mA/m}^2}$$

c) Find $\mathbf{E}_{t2}$, $\mathbf{D}_{t2}$, and $\mathbf{J}_{t2}$ at $P_1$: By continuity of tangential $\mathbf{E}$,

$$\mathbf{E}_{t2} = \mathbf{E}_{t1} = \underline{(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) \text{ V/m}}$$

Then

$$\mathbf{D}_{t2} = \epsilon_2 \mathbf{E}_{t2} = (10^{-11})(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) = \underline{(300\mathbf{a}_x + 200\mathbf{a}_y)\cos(10^9 t) \text{ pC/m}^2}$$

$$\mathbf{J}_{t2} = \sigma_2 \mathbf{E}_{t2} = (10^{-3})(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) = \underline{(30\mathbf{a}_x + 20\mathbf{a}_y)\cos(10^9 t) \text{ mA/m}^2}$$

d) (Harder) Use the continuity equation to help show that $J_{N1} - J_{N2} = \partial D_{N2}/\partial t - \partial D_{N1}/\partial t$ and then determine $\mathbf{E}_{N2}$, $\mathbf{D}_{N2}$, and $\mathbf{J}_{N2}$: We assume the existence of a surface charge layer at the boundary having density $\rho_s$ C/m$^2$. If we draw a cylindrical "pillbox" whose top and bottom surfaces (each of area $\Delta a$) are on either side of the interface, we may use the continuity condition to write

$$(J_{N2} - J_{N1})\Delta a = -\frac{\partial \rho_s}{\partial t}\Delta a$$

where $\rho_s = D_{N2} - D_{N1}$. Therefore,

$$J_{N1} - J_{N2} = \frac{\partial}{\partial t}(D_{N2} - D_{N1})$$

In terms of the normal electric field components, this becomes

$$\sigma_1 E_{N1} - \sigma_2 E_{N2} = \frac{\partial}{\partial t}(\epsilon_2 E_{N2} - \epsilon_1 E_{N1})$$

Now let $E_{N2} = A\cos(10^9 t) + B\sin(10^9 t)$, while from before, $E_{N1} = 10\cos(10^9 t)$.

**9.23d** (continued)

These, along with the permittivities and conductivities, are substituted to obtain

$$
\begin{aligned}
&(4 \times 10^{-3})(10)\cos(10^9 t) - 10^{-3}[A\cos(10^9 t) + B\sin(10^9 t)] \\
&= \frac{\partial}{\partial t}\left[10^{-11}[A\cos(10^9 t) + B\sin(10^9 t)] - (2 \times 10^{-11})(10)\cos(10^9 t)\right] \\
&= -(10^{-2} A\sin(10^9 t) + 10^{-2} B\cos(10^9 t) + (2 \times 10^{-1})\sin(10^9 t)
\end{aligned}
$$

We now equate coefficients of the sin and cos terms to obtain two equations:

$$4 \times 10^{-2} - 10^{-3} A = 10^{-2} B$$

$$-10^{-3} B = -10^{-2} A + 2 \times 10^{-1}$$

These are solved together to find $A = 20.2$ and $B = 2.0$. Thus

$$\mathbf{E}_{N2} = \left[20.2\cos(10^9 t) + 2.0\sin(10^9 t)\right]\mathbf{a}_z = \underline{20.3\cos(10^9 t + 5.6^\circ)\mathbf{a}_z \text{ V/m}}$$

Then

$$\mathbf{D}_{N2} = \epsilon_2 \mathbf{E}_{N2} = \underline{203\cos(10^9 t + 5.6^\circ)\mathbf{a}_z \text{ pC/m}^2}$$

and

$$\mathbf{J}_{N2} = \sigma_2 \mathbf{E}_{N2} = \underline{20.3\cos(10^9 t + 5.6^\circ)\mathbf{a}_z \text{ mA/m}^2}$$

**9.24.** A vector potential is given as $\mathbf{A} = A_0 \cos(\omega t - kz)\,\mathbf{a}_y$. a) Assuming as many components as possible are zero, find **H**, **E**, and $V$;

With **A** $y$-directed only, and varying spatially only with $z$, we find

$$\mathbf{H} = \frac{1}{\mu}\nabla\times\mathbf{A} = -\frac{1}{\mu}\frac{\partial A_y}{\partial z}\,\mathbf{a}_x = \underline{-\frac{kA_0}{\mu}\sin(\omega t - kz)\,\mathbf{a}_x \text{ A/m}}$$

Now, in a lossless medium we will have zero conductivity, so that the point form of Ampere's circuital law involves only the displacement current term:

$$\nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t} = \epsilon\frac{\partial\mathbf{E}}{\partial t}$$

Using the magnetic field as found above, we find

$$\nabla\times\mathbf{H} = \frac{\partial H_x}{\partial z}\,\mathbf{a}_y = \frac{k^2A_0}{\mu}\cos(\omega t - kz)\,\mathbf{a}_y = \epsilon\frac{\partial\mathbf{E}}{\partial t} \;\Rightarrow\; \mathbf{E} = \underline{\frac{k^2A_0}{\omega\mu\epsilon}\sin(\omega t - kz)\,\mathbf{a}_y \text{ V/m}}$$

Now,

$$\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t} \;\Rightarrow\; \nabla V = -\left[\frac{\partial\mathbf{A}}{\partial t} + \mathbf{E}\right]$$

or

$$\nabla V = A_0\omega\left[1 - \frac{k^2}{\omega^2\mu\epsilon}\right]\sin(\omega t - kz)\,\mathbf{a}_y = \frac{\partial V}{\partial y}\,\mathbf{a}_y$$

Integrating over $y$ we find

$$V = A_0\,\omega\,y\left[1 - \frac{k^2}{\omega^2\mu\epsilon}\right]\sin(\omega t - kz) + C$$

where $C$, the integration constant, can be taken as zero. In part $b$, it will be shown that $k = \omega\sqrt{\mu\epsilon}$, which means that $\underline{V = 0}$.

b) Specify $k$ in terms of $A_0$, $\omega$, and the constants of the lossless medium, $\epsilon$ and $\mu$. Use the other Maxwell curl equation:

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} = -\mu\frac{\partial\mathbf{H}}{\partial t}$$

so that

$$\frac{\partial\mathbf{H}}{\partial t} = -\frac{1}{\mu}\,\nabla\times\mathbf{E} = \frac{1}{\mu}\frac{\partial E_y}{\partial z}\,\mathbf{a}_x = -\frac{k^3A_0}{\omega\mu^2\epsilon}\cos(\omega t - kz)\,\mathbf{a}_x$$

Integrate over $t$ (and set the integration constant to zero) and require the result to be consistant with part $a$:

$$\mathbf{H} = -\frac{k^3A_0}{\omega^2\mu^2\epsilon}\sin(\omega t - kz)\,\mathbf{a}_x = \underbrace{-\frac{kA_0}{\mu}\sin(\omega t - kz)\,\mathbf{a}_x}_{\text{from part a}}$$

We identify

$$\underline{k = \omega\sqrt{\mu\epsilon}}$$

**9.25.** In a region where $\mu_r = \epsilon_r = 1$ and $\sigma = 0$, the retarded potentials are given by $V = x(z - ct)$ V and $\mathbf{A} = x[(z/c) - t]\mathbf{a}_z$ Wb/m, where $c = 1/\sqrt{\mu_0 \epsilon_0}$.

a) Show that $\nabla \cdot \mathbf{A} = -\mu\epsilon(\partial V/\partial t)$:

First,

$$\nabla \cdot \mathbf{A} = \frac{\partial A_z}{\partial z} = \frac{x}{c} = x\sqrt{\mu_0 \epsilon_0}$$

Second,

$$\frac{\partial V}{\partial t} = -cx = -\frac{x}{\sqrt{\mu_0 \epsilon_0}}$$

so we observe that $\nabla \cdot \mathbf{A} = -\mu_0 \epsilon_0 (\partial V/\partial t)$ in free space, implying that the given statement would hold true in general media.

b) Find **B**, **H**, **E**, and **D**:

Use

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A_x}{\partial x}\mathbf{a}_y = \underline{\left(t - \frac{z}{c}\right)\mathbf{a}_y \text{ T}}$$

Then

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \underline{\frac{1}{\mu_0}\left(t - \frac{z}{c}\right)\mathbf{a}_y \text{ A/m}}$$

Now,

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -(z - ct)\mathbf{a}_x - x\mathbf{a}_z + x\mathbf{a}_z = \underline{(ct - z)\mathbf{a}_x \text{ V/m}}$$

Then

$$\mathbf{D} = \epsilon_0 \mathbf{E} = \underline{\epsilon_0 (ct - z)\mathbf{a}_x \text{ C/m}^2}$$

c) Show that these results satisfy Maxwell's equations if **J** and $\rho_v$ are zero:

i. $\nabla \cdot \mathbf{D} = \nabla \cdot \epsilon_0 (ct - z)\mathbf{a}_x = 0$

ii. $\nabla \cdot \mathbf{B} = \nabla \cdot (t - z/c)\mathbf{a}_y = 0$

iii.

$$\nabla \times \mathbf{H} = -\frac{\partial H_y}{\partial z}\mathbf{a}_x = \frac{1}{\mu_0 c}\mathbf{a}_x = \sqrt{\frac{\epsilon_0}{\mu_0}}\mathbf{a}_x$$

which we require to equal $\partial \mathbf{D}/\partial t$:

$$\frac{\partial \mathbf{D}}{\partial t} = \epsilon_0 c \mathbf{a}_x = \sqrt{\frac{\epsilon_0}{\mu_0}}\mathbf{a}_x$$

iv.

$$\nabla \times \mathbf{E} = \frac{\partial E_x}{\partial z}\mathbf{a}_y = -\mathbf{a}_y$$

which we require to equal $-\partial \mathbf{B}/\partial t$:

$$\frac{\partial \mathbf{B}}{\partial t} = \mathbf{a}_y$$

So all four Maxwell equations are satisfied.

**9.26.** Write Maxwell's equations in point form in terms of **E** and **H** as they apply to a sourceless medium, where **J** and $\rho_v$ are both zero. Replace $\epsilon$ by $\mu$, $\mu$ by $\epsilon$, **E** by **H**, and **H** by $-$**E**, and show that the equations are unchanged. This is a more general expression of the *duality principle* in circuit theory.

Maxwell's equations in sourceless media can be written as:

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \tag{1}$$

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} \tag{2}$$

$$\nabla \cdot \epsilon \mathbf{E} = 0 \tag{3}$$

$$\nabla \cdot \mu \mathbf{H} = 0 \tag{4}$$

In making the above substitutions, we find that (1) converts to (2), (2) converts to (1), and (3) and (4) convert to each other.

## CHAPTER 10

**10.1.** The parameters of a certain transmission line operating at $\omega = 6 \times 10^8$ rad/s are $L = 0.35$ $\mu$H/m, $C = 40$ pF/m, $G = 75$ $\mu$S/m, and $R = 17$ $\Omega$/m. Find $\gamma$, $\alpha$, $\beta$, $\lambda$, and $Z_0$: We use

$$\begin{aligned}\gamma &= \sqrt{ZY} = \sqrt{(R + j\omega L)(G + j\omega C)} \\ &= \sqrt{[17 + j(6 \times 10^8)(0.35 \times 10^{-6})][75 \times 10^{-6} + j(6 \times 10^8)(40 \times 10^{-12})]} \\ &= \underline{0.094 + j2.25 \text{ m}^{-1}} = \alpha + j\beta\end{aligned}$$

Therefore, $\alpha = \underline{0.094 \text{ Np/m}}$, $\beta = \underline{2.25 \text{ rad/m}}$, and $\lambda = 2\pi/\beta = \underline{2.79 \text{ m}}$. Finally,

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{17 + j2.1 \times 10^2}{75 \times 10^{-6} + j2.4 \times 10^{-2}}} = \underline{93.6 - j3.64 \ \Omega}$$

**10.2.** A sinusoidal wave on a transmission line is specified by voltage and current in phasor form:

$$V_s(z) = V_0 \, e^{\alpha z} \, e^{j\beta z} \quad \text{and} \quad I_s(z) = I_0 \, e^{\alpha z} \, e^{j\beta z} \, e^{j\phi}$$

where $V_0$ and $I_0$ are both real.

a) In which direction does this wave propagate and why? Propagation is in the *backward z* direction, because of the factor $e^{+j\beta z}$.

b) It is found that $\alpha = 0$, $Z_0 = 50\,\Omega$, and the wave velocity is $v_p = 2.5 \times 10^8$ m/s, with $\omega = 10^8\,\text{s}^{-1}$. Evaluate $R$, $G$, $L$, $C$, $\lambda$, and $\phi$: First, the fact that $\alpha = 0$ means that the line is lossless, from which we immediately conclude that $\underline{R = G = 0}$. As this is true it follows that $Z_0 = \sqrt{L/C}$ and $v_p = 1/\sqrt{LC}$, from which

$$C = \frac{1}{Z_0 v_p} = \frac{1}{50(2.5 \times 10^8)} = 8.0 \times 10^{-11} \text{ F} = \underline{80 \text{ pF}}$$

Then

$$L = CZ_0^2 = (8.0 \times 10^{-11})(50)^2 = 2.0 \times 10^{-7} = \underline{0.20 \ \mu\text{H}}$$

Now,

$$\lambda = \frac{v_p}{f} = \frac{2\pi v_p}{\omega} = \frac{2\pi(2.5 \times 10^8)}{10^8} = \underline{15.7 \text{ m}}$$

Finally, the current phase is found through

$$I_0 e^{j\phi} = \frac{V_0}{Z_0}$$

Since $V_0$, $I_0$, and $Z_0$ are all real, it follows that $\underline{\phi = 0}$.

**10.3.** The characteristic impedance of a certain lossless transmission line is 72 Ω. If $L = 0.5\,\mu$H/m, find:

a) $C$: Use $Z_0 = \sqrt{L/C}$, or

$$C = \frac{L}{Z_0^2} = \frac{5 \times 10^{-7}}{(72)^2} = 9.6 \times 10^{-11}\,\text{F/m} = \underline{96\,\text{pF/m}}$$

$v_p$:

$$v_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(5 \times 10^{-7})(9.6 \times 10^{-11})}} = \underline{1.44 \times 10^8 \text{ m/s}}$$

c) $\beta$ if $f = 80$ MHz:

$$\beta = \omega\sqrt{LC} = \frac{2\pi \times 80 \times 10^6}{1.44 \times 10^8} = \underline{3.5 \text{ rad/m}}$$

d) The line is terminated with a load of 60 Ω. Find Γ and $s$:

$$\Gamma = \frac{60 - 72}{60 + 72} = \underline{-0.09} \qquad\qquad s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + .09}{1 - .09} = \underline{1.2}$$

**10.4.** A sinusoidal voltage wave of amplitude $V_0$, frequency $\omega$, and phase constant, $\beta$, propagates in the forward $z$ direction toward the open load end in a lossless transmission line of characteristic impedance $Z_0$. At the end, the wave totally reflects with zero phase shift, and the reflected wave now interferes with the incident wave to yield a standing wave pattern over the line length (as per Example 10.1). Determine the standing wave pattern for the *current* in the line. Express the result in real instantaneous form and simplify.

In phasor form, the forward and backward waves are:

$$V_{sT}(z) = V_0 e^{-j\beta z} + V_0 e^{j\beta z}$$

The current is found from the voltage by dividing by $Z_0$ (while incorporating the proper sign for forward and backward waves):

$$I_{sT}(z) = \frac{V_0}{Z_0} e^{-j\beta z} - \frac{V_0}{Z_0} e^{j\beta z} = -\frac{V_0}{Z_0}\left(e^{j\beta z} - e^{-j\beta z}\right) = -j\frac{2V_0}{Z_0}\sin(\beta z)$$

The real instantaneous current is now

$$\mathcal{I}(z,t) = \mathcal{R}e\left\{I_{sT}(z)e^{j\omega t}\right\} = \mathcal{R}e\left\{-j\frac{2V_0}{Z_0}\sin(\beta z)[\underbrace{\cos(\omega t) + j\sin(\omega t)}_{e^{j\omega t}}]\right\}$$

$$= \frac{2V_0}{Z_0}\sin(\beta z)\sin(\omega t)$$

**10.5.** Two characteristics of a certain lossless transmission line are $Z_0 = 50\ \Omega$ and $\gamma = 0 + j0.2\pi\ \text{m}^{-1}$ at $f = 60$ MHz.

a) Find $L$ and $C$ for the line: We have $\beta = 0.2\pi = \omega\sqrt{LC}$ and $Z_0 = 50 = \sqrt{L/C}$. Thus

$$\frac{\beta}{Z_0} = \omega C \quad \Rightarrow \quad C = \frac{\beta}{\omega Z_0} = \frac{0.2\pi}{(2\pi \times 60 \times 10^6)(50)} = \frac{1}{3} \times 10^{10} = \underline{33.3\ \text{pF/m}}$$

Then $L = CZ_0^2 = (33.3 \times 10^{-12})(50)^2 = 8.33 \times 10^{-8}\ \text{H/m} = \underline{83.3\ \text{nH/m}}$.

b) A load, $Z_L = 60 + j80\ \Omega$ is located at $z = 0$. What is the shortest distance from the load to a point at which $Z_{in} = R_{in} + j0$? I will do this using two different methods:

*The Hard Way*: We use the general expression

$$Z_{in} = Z_0 \left[ \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L \tan(\beta l)} \right]$$

We can then normalize the impedances with respect to $Z_0$ and write

$$z_{in} = \frac{Z_{in}}{Z_0} = \left[ \frac{(Z_L/Z_0) + j\tan(\beta l)}{1 + j(Z_L/Z_0)\tan(\beta l)} \right] = \left[ \frac{z_L + j\tan(\beta l)}{1 + jz_L \tan(\beta l)} \right]$$

where $z_L = (60 + j80)/50 = 1.2 + j1.6$. Using this, and defining $x = \tan(\beta l)$, we find

$$z_{in} = \left[ \frac{1.2 + j(1.6 + x)}{(1 - 1.6x) + j1.2x} \right] \left[ \frac{(1 - 1.6x) - j1.2x}{(1 - 1.6x) - j1.2x} \right]$$

The second bracketed term is a factor of one, composed of the complex conjugate of the denominator of the first term, divided by itself. Carrying out this product, we find

$$z_{in} = \left[ \frac{1.2(1 - 1.6x) + 1.2x(1.6 + x) - j[(1.2)^2 x - (1.6 + x)(1 - 1.6x)]}{(1 - 1.6x)^2 + (1.2)^2 x^2} \right]$$

We require the imaginary part to be zero. Thus

$$(1.2)^2 x - (1.6 + x)(1 - 1.6x) = 0 \quad \Rightarrow \quad 1.6x^2 + 3x - 1.6 = 0$$

$$\text{So} \ \ x = \tan(\beta l) = \frac{-3 \pm \sqrt{9 + 4(1.6)^2}}{2(1.6)} = (.433, -2.31)$$

We take the positive root, and find

$$\beta l = \tan^{-1}(.433) = 0.409 \quad \Rightarrow \quad l = \frac{0.409}{0.2\pi} = 0.65\ \text{m} = \underline{65\ \text{cm}}$$

*The Easy Way*: We find

$$\Gamma = \frac{60 + j80 - 50}{60 + j80 + 50} = 0.405 + j0.432 = 0.59\angle 0.818$$

Thus $\phi = 0.818$ rad, and we use the fact that the input impedance will be purely real at the location of a voltage minimum or maximum. The first voltage maximum will occur at a distance in front of the load given by

$$z_{max} = \frac{\phi}{2\beta} = \frac{0.818}{2(0.2\pi)} = 0.65\ \text{m}$$

**10.6.** A 50-ohm load is attached to a 50m section of the transmission line of Problem 10.1, and a 100-W signal is fed to the input end of the line.

a) Evaluate the distributed line loss in dB/m: From Problem 10.1 (or from the answer in Appendix F) we have $\alpha = 0.094$ Np/m. Then

$$\text{Loss[dB/m]} = 8.69\alpha = 8.69(0.094) = \underline{0.82\text{ dB/m}}$$

b) Evaluate the reflection coefficient at the load: We need the characteristic impedance of the line. Again, in solving Problem 10.1 (or looking up the answer in the appendix), we have $Z_0 = 93.6 - j3.64$ ohms. The reflection coefficient is

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50 - (93.6 - j3.64)}{50 + (93.6 - j3.64)} = \underline{-0.304 + j0.0176 \;\; = \;\; 0.305\angle 177^\circ}$$

c) Evaluate the power that is dissipated by the load resistor: This will be

$$P_d = 100\text{W} \times e^{-2\alpha L} \times (1 - |\Gamma_L|^2) = 100\, e^{-2(0.094)(50)} \left[1 - (0.305)^2\right] = \underline{7.5\text{ mW}}$$

d) What power drop in dB does the dissipated power in the load represent when compared to the original input power? This we find as a positive number through

$$P_d[\text{dB}] = 10\log_{10}\left[\frac{P_{in}}{P_d}\right] = 10\log_{10}\left[\frac{100}{0.0075}\right] = \underline{41.2\text{ dB}}$$

e) on partial reflection from the load, how much power returns to the input and what dB drop does this represent when compared to the original 100-W input power? After one round trip plus a reflection at the load, the power returning to the input is expressed as

$$P_{out} = P_{in} \times e^{-2\alpha(2L)} \times |\Gamma_L|^2 = 100\, e^{200(0.094)} (0.305)^2 = 6.37 \times 10^{-10}\text{ W} = \underline{637\text{ pW}}$$

As a decibel reduction from the original input power, this becomes

$$P_{out}[\text{dB}] = 10\log_{10}\left[\frac{P_{in}}{P_{out}}\right] = 10\log_{10}\left[\frac{100}{6.37 \times 10^{-10}}\right] = \underline{112\text{ dB}}$$

**10.7.** A transmitter and receiver are connected using a cascaded pair of transmission lines. At the operating frequency, Line 1 has a measured loss of 0.1 dB/m, and Line 2 is rated at 0.2 dB/m. The link is composed of 40m of Line 1, joined to 25m of Line 2. At the joint, a splice loss of 2 dB is measured. If the transmitted power is 100mW, what is the received power?

The total loss in the link in dB is $40(0.1) + 25(0.2) + 2 = 11$ dB. Then the received power is $P_r = 100\text{mW} \times 10^{-0.1(11)} = \underline{7.9\text{ mW}}$.

**10.8.** An absolute measure of power is the dBm scale, in which power is specified in decibels relative to one milliwatt. Specifically, $P(\text{dBm}) = 10\log_{10}[P(\text{mW})/1\,\text{mW}]$. Suppose that a receiver is rated as having a *sensitivity* of −20 dBm, indicating the *mimimum* power that it must receive in order to adequately interpret the transmitted electronic data. Suppose this receiver is at the load end of a 50-ohm transmission line having 100-m length and loss rating of 0.09 dB/m. The receiver impedance is 75 ohms, and so is not matched to the line. What is the minimum required input power to the line in a) dBm, b) mW?

*Method 1 – using decibels*: The total loss in dB will be the sum of the transit loss in the line and the loss arising from partial transmission into the load. The latter will be

$$\text{Loss}_{load}\,[\text{dB}] = 10\log_{10}\left(\frac{1}{1-|\Gamma_L|^2}\right)$$

where $\Gamma_L = (75-50)/(75+50) = 0.20$. So

$$\text{Loss}_{load} = 10\log_{10}\left(\frac{1}{1-(0.20)^2}\right) = 0.18 \text{ dB}$$

The transit loss will be

$$\text{Loss}_{trans} = 10\log_{10}\left(\frac{1}{e^{-2\alpha L}}\right) = (0.09\,\text{dB/m})(100\,\text{m}) = 9.0\,\text{dB}$$

The total loss in dB is then $\text{Loss}_{tot} = 9.0 + 0.18 = 9.2$ dB. The minimum required input power is now

$$P_{in}\,[\text{dBm}] = -20\,\text{dBm} + 9.2\,\text{dB} = \underline{-10.8 \text{ dBm}}$$

In milliwatts, this is

$$P_{in}\,[\text{mW}] = 10^{-1.08} = \underline{8.3\times 10^{-2} \text{ mW} = 83\ \mu\text{W}}$$

*Method 2 – using loss factors*: The 0.09 dB/m line loss corresponds to an exponential voltage attenuation coefficient of $\alpha = 0.09/8.69 = 1.04\times 10^{-2}$ Np/m. Now, the power dropped at the load will be

$$P_{load} = P_{in}e^{-2\alpha L}(1-|\Gamma_L|^2) = P_{in}\exp\left[-2(1.04\times 10^{-2})(100)\right]\left[1-(0.2)^2\right] = 0.12P_{in}$$

Since the minimum power at the load of -20 dBm in mW is $10^{-2}$, the minimum input power will be

$$P_{in}[\text{mW}] = \frac{10^{-2}}{0.12} = 8.3\times 10^{-2}\,\text{mW} = 83\ \mu\text{W} \ \text{ as before}$$

In dBm this is $P_{in} = 10\log_{10}\left(8.3\times 10^{-2}\right) = -10.8$ dBm.

**10.9.** A sinusoidal voltage source drives the series combination of an impedance, $Z_g = 50 - j50\ \Omega$, and a lossless transmission line of length $L$, shorted at the load end. The line characteristic impedance is 50 $\Omega$, and wavelength $\lambda$ is measured on the line.

a) Determine, in terms of wavelength, the shortest line length that will result in the voltage source driving a total impedance of 50 $\Omega$: Using Eq. (98), with $Z_L = 0$, we find the input impedance, $Z_{in} = jZ_0 \tan(\beta L)$, where $Z_0 = 50$ ohms. This input inpedance is in series with the generator impedance, giving a total of $Z_{tot} = 50 - j50 + j50 \tan(\beta L)$. For this impedance to equal 50 ohms, the imaginary parts must cancel. Therefore, $\tan(\beta L) = 1$, or $\beta L = \pi/4$, at minimum. So $L = \pi/(4\beta) = \pi/(4 \times 2\pi/\lambda) = \underline{\lambda/8}$.

b) Will other line lengths meet the requirements of part *a*? If so what are they? Yes, the requirement being $\beta L = \pi/4 + m\pi$, where $m$ is an integer. Therefore

$$L = \frac{\pi/4 + m\pi}{\beta} = \frac{\pi(1+4m)}{4 \times 2\pi/\lambda} = \frac{\lambda}{8} + m\frac{\lambda}{2}$$

**10.10.** Two lossless transmission lines having different characteristic impedances are to be joined end-to-end. The impedances are $Z_{01} = 100$ ohms and $Z_{03} = 25$ ohms. The operating frequency is 1 GHz.

a) Find the required characteristic impedance, $Z_{02}$, of a quarter-wave section to be inserted between the two, which will impedance-match the joint, thus allowing total power transmission through the three lines: The required inpedance will be $Z_{02} = \sqrt{Z_{01}Z_{03}} = \sqrt{(100)(25)} = \underline{50 \text{ ohms}}$.

b) The capacitance per unit length of the intermediate line is found to be 100 pF/m. Find the shortest length in meters of this line that is needed to satisfy the impedance-matching condition: For the lossless intermediate line,

$$Z_{02} = \sqrt{\frac{L_2}{C_2}} \;\Rightarrow\; L_2 = C_2 Z_{02}^2 \quad \text{Then} \quad \beta_2 = \omega\sqrt{L_2 C_2} = 2\pi f\, C_2\, Z_{02}$$

The line length at $\lambda/4$ (the shortest length that will work) is then

$$\ell_2 = \frac{\lambda_2}{4} = \frac{1}{4}\left(\frac{2\pi}{\beta_2}\right) = \frac{1}{4fC_2Z_{02}} = \frac{1}{(4 \times 10^9)(10^{-10})(50)} = \underline{0.05 \text{ m}}$$

c) With the three-segment setup as found in parts *a* and *b*, the frequency is now doubled to 2 GHz. Find the input impedance at the Line 1-to-Line 2 junction, seen by waves incident from Line 1: With the frequency doubled, the wavelength is cut in half, which means that the intermediate section is now a half-wavelength long. In that case, the input impedance is just the impedance of the far line, or $Z_{in} = Z_{03} = \underline{25 \text{ ohms}}$.

d) Under the conditions of part *c*, and with power incident from Line 1, evaluate the standing wave ratio that will be measured in Line 1, and the fraction of the incident power from Line 1 that is reflected and propagates back to the Line 1 input. The reflection coefficient at the junction is $\Gamma_{in} = (25-100)/(25+100) = -3/5$. So the VSWR $= (1+3/5)/(1-3/5) = \underline{4}$. The fraction of the power reflected at the junction is $|\Gamma|^2 = (3/5)^2 = \underline{0.36}$, or 36%.

**10.11.** A transmission line having primary constants $L$, $C$, $R$, and $G$, has length $\ell$ and is terminated by a load having complex impedance $R_L + jX_L$. At the input end of the line, a $DC$ voltage source, $V_0$, is connected. Assuming all parameters are known at zero frequency, find the steady state power dissipated by the load if

a) $R = G = 0$: Here, the line just acts as a pair of lossless leads to the impedance. At zero frequency, the dissipated power is just $P_d = \underline{V_0^2/R_L}$.

b) $R \neq 0$, $G = 0$: In this case, the load is effectively in series with a resistance of value $R\ell$. The voltage at the load is therefore $V_L = V_0 R_L/(R\ell + R_L)$, and the dissipated power is $P_d = V_L^2/R_L = \underline{V_0^2 R_L/(R\ell + R_L)^2}$.

c) $R = 0$, $G \neq 0$: Now, the load is in parallel with a resistance, $1/(G\ell)$, but the voltage at the load is still $V_0$. Dissipated power by the load is $P_d = \underline{V_0^2/R_L}$.

d) $R \neq 0$, $G \neq 0$: One way to approach this problem is to think of the power at the load as arising from an incident voltage wave of vanishingly small frequency, and to assume that losses in the line are sufficient to allow steady state conditions to be reached after a single reflection from the load. The "forward-traveling" voltage as a function of $z$ is given by $V(z) = V_0 \exp(-\gamma z)$, where $\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \rightarrow \sqrt{RG}$ as frequency approaches zero. Considering a single reflection only, the voltage at the load is then $V_L = (1 + \Gamma)V_0 \exp\left(-\sqrt{RG}\,\ell\right)$. The reflection coefficient requires the line characteristic impedance, given by $Z_0 = [(R + j\omega L)/(G + j\omega C)]^{1/2} \rightarrow \sqrt{R/G}$ as $\omega \rightarrow 0$. The reflection coefficient is then $\Gamma = (R_L - \sqrt{R/G})/(R_L + \sqrt{R/G})$, and so the load voltage becomes:

$$V_L = \frac{2R_L}{R_L + \sqrt{R/G}} \exp\left(-\sqrt{RG}\,\ell\right)$$

The dissipated power is then

$$P_d = \frac{V_L^2}{R_L} = \frac{4R_L V_0^2}{\left(R_L + \sqrt{R/G}\right)^2} \exp\left(-2\sqrt{RG}\,\ell\right) \text{ W}$$

**10.12.** In a circuit in which a sinusoidal voltage source drives its internal impedance in series with a load impedance, it is known that maximum power transfer to the load occurs when the source and load impedances form a complex conjugate pair. Suppose the source (with its internal impedance) now drives a complex load of impedance $Z_L = R_L + jX_L$ that has been moved to the end of a lossless transmission line of length $\ell$ having characteristic impedance $Z_0$. If the source impedance is $Z_g = R_g + jX_g$, write an equation that can be solved for the required line length, $\ell$, such that the displaced load will receive the maximum power.

The condition of maximum power transfer will be met if the *input impedance* to the line is the conjugate of the internal impedance. Using Eq. (98), we write

$$Z_{in} = Z_0 \left[ \frac{(R_L + jX_L)\cos(\beta\ell) + jZ_0 \sin(\beta\ell)}{Z_0 \cos(\beta\ell) + j(R_L + jX_L)\sin(\beta\ell)} \right] = R_g - jX_g$$

This is the equation that we have to solve for $\ell$ – assuming that such a solution exists. To find out, we need to work with the equation a little. Multiplying both sides by the denominator of the left side gives

$$Z_0(R_L + jX_L)\cos(\beta\ell) + jZ_0^2 \sin(\beta\ell) = (R_g - jX_g)[Z_0 \cos(\beta\ell) + j(R_L + jX_L)\sin(\beta\ell)]$$

We next separate the equation by equating the real parts of both sides and the imaginary parts of both sides, giving

$$(R_L - R_g)\cos(\beta\ell) = \frac{X_L X_g}{Z_0} \sin(\beta\ell) \quad \text{(real parts)}$$

and

$$(X_L + X_g)\cos(\beta\ell) = \frac{R_g R_L - Z_0^2}{Z_0} \sin(\beta\ell) \quad \text{(imaginary parts)}$$

Using the two equations, we find two conditions on the tangent of $\beta\ell$:

$$\tan(\beta\ell) = \frac{Z_0(R_L - R_g)}{X_g X_L} = \frac{Z_0(X_L + X_g)}{R_g R_L - Z_0^2}$$

For a viable solution to exist for $\ell$, both equalities must be satisfied, thus limiting the possible choices of the two impedances.

**10.13.** The incident voltage wave on a certain lossless transmission line for which $Z_0 = 50\,\Omega$ and $v_p = 2 \times 10^8$ m/s is $V^+(z,t) = 200\cos(\omega t - \pi z)$ V.

a) Find $\omega$: We know $\beta = \pi = \omega/v_p$, so $\omega = \pi(2 \times 10^8) = \underline{6.28 \times 10^8 \text{ rad/s}}$.

b) Find $I^+(z,t)$: Since $Z_0$ is real, we may write

$$I^+(z,t) = \frac{V^+(z,t)}{Z_0} = \underline{4\cos(\omega t - \pi z) \text{ A}}$$

The section of line for which $z > 0$ is replaced by a load $Z_L = 50 + j30\ \Omega$ at $z = 0$. Find

c) $\Gamma_L$: This will be

$$\Gamma_L = \frac{50 + j30 - 50}{50 + j30 + 50} = .0825 + j0.275 = \underline{0.287\angle 1.28 \text{ rad}}$$

d) $V_s^-(z) = \Gamma_L V_s^+(z) e^{j2\beta z} = 0.287(200)e^{j\pi z}e^{j1.28} = \underline{57.5e^{j(\pi z + 1.28)}}$

e) $V_s$ at $z = -2.2$ m:

$$V_s(-2.2) = V_s^+(-2.2) + V_s^-(-2.2) = 200e^{j2.2\pi} + 57.5e^{-j(2.2\pi - 1.28)} = 257.5e^{j0.63}$$
$$= \underline{257.5\angle 36^\circ}$$

**10.14.** A lossless transmission line having characteristic impedance $Z_0 = 50$ ohms is driven by a source at the input end that consists of the series combination of a 10-V sinusoidal generator and a 50-ohm resistor. The line is one-quarter wavelength long. At the other end of the line, a load impedance, $Z_L = 50 - j50$ ohms is attached.

a) Evaluate the input impedance to the line seen by the voltage source-resistor combination: For a quarter-wave section,

$$Z_{in} = \frac{Z_0^2}{Z_L} = \frac{(50)^2}{50 - j50} = \underline{25 + j25 \text{ ohms}}$$

b) Evaluate the power that is dissipated by the load: This will be the same as the power dissipated by $Z_{in}$, assuming we replace the line-load section by a lumped element of impedance $Z_{in}$. The voltage across $Z_{in}$ will be

$$V_{in} = V_{s0}\frac{Z_{in}}{Z_g + Z_{in}} = 10\left[\frac{25 + j25}{50 + 25 + j25}\right] = 10 + j5$$

The power will be

$$P_{in} = P_L = \frac{1}{2}\mathcal{R}e\left\{\frac{V_{in}V_{in}^*}{Z_{in}^*}\right\} = \frac{1}{2}\mathcal{R}e\left\{\frac{(10 + j5)(10 - j5)}{25 - j25}\right\} = \underline{1.25 \text{ W}}$$

c) Evaluate the voltage amplitude that appears across the load: The phasor voltage at any point in the line is given by the sum of forward and backward waves:

$$V_s(z) = V_0^+ e^{-j\beta z} + V_0^- e^{+j\beta z}$$

where $V_0^- = \Gamma_L V_0^+$, and where

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50 - j50 - 50}{50 - j50 + 50} = 0.2 - j0.4$$

By our convention, the load is located at $z = 0$. The voltage at the line input, $V_{in}$, is therefore given by the above voltage expression evaluated at $z = -\ell$, where $\ell = -\lambda/4$. Thus $\beta\ell = \pi/2$, and

$$V_{in} = V_s(-\ell) = V_0^+\left[e^{j\beta\ell} + \Gamma_L\, e^{-j\beta\ell}\right] = V_0^+\left[j + (0.2 - j0.4)(-j)\right] = V_0^+(-0.4 + j0.8)$$

Using $V_{in}$ from part $b$, we have

$$V_0^+ = \frac{(10 + j5)}{(-0.4 + j0.8)}$$

Now, the voltage at the load will be

$$V_L = V_0^+(1 + \Gamma_L) = \frac{(10 + j5)}{(-0.4 + j0.8)}(1 + 0.2 - j0.4) = \underline{-5 + j15 \text{ V}}$$

As a check,

$$P_L = \frac{1}{2}\mathcal{R}e\left\{\frac{V_L V_L^*}{Z_L^*}\right\} = \frac{1}{2}\mathcal{R}e\left\{\frac{(-5 + j15)(-5 - j15)}{50 + j50}\right\} = 1.25 \text{ W}$$

which is in agreement with part $b$.

**10.15.** For the transmission line represented in Fig. 10.29, find $V_{s,out}$ if $f =$:

a) 60 Hz: At this frequency,

$$\beta = \frac{\omega}{v_p} = \frac{2\pi \times 60}{(2/3)(3 \times 10^8)} = 1.9 \times 10^{-6} \text{ rad/m So } \beta l = (1.9 \times 10^{-6})(80) = 1.5 \times 10^{-4} << 1$$

The line is thus essentially a lumped circuit, where $Z_{in} \doteq Z_L = 80\ \Omega$. Therefore

$$V_{s,out} = 120 \left[ \frac{80}{12 + 80} \right] = \underline{104 \text{ V}}$$

b) 500 kHz: In this case

$$\beta = \frac{2\pi \times 5 \times 10^5}{2 \times 10^8} = 1.57 \times 10^{-2} \text{ rad/s So } \beta l = 1.57 \times 10^{-2}(80) = 1.26 \text{ rad}$$

Now

$$Z_{in} = 50 \left[ \frac{80 \cos(1.26) + j50 \sin(1.26)}{50 \cos(1.26) + j80 \sin(1.26)} \right] = 33.17 - j9.57 = 34.5\angle - .28$$

The equivalent circuit is now the voltage source driving the series combination of $Z_{in}$ and the 12 ohm resistor. The voltage across $Z_{in}$ is thus

$$V_{in} = 120 \left[ \frac{Z_{in}}{12 + Z_{in}} \right] = 120 \left[ \frac{33.17 - j9.57}{12 + 33.17 - j9.57} \right] = 89.5 - j6.46 = 89.7\angle - .071$$

The voltage at the line input is now the sum of the forward and backward-propagating waves just to the right of the input. We reference the load at $z = 0$, and so the input is located at $z = -80$ m. In general we write $V_{in} = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z}$, where

$$V_0^- = \Gamma_L V_0^+ = \frac{80 - 50}{80 + 50} V_0^+ = \frac{3}{13} V_0^+$$

At $z = -80$ m we thus have

$$V_{in} = V_0^+ \left[ e^{j1.26} + \frac{3}{13} e^{-j1.26} \right] \Rightarrow V_0^+ = \frac{89.5 - j6.46}{e^{j1.26} + (3/13)e^{-j1.26}} = 42.7 - j100 \text{ V}$$

Now

$$V_{s,out} = V_0^+(1 + \Gamma_L) = (42.7 - j100)(1 + 3/(13)) = 134\angle - 1.17 \text{ rad} = \underline{52.6 - j123 \text{ V}}$$

As a check, we can evaluate the average power reaching the load:

$$P_{avg,L} = \frac{1}{2} \frac{|V_{s,out}|^2}{R_L} = \frac{1}{2} \frac{(134)^2}{80} = 112 \text{ W}$$

This must be the same power that occurs at the input impedance:

$$P_{avg,in} = \frac{1}{2} \text{Re}\left\{ V_{in} I_{in}^* \right\} = \frac{1}{2} \text{Re}\left\{ (89.5 - j6.46)(2.54 + j0.54) \right\} = 112 \text{ W}$$

where $I_{in} = V_{in}/Z_{in} = (89.5 - j6.46)/(33.17 - j9.57) = 2.54 + j0.54$.

**10.16.** A 100-Ω lossless transmission line is connected to a second line of 40-Ω impedance, whose length is $\lambda/4$. The other end of the short line is terminated by a 25-Ω resistor. A sinusoidal wave (of frequency $f$) having 50 W average power is incident from the 100-Ω line.

a) Evaluate the input impedance to the quarter-wave line: For the quarter-wave section,

$$Z_{in} = \frac{Z_{02}^2}{Z_L} = \frac{(40)^2}{25} = \underline{64 \text{ ohms}}$$

b) Determine the steady state power that is dissipated by the resistor: This will be the same as the power dropped across a lumped element of impedance $Z_{in}$ at the junction, which replaces the termimated 40-ohm line. The reflection coefficient at the junction is

$$\Gamma_{in} = \frac{Z_{in} - Z_{01}}{Z_{in} + Z_{01}} = \frac{64 - 100}{64 + 100} = -\frac{9}{41}$$

The dissipated power there is then

$$P_{in} = P_L = 50\left(1 - |\Gamma_{in}|^2\right) = 50\left(1 - \left(\frac{9}{41}\right)^2\right) = \underline{47.6 \text{ W}}$$

c) Now suppose the operating frequency is lowered to one-half its original value. Determine the new input impedance, $Z'_{in}$, for this case: Halving the frequency doubles the wavelength, so that now the 40-ohm section is of length $\ell = \lambda/8$. $\beta\ell$ is now $\pi/4$, and the input impedance, from Eq. (98) is:

$$Z'_{in} = 40\left[\frac{25\cos(\pi/4) + j40\sin(\pi/4)}{40\cos(\pi/4) + j25\sin(\pi/4)}\right] = \underline{36.0 + j17.5 \text{ ohms}}$$

d) For the new frequency, calculate the power in watts that returns to the input end of the line after reflection: The new reflection coefficient is

$$\Gamma'_{in} = \frac{Z'_{in} - Z_{01}}{Z'_{in} + Z_{01}} = \frac{36.0 + j17.5 - 100}{36.0 + j17.5 + 100} = -0.447 + j0.186$$

The reflected power (all of which returns to the input) is

$$P_{ref} = 50|\Gamma'_{in}|^2 = 50(0.234) = \underline{11.7 \text{ W}}$$

**10.17.** Determine the average power absorbed by each resistor in Fig. 10.30: The problem is made easier by first converting the current source/100 ohm resistor combination to its Thevenin equivalent. This is a $50\angle 0$ V voltage source in series with the 100 ohm resistor. The next step is to determine the input impedance of the $2.6\lambda$ length line, terminated by the 25 ohm resistor: We use $\beta l = (2\pi/\lambda)(2.6\lambda) = 16.33$ rad. This value, modulo $2\pi$ is (by subtracting $2\pi$ twice) 3.77 rad. Now

$$Z_{in} = 50\left[\frac{25\cos(3.77) + j50\sin(3.77)}{50\cos(3.77) + j25\sin(3.77)}\right] = 33.7 + j24.0$$

The equivalent circuit now consists of the series combination of 50 V source, 100 ohm resistor, and $Z_{in}$, as calculated above. The current in this circuit will be

$$I = \frac{50}{100 + 33.7 + j24.0} = 0.368\angle - .178$$

The power dissipated by the 25 ohm resistor is the same as the power dissipated by the real part of $Z_{in}$, or

$$P_{25} = P_{33.7} = \frac{1}{2}|I|^2 R = \frac{1}{2}(.368)^2(33.7) = \underline{2.28 \text{ W}}$$

To find the power dissipated by the 100 ohm resistor, we need to return to the Norton configuration, with the original current source in parallel with the 100 ohm resistor, and in parallel with $Z_{in}$. The voltage across the 100 ohm resistor will be the same as that across $Z_{in}$, or $V = IZ_{in} = (.368\angle - .178)(33.7 + j24.0) = 15.2\angle 0.44$. The power dissipated by the 100 ohm resistor is now

$$P_{100} = \frac{1}{2}\frac{|V|^2}{R} = \frac{1}{2}\frac{(15.2)^2}{100} = \underline{1.16 \text{ W}}$$

**10.18** The line shown in Fig. 10.31 is lossless. Find $s$ on both sections 1 and 2: For section 2, we consider the propagation of one forward and one backward wave, comprising the superposition of all reflected waves from both ends of the section. The ratio of the backward to the forward wave amplitude is given by the reflection coefficient at the load, which is

$$\Gamma_L = \frac{50 - j100 - 50}{50 - j100 + 50} = \frac{-j}{1-j} = \frac{1}{2}(1-j)$$

Then $|\Gamma_L| = (1/2)\sqrt{(1-j)(1+j)} = 1/\sqrt{2}$. Finally

$$s_2 = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} = \frac{1+1/\sqrt{2}}{1-1/\sqrt{2}} = \underline{5.83}$$

For section 1, we need the reflection coefficient at the junction (location of the 100 Ω resistor) seen by waves incident from section 1: We first need the input impedance of the $.2\lambda$ length of section 2:

$$Z_{in2} = 50\left[\frac{(50-j100)\cos(\beta_2 l) + j50\sin(\beta_2 l)}{50\cos(\beta_2 l) + j(50-j100)\sin(\beta_2 l)}\right] = 50\left[\frac{(1-j2)(0.309) + j0.951}{0.309 + j(1-j2)(0.951)}\right]$$
$$= 8.63 + j3.82 = 9.44\angle 0.42\,\text{rad}$$

Now, this impedance is in parallel with the 100Ω resistor, leading to a net junction impedance found by

$$\frac{1}{Z_{inT}} = \frac{1}{100} + \frac{1}{8.63 + j3.82} \quad \Rightarrow \quad Z_{inT} = 8.06 + j3.23 = 8.69\angle 0.38\,\text{rad}$$

The reflection coefficient will be

$$\Gamma_j = \frac{Z_{inT} - 50}{Z_{inT} + 50} = -0.717 + j0.096 = 0.723\angle 3.0\,\text{rad}$$

and the standing wave ratio is $s_1 = (1+0.723)/(1-0.723) = \underline{6.22}$.

**10.19.** A lossless transmission line is 50 cm in length and operating at a frequency of 100 MHz. The line parameters are $L = 0.2\ \mu\text{H/m}$ and $C = 80$ pF/m. The line is terminated by a short circuit at $z = 0$, and there is a load, $Z_L = 50 + j20$ ohms across the line at location $z = -20$ cm. What average power is delivered to $Z_L$ if the input voltage is $100\angle 0$ V? With the given capacitance and inductance, we find

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{2 \times 10^{-7}}{8 \times 10^{-11}}} = 50\ \Omega$$

and

$$v_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(2 \times 10^{-7})(9 \times 10^{-11})}} = 2.5 \times 10^8 \text{ m/s}$$

Now $\beta = \omega/v_p = (2\pi \times 10^8)/(2.5 \times 10^8) = 2.5$ rad/s. We then find the input impedance to the shorted line section of length 20 cm (putting this impedance at the location of $Z_L$, so we can combine them): We have $\beta l = (2.5)(0.2) = 0.50$, and so, using the input impedance formula with a zero load impedance, we find $Z_{in1} = j50\tan(0.50) = j27.4$ ohms. Now, at the location of $Z_L$, the net impedance there is the parallel combination of $Z_L$ and $Z_{in1}$: $Z_{net} = (50+j20)||(j27.4) = 7.93+j19.9$. We now transform this impedance to the line input, 30 cm to the left, obtaining (with $\beta l = (2.5)(.3) = 0.75$):

$$Z_{in2} = 50\left[\frac{(7.93 + j19.9)\cos(.75) + j50\sin(.75)}{50\cos(.75) + j(7.93 + j19.9)\sin(.75)}\right] = 35.9 + j98.0 = 104.3\angle 1.22$$

The power delivered to $Z_L$ is the same as the power delivered to $Z_{in2}$: The current magnitude is $|I| = (100)/(104.3) = 0.96$ A. So finally,

$$P = \frac{1}{2}|I|^2 R = \frac{1}{2}(0.96)^2(35.9) = \underline{16.5 \text{ W}}$$

**10.20** a) Determine $s$ on the transmission line of Fig. 10.32. Note that the dielectric is air: The reflection coefficient at the load is

$$\Gamma_L = \frac{40 + j30 - 50}{40 + j30 + 50} = j0.333 = 0.333\angle 1.57\,\text{rad} \quad \text{Then} \quad s = \frac{1 + .333}{1 - .333} = \underline{2.0}$$

b) Find the input impedance: With the length of the line at $2.7\lambda$, we have $\beta l = (2\pi)(2.7) = 16.96$ rad. The input impedance is then

$$Z_{in} = 50 \left[ \frac{(40 + j30)\cos(16.96) + j50\sin(16.96)}{50\cos(16.96) + j(40 + j30)\sin(16.96)} \right] = 50 \left[ \frac{-1.236 - j5.682}{1.308 - j3.804} \right] = \underline{61.8 - j37.5\ \Omega}$$

c) If $\omega L = 10\,\Omega$, find $I_s$: The source drives a total impedance given by $Z_{net} = 20 + j\omega L + Z_{in} = 20 + j10 + 61.8 - j37.5 = 81.8 - j27.5$. The current is now $I_s = 100/(81.8 - j27.5) = \underline{1.10 + j0.37\,\text{A}}$.

d) What value of $L$ will produce a maximum value for $|I_s|$ at $\omega = 1$ Grad/s? To achieve this, the imaginary part of the total impedance of part $c$ must be reduced to zero (so we need an inductor). The inductor impedance must be equal to negative the imaginary part of the line input impedance, or $\omega L = 37.5$, so that $L = 37.5/\omega = \underline{37.5\,\text{nH}}$. Continuing, for this value of $L$, calculate the average power:

e) supplied by the source: $P_s = (1/2)\text{Re}\{V_s I_s^*\} = (1/2)(100)(1.10) = \underline{55.0\,\text{W}}$.

f) delivered to $Z_L = 40 + j30\,\Omega$: The power delivered to the load will be the same as the power delivered to the input impedance. We write

$$P_L = \frac{1}{2}\text{Re}\{Z_{in}\}|I_s|^2 = \frac{1}{2}(61.8)[(1.10 + j.37)(1.10 - j.37)] = \underline{41.6\,\text{W}}$$

**10.21.** A lossless line having an air dielectric has a characteristic impedance of 400 $\Omega$. The line is operating at 200 MHz and $Z_{in} = 200 - j200\ \Omega$. Use analytic methods or the Smith chart (or both) to find: (a) $s$; (b) $Z_L$ if the line is 1 m long; (c) the distance from the load to the nearest voltage maximum: I will first use the analytic approach. Using normalized impedances, Eq. (13) becomes

$$z_{in} = \frac{Z_{in}}{Z_0} = \left[ \frac{z_L\cos(\beta L) + j\sin(\beta L)}{\cos(\beta L) + jz_L\sin(\beta L)} \right] = \left[ \frac{z_L + j\tan(\beta L)}{1 + jz_L\tan(\beta L)} \right]$$

Solve for $z_L$:

$$z_L = \left[ \frac{z_{in} - j\tan(\beta L)}{1 - jz_{in}\tan(\beta L)} \right]$$

where, with $\lambda = c/f = 3 \times 10^8/2 \times 10^8 = 1.50$ m, we find $\beta L = (2\pi)(1)/(1.50) = 4.19$, and so $\tan(\beta L) = 1.73$. Also, $z_{in} = (200 - j200)/400 = 0.5 - j0.5$. So

$$z_L = \frac{0.5 - j0.5 - j1.73}{1 - j(0.5 - j0.5)(1.73)} = 2.61 + j0.174$$

Finally, $Z_L = z_L(400) = \underline{1.04 \times 10^3 + j69.8\ \Omega}$. Next

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{6.42 \times 10^2 + j69.8}{1.44 \times 10^3 + j69.8} = .446 + j2.68 \times 10^{-2} = .447\angle 6.0 \times 10^{-2}\ \text{rad}$$

**10.21.** (continued) Now

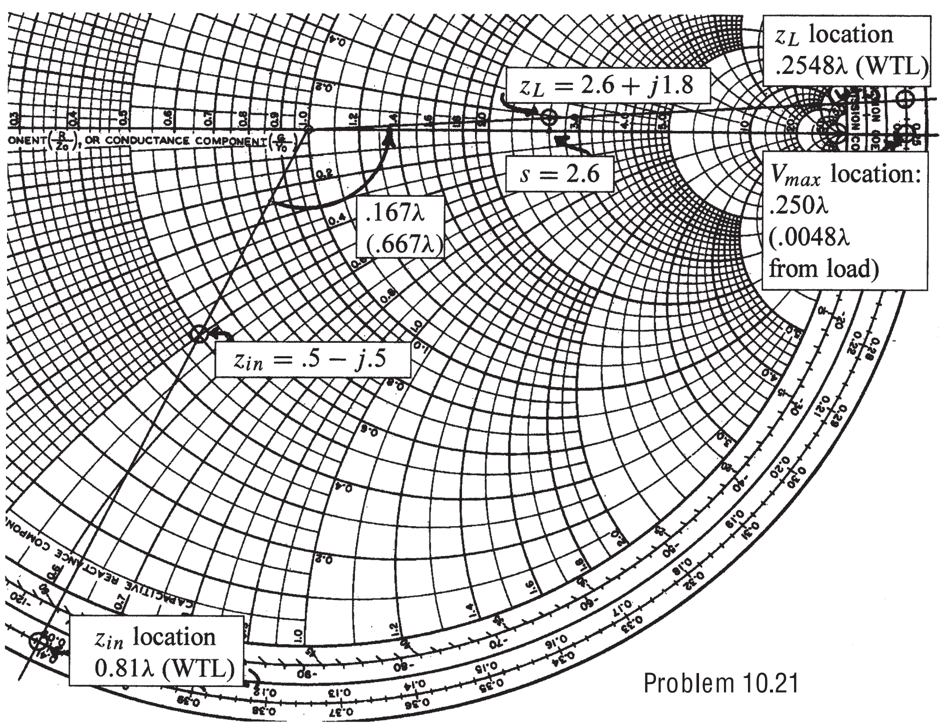
$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + .447}{1 - .447} = \underline{2.62}$$

Finally

$$z_{max} = -\frac{\phi}{2\beta} = -\frac{\lambda\phi}{4\pi} = -\frac{(6.0 \times 10^{-2})(1.50)}{4\pi} = -7.2 \times 10^{-3} \text{ m} = \underline{-7.2 \text{ mm}}$$

We next solve the problem using the Smith chart. Referring to the figure below, we first locate and mark the normalized input impedance, $z_{in} = 0.5 - j0.5$. A line drawn from the origin through this point intersects the outer chart boundary at the position $0.0881\,\lambda$ on the wavelengths toward load (WTL) scale. With a wavelength of 1.5 m, the 1 meter line is 0.6667 wavelengths long. On the WTL scale, we add $0.6667\lambda$, or equivalently, $0.1667\lambda$ (since $0.5\lambda$ is once around the chart), obtaining $(0.0881 + 0.1667)\lambda) = 0.2548\lambda$, which is the position of the load. A straight line is now drawn from the origin though the $0.2548\lambda$ position. A compass is then used to measure the distance between the origin and $z_{in}$. With this distance set, the compass is then used to scribe off the same distance from the origin to the load impedance, along the line between the origin and the $0.2548\lambda$ position. That point is the normalized load impedance, which is read to be $z_L = 2.6 + j0.18$. Thus $Z_L = z_L(400) = 1040 + j72$. This is in reasonable agreement with the analytic result of $1040 + j69.8$. The difference in imaginary parts arises from uncertainty in reading the chart in that region.

In transforming from the input to the load positions, we cross the $r > 1$ real axis of the chart at r=2.6. This is close to the value of the VSWR, as we found earlier. We also see that the $r > 1$ real axis (at which the first $V_{max}$ occurs) is a distance of $0.0048\lambda$ (marked as $.005\lambda$ on the chart) in front of the load. The actual distance is $z_{max} = -0.0048(1.5)\,\text{m} = -0.0072\,\text{m} = -7.2\,\text{mm}$.



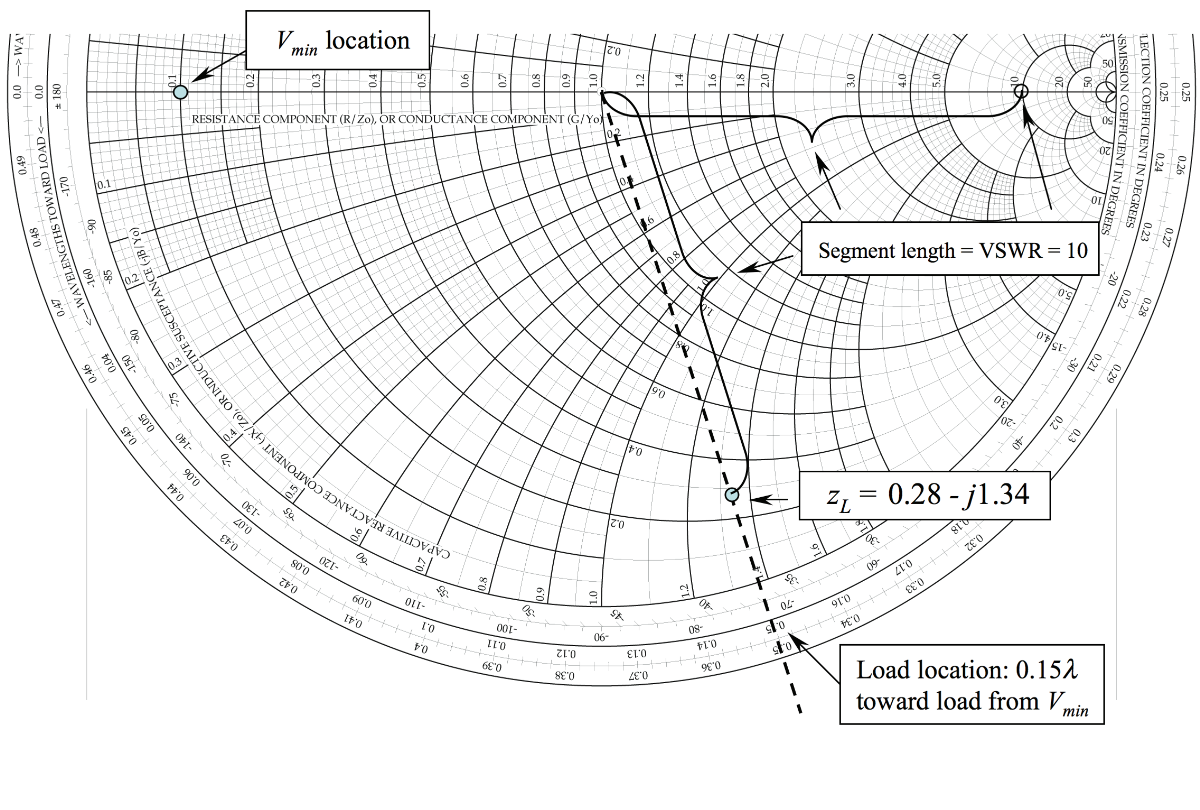

Problem 10.21

**10.22.** A lossless 75-ohm line is terminated by an unknown load impedance. A VSWR of 10 is measured, and the first voltage minimum occurs at a 0.15 wavelengths in front of the load. Using the Smith chart, find

a) The load impedance: Referring to the Smith chart section below, first mark the VSWR on the positive real axis and set the compass to that length. The voltage minimum will be located on the negative real axis and will have normalized impedance of the reciprocal of the VSWR, or 0.1. This value is marked and is labled as the $V_{min}$ location. Now, move toward the load by a distance of 0.15 wavelenths (using the wavelenths toward load scale). The dashed line is drawn from the origin through the 0.15$\lambda$ mark on the scale. Use the compass (set to the VSWR length) to scribe the point on the dashed line that is labled $z_L$. We identify that as the normalized load impedance, $z_L = 0.28 - j1.34$. The load impedance is then $Z_L = 75z_L = \underline{21.0 - j100}$ ohms

b) The magnitude and phase of the reflection coefficient: The magnitude of $\Gamma_L$ can be found by measuring the compass span on the linear "Ref. coeff. E or I" scale on the bottom of the chart. Set the compass point at the center position, and then scribe on the scale to the left to find $|\Gamma_L| = 0.82$. The phase is the angle of the dashed line from the positive real axis, which is read from the "angle of reflection coefficient" scale as $\phi = -72°$. In summary, $\Gamma_L = \underline{0.82\angle - 72°}$

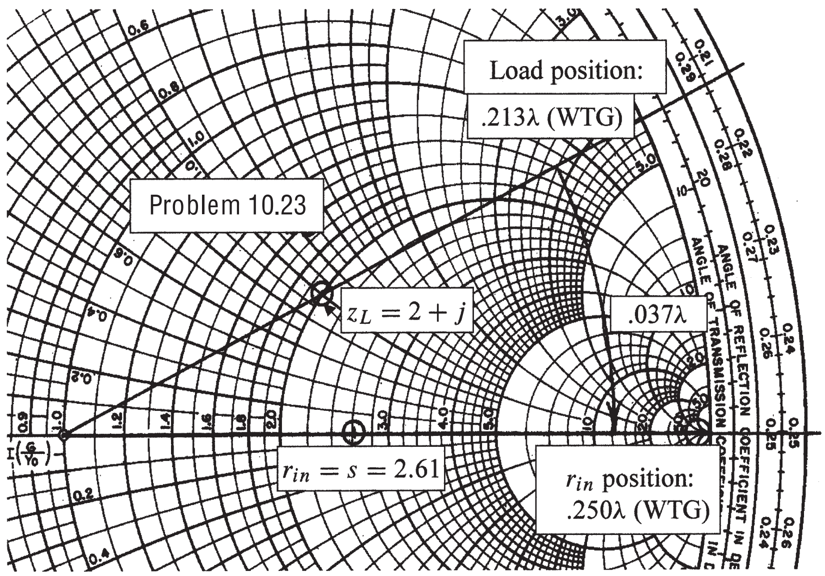
c) The shortest length of line necessary to achieve an entirely resistive input impedance: In moving toward the generator from the load, we look for the first real axis crossing. This occurs simply at the $V_{min}$ location, and so we identify the shortest length as just $\underline{0.15\lambda}$.



**10.23.** The normalized load on a lossless transmission line is $z_L = 2 + j1$. Let $\lambda = 20$ m Make use of the Smith chart to find:

a) the shortest distance from the load to the point at which $z_{in} = r_{in} + j0$, where $r_{in} > 1$ (not greater than 0 as stated): Referring to the figure below, we start by marking the given $z_L$ on the chart and drawing a line from the origin through this point to the outer boundary. On the WTG scale, we read the $z_L$ location as $0.213\lambda$. Moving from here toward the generator, we cross the positive $\Gamma_R$ axis (at which the impedance is purely real and greater than 1) at $0.250\lambda$. The distance is then $(0.250 - 0.213)\lambda = \underline{0.037\lambda}$ from the load. With $\lambda = 20$ m, the actual distance is $20(0.037) = 0.74$ m.

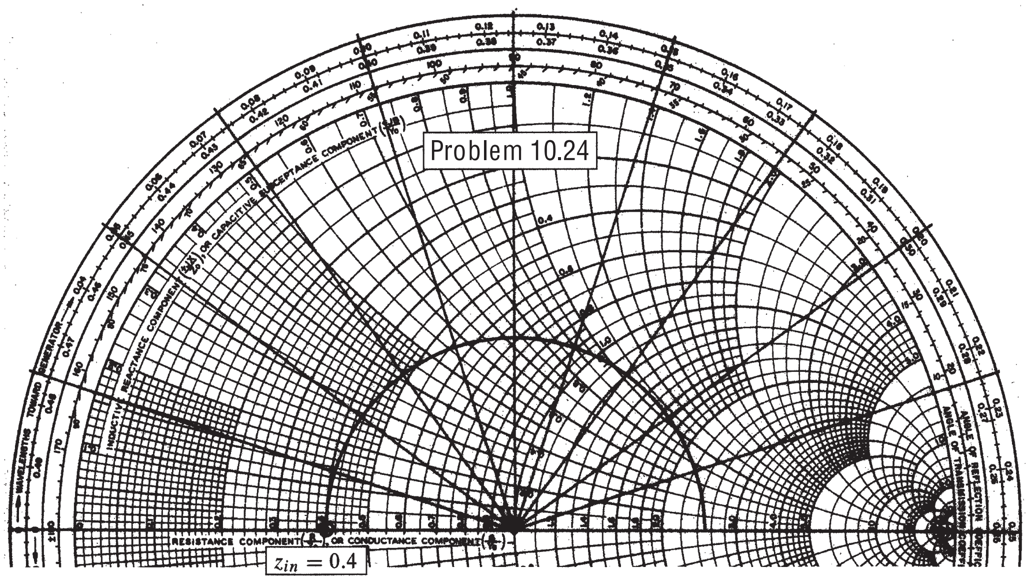
b) Find $z_{in}$ at the point found in part $a$: Using a compass, we set its radius at the distance between the origin and $z_L$. We then scribe this distance along the real axis to find $z_{in} = r_{in} = \underline{2.61}$.




c) The line is cut at this point and the portion containing $z_L$ is thrown away. A resistor $r = r_{in}$ of part $a$ is connected across the line. What is $s$ on the remainder of the line? This will be just $s$ for the line as it was before. As we know, $s$ will be the positive real axis value of the normalized impedance, or $s = \underline{2.61}$.

d) What is the shortest distance from this resistor to a point at which $z_{in} = 2 + j1$? This would return us to the original point, requiring a complete circle around the chart (one-half wavelength distance). The distance from the resistor will therefore be: $d = 0.500\,\lambda - 0.037\,\lambda = \underline{0.463\,\lambda}$. With $\lambda = 20$ m, the actual distance would be $20(0.463) = 9.26$ m.

**10.24.** With the aid of the Smith chart, plot a curve of $|Z_{in}|$ vs. $l$ for the transmission line shown in Fig. 10.33. Cover the range $0 < l/\lambda < 0.25$. The required input impedance is that at the actual line input (to the left of the two 20Ω resistors. The input to the line section occurs just to the right of the 20Ω resistors, and the input impedance there we first find with the Smith chart. This impedance is in series with the two 20Ω resistors, so we add 40Ω to the calculated impedance from the Smith chart to find the net line input impedance. To begin, the 20Ω load resistor represents a normalized impedance of $z_l = 0.4$, which we mark on the chart (see below). Then, using a compass, draw a circle beginning at $z_L$ and progressing clockwise to the positive real axis. The circle traces the locus of $z_{in}$ values for line lengths over the range $0 < l < \lambda/4$.




On the chart, radial lines are drawn at positions corresponding to $.025\lambda$ increments on the WTG scale. The intersections of the lines and the circle give a total of 11 $z_{in}$ values. To these we add normalized impedance of $40/50 = 0.8$ to add the effect of the 40Ω resistors and obtain the normalized impedance at the line input. The magnitudes of these values are then found, and the results are multiplied by 50Ω. The table below summarizes the results.

| $l/\lambda$ | $z_{inl}$ (to right of 40Ω) | $z_{in} = z_{inl} + 0.8$ | $\|Z_{in}\| = 50\|z_{in}\|$ |
|---|---|---|---|
| 0 | 0.40 | 1.20 | 60 |
| .025 | 0.41 + j.13 | 1.21 + j.13 | 61 |
| .050 | 0.43 + j.27 | 1.23 + j.27 | 63 |
| .075 | 0.48 + j.41 | 1.28 + j.41 | 67 |
| .100 | 0.56 + j.57 | 1.36 + j.57 | 74 |
| .125 | 0.68 + j.73 | 1.48 + j.73 | 83 |
| .150 | 0.90 + j.90 | 1.70 + j.90 | 96 |
| .175 | 1.20 + j1.05 | 2.00 + j1.05 | 113 |
| .200 | 1.65 + j1.05 | 2.45 + j1.05 | 134 |
| .225 | 2.2 + j.7 | 3.0 + j.7 | 154 |
| .250 | 2.5 | 3.3 | 165 |

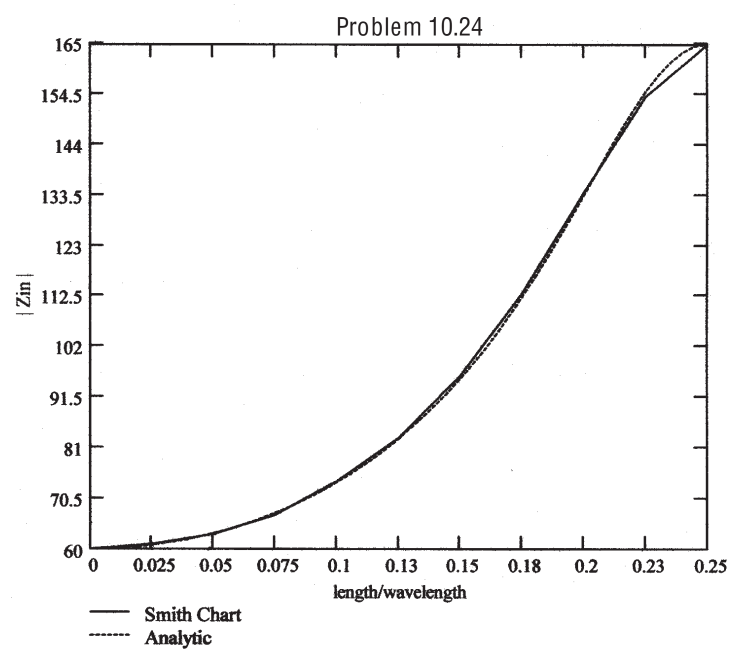
**10.24.** (continued) As a check, the line input input impedance can be found analytically through

$$Z_{in} = 40 + 50\left[\frac{20\cos(2\pi l/\lambda) + j50\sin(2\pi l/\lambda)}{50\cos(2\pi l/\lambda) + j20\sin(2\pi l/\lambda)}\right] = 50\left[\frac{60\cos(2\pi l/\lambda) + j66\sin(2\pi l/\lambda)}{50\cos(2\pi l/\lambda) + j20\sin(2\pi l/\lambda)}\right]$$

from which

$$|Z_{in}| = 50\left[\frac{36\cos^2(2\pi l/\lambda) + 43.6\sin^2(2\pi l/\lambda)}{25\cos^2(2\pi l/\lambda) + 4\sin^2(2\pi l/\lambda)}\right]^{1/2}$$

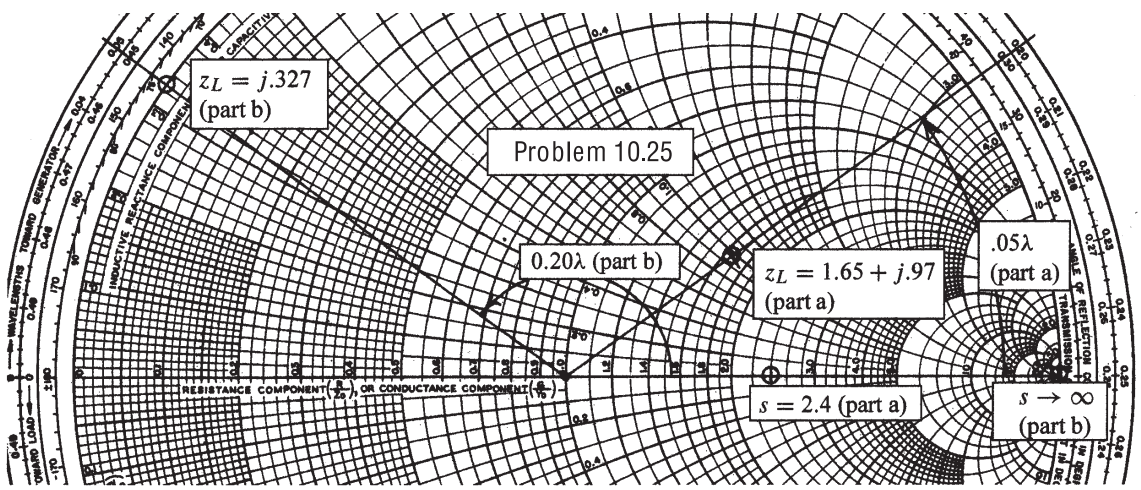
This function is plotted below along with the results obtained from the Smith chart. A fairly good comparison is obtained.



**10.25.** A 300-ohm transmission line is short-circuited at $z = 0$. A voltage maximum, $|V|_{max} = 10$ V, is found at $z = -25$ cm, and the minimum voltage, $|V|_{min} = 0$, is found at $z = -50$ cm. Use the Smith chart to find $Z_L$ (with the short circuit replaced by the load) if the voltage readings are:

a) $|V|_{max} = 12$ V at $z = -5$ cm, and $|V|_{min} = 5$ V: First, we know that the maximum and minimum voltages are spaced by $\lambda/4$. Since this distance is given as 25 cm, we see that $\lambda = 100$ cm $= 1$ m. Thus the maximum voltage location is $5/100 = 0.05\lambda$ in front of the load. The standing wave ratio is $s = |V|_{max}/|V|_{min} = 12/5 = 2.4$. We mark this on the positive real axis of the chart (see next page). The load position is now 0.05 wavelengths toward the load from the $|V|_{max}$ position, or at 0.30 $\lambda$ on the WTL scale. A line is drawn from the origin through this point on the chart, as shown. We next set the compass to the distance between the origin and the $z = r = 2.4$ point on the real axis. We then scribe this same distance along the line drawn through the .30 $\lambda$ position. The intersection is the value of $z_L$, which we read as $z_L = 1.65 + j.97$. The actual load impedance is then $Z_L = 300z_L = \underline{495 + j290\ \Omega}$.

b) $|V|_{max} = 17$ V at $z = -20$ cm, and $|V|_{min} = 0$. In this case the standing wave ratio is infinite, which puts the starting point on the $r \to \infty$ point on the chart. The distance of 20 cm corresponds to $20/100 = 0.20\,\lambda$, placing the load position at $0.45\,\lambda$ on the WTL scale. A line is drawn from the origin through this location on the chart. An infinite standing wave ratio places us on the outer boundary of the chart, so we read $z_L = j0.327$ at the $0.45\,\lambda$ WTL position. Thus $Z_L = j300(0.327) \doteq \underline{j98\,\Omega}$.



**10.26.** A 50-ohm lossless line is of length $1.1\lambda$. It is terminated by an unknown load impedance. The input end of the 50-ohm line is attached to the load end of a lossless 75-ohm line. A VSWR of 4 is measured on the 75-ohm line, on which the first voltage maximum occurs at a distance of $0.2\lambda$ in front of the junction between the two lines. Use the Smith chart to find the unknown load impedance.

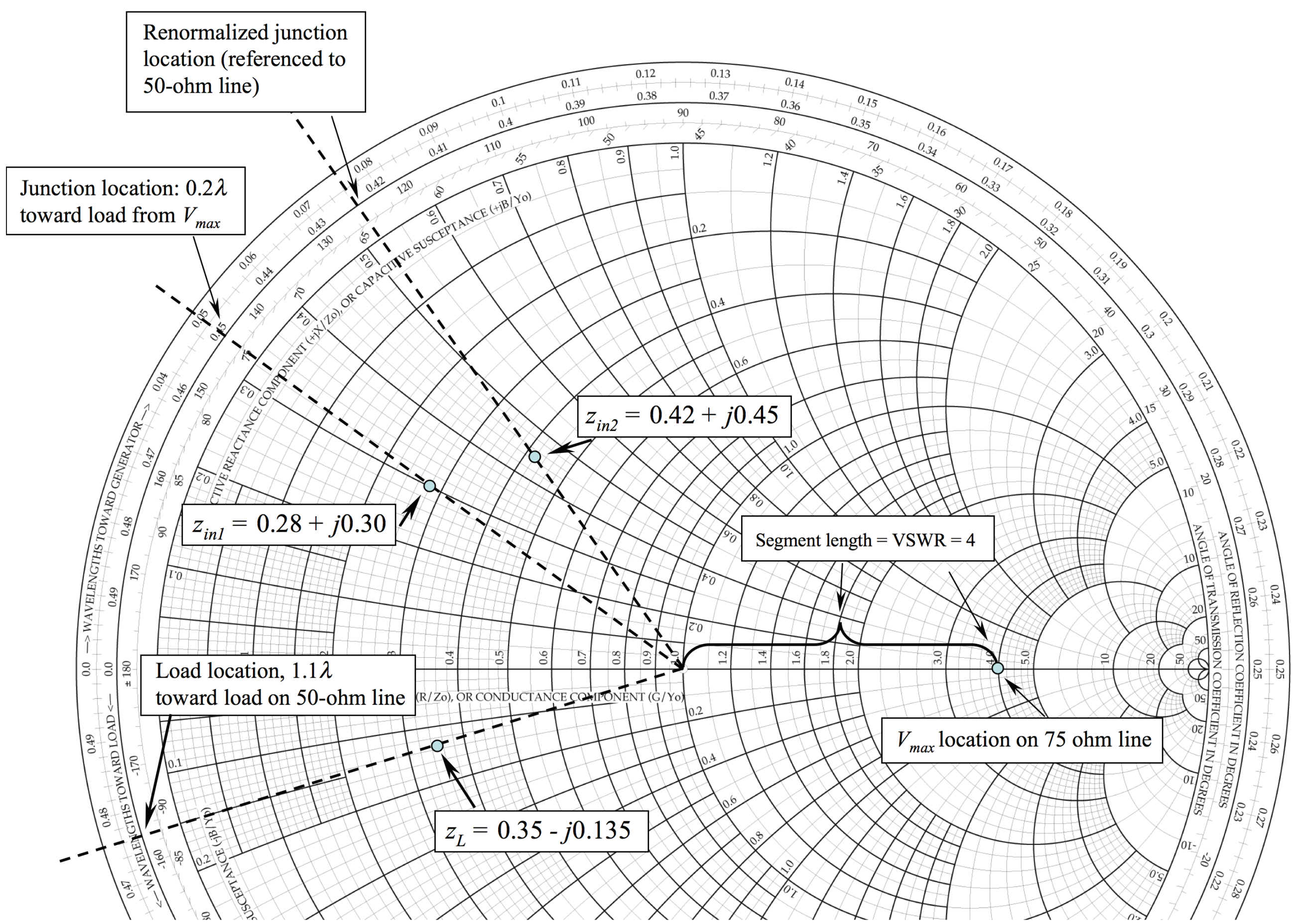
First, mark the VSWR on the positive real axis, which gives the magnitude of $\Gamma$ as determined on the 75-ohm line. The starting point is thus $r = 4, x = 0$, which is the location of the first voltage maximum. From there, move toward the load by 0.2 wavelengths, and note the normalized impedance there, marked as $z_{in1} = 0.28 + j0.30$. This is the normalized load impedance at the junction, as seen by the 75-ohm line.

The next step is to re-normalize $z_{in1}$ to the 50-ohm line to find $z_{in2}$. This will be

$$z_{in2} = z_{in1}\,\frac{75}{50} = 0.42 + j0.45$$

which is marked on the chart as shown. Now, this point is translated toward the load by $1.1\lambda$ (equivalent to $0.1\lambda$) to obtain the normlized load impedance, $z_L = 0.35 - j0.135$, marked on the chart. The load impedance is thus

$$Z_L = 50(0.35 - j0.135) = \underline{17.5 - j6.8} \text{ ohms}$$

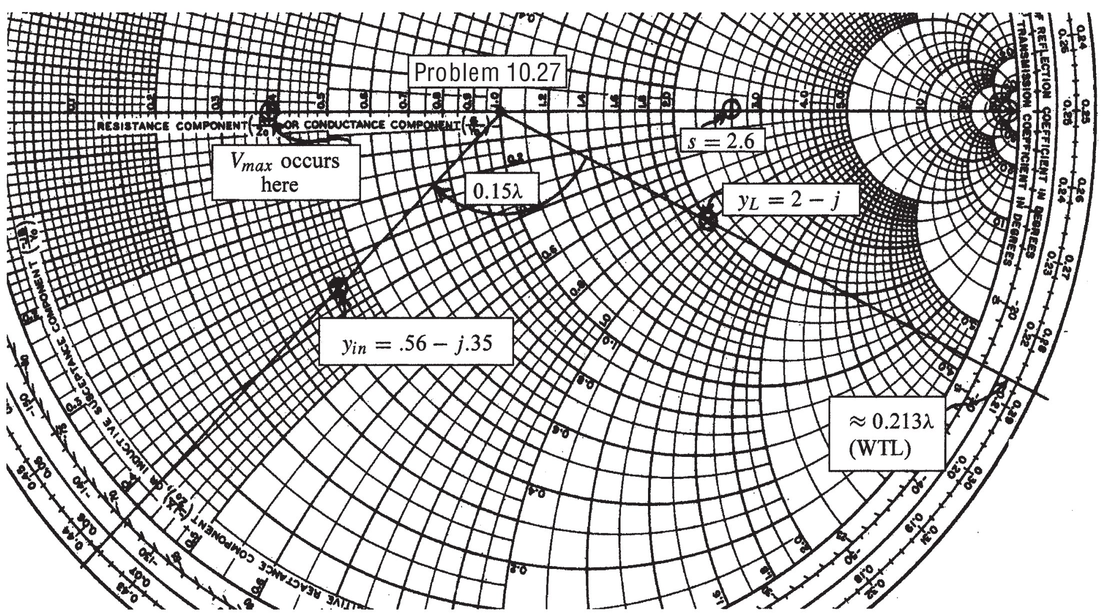


**10.27.** The characteristic admittance ($Y_0 = 1/Z_0$) of a lossless transmission line is 20 mS. The line is terminated in a load $Y_L = 40 - j20$ mS. Make use of the Smith chart to find:

a) $s$: We first find the normalized load admittance, which is $y_L = Y_L/Y_0 = 2 - j1$. This is plotted on the Smith chart below. We then set on the compass the distance between $y_L$ and the origin. The same distance is then scribed along the positive real axis, and the value of $s$ is read as 2.6.

b) $Y_{in}$ if $l = 0.15\,\lambda$: First we draw a line from the origin through $z_L$ and note its intersection with the WTG scale on the chart outer boundary. We note a reading on that scale of about 0.287 $\lambda$. To this we add 0.15 $\lambda$, obtaining about 0.437 $\lambda$, which we then mark on the chart (0.287 $\lambda$ is not the precise value, but I have added 0.15 $\lambda$ to that mark to obtain the point shown on the chart that is near to 0.437 $\lambda$. This "eyeballing" method increases the accuracy a little). A line drawn from the 0.437 $\lambda$ position on the WTG scale to the origin passes through the input admittance. Using the compass, we scribe the distance found in part $a$ across this line to find $y_{in} = 0.56 - j0.35$, or $Y_{in} = 20 y_{in} = 11 - j7.0$ mS.

c) the distance in wavelengths from $Y_L$ to the nearest voltage maximum: On the admittance chart, the $V_{max}$ position is on the negative $\Gamma_r$ axis. This is at the zero position on the WTL scale. The load is at the approximate $0.213\,\lambda$ point on the WTL scale, so this distance is the one we want.

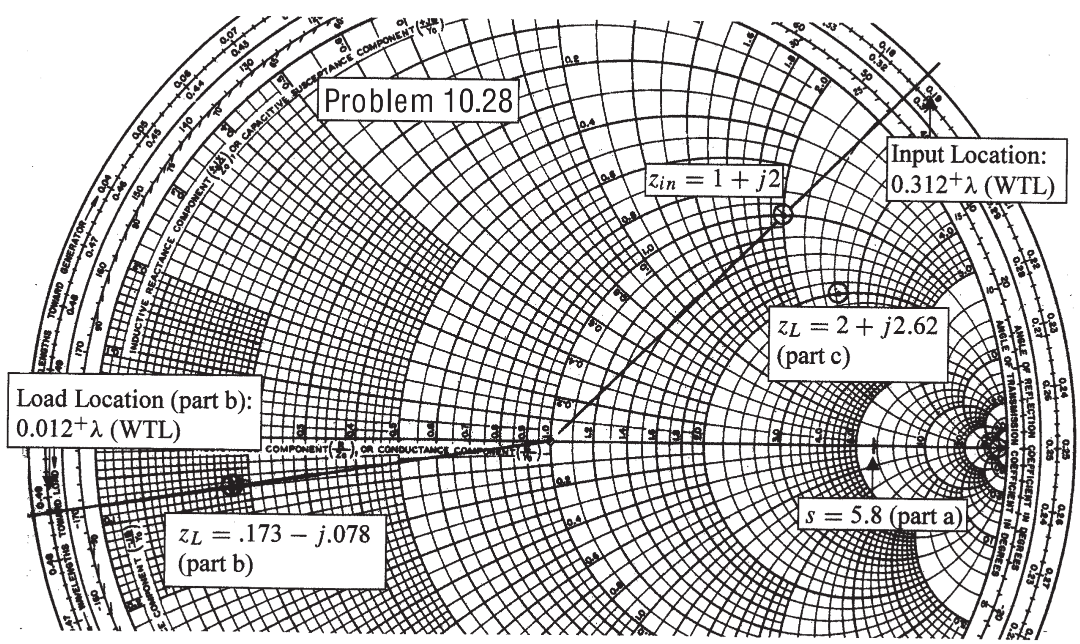


**10.28.** The wavelength on a certain lossless line is 10cm. If the normalized input impedance is $z_{in} = 1 + j2$, use the Smith chart to determine:

a) $s$: We begin by marking $z_{in}$ on the chart (see below), and setting the compass at its distance from the origin. We then use the compass at that setting to scribe a mark on the positive real axis, noting the value there of $s = \underline{5.8}$.

b) $z_L$, if the length of the line is 12 cm: First, use a straight edge to draw a line from the origin through $z_{in}$, and through the outer scale. We read the input location as slightly more than $0.312\lambda$ on the WTL scale (this additional distance beyond the .312 mark is not measured, but is instead used to add a similar distance when the impedance is transformed). The line length of 12cm corresponds to 1.2 wavelengths. Thus, to transform to the load, we go counter-clockwise twice around the chart, plus $0.2\lambda$, finally arriving at (again) slightly more than $0.012\lambda$ on the WTL scale. A line is drawn to the origin from that position, and the compass (with its previous setting) is scribed through the line. The intersection is the normalized load impedance, which we read as $z_L = \underline{0.173 - j0.078}$.

c) $x_L$, if $z_L = 2 + jx_L$, where $x_L > 0$. For this, use the compass at its original setting to scribe through the $r = 2$ circle in the upper half plane. At that point we read $x_L = \underline{2.62}$.



**10.29.** A standing wave ratio of 2.5 exists on a lossless 60 Ω line. Probe measurements locate a voltage minimum on the line whose location is marked by a small scratch on the line. When the load is replaced by a short circuit, the minima are 25 cm apart, and one minimum is located at a point 7 cm toward the source from the scratch. Find $Z_L$: We note first that the 25 cm separation between minima imply a wavelength of twice that, or $\lambda = 50$ cm. Suppose that the scratch locates the first voltage minimum. With the short in place, the first minimum occurs at the load, and the second at 25 cm in front of the load. The effect of replacing the short with the load is to move the minimum at 25 cm to a new location 7 cm toward the load, or at 18 cm. This is a possible location for the scratch, which would otherwise occur at multiples of a half-wavelength farther away from that point, toward the generator. Our assumed scratch position will be 18 cm or 18/50 = 0.36 wavelengths from the load. Using the Smith chart (see below) we first draw a line from the origin through the $0.36\lambda$ point on the wavelengths toward load scale. We set the compass to the length corresponding to the $s = r = 2.5$ point on the chart, and then scribe this distance through the straight line. We read $z_L = 0.79 + j0.825$, from which $Z_L = 47.4 + j49.5\ \Omega$. As a check, I will do the problem analytically. First, we use

$$z_{min} = -18\,\text{cm} = -\frac{1}{2\beta}(\phi + \pi) \quad \Rightarrow \quad \phi = \left[\frac{4(18)}{50} - 1\right]\pi = 1.382\,\text{rad} = 79.2^\circ$$

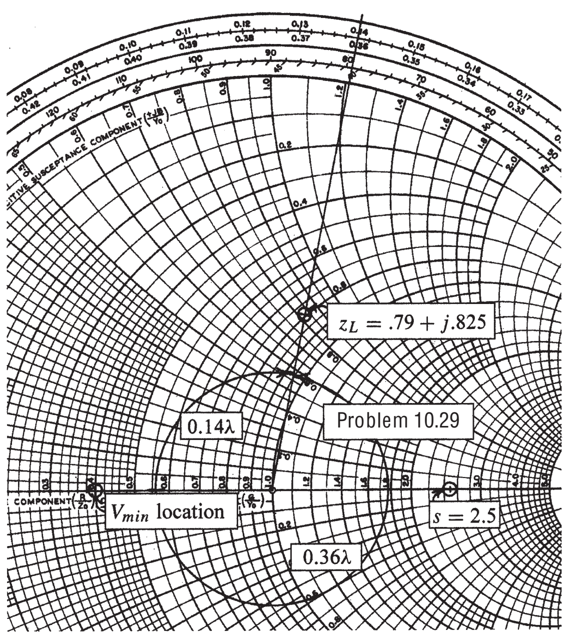
Now

$$|\Gamma_L| = \frac{s-1}{s+1} = \frac{2.5-1}{2.5+1} = 0.4286$$

and so $\Gamma_L = 0.4286\angle 1.382$. Using this, we find

$$z_L = \frac{1+\Gamma_L}{1-\Gamma_L} = 0.798 + j0.823$$

and thus $Z_L = z_L(60) = \underline{47.8 + j49.3\ \Omega}$.

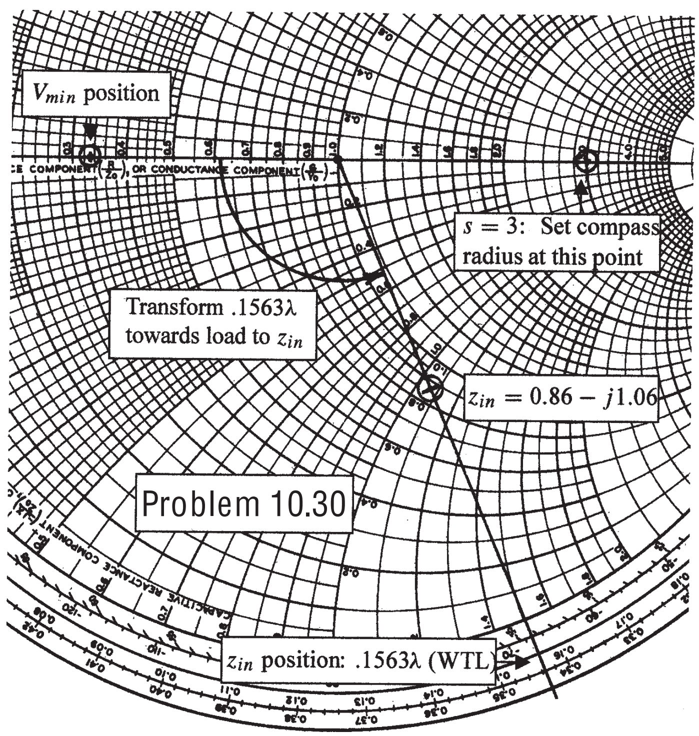


**10.30.** A 2-wire line, constructed of lossless wire of circular cross-section is gradually flared into a coupling loop that looks like an egg beater. At the point $X$, indicated by the arrow in Fig. 10.34, a short circuit is placed across the line. A probe is moved along the line and indicates that the first voltage minimum to the left of $X$ is 16cm from $X$. With the short circuit removed, a voltage minimum is found 5cm to the left of $X$, and a voltage maximum is located that is 3 times voltage of the minimum. Use the Smith chart to determine:

a) $f$: No Smith chart is needed to find $f$, since we know that the first voltage minimum in front of a short circuit is one-half wavelength away. Therefore, $\lambda = 2(16) = 32$cm, and (assuming an air-filled line), $f = c/\lambda = 3 \times 10^8/0.32 = \underline{0.938\,\text{GHz}}$.

b) $s$: Again, no Smith chart is needed, since $s$ is the ratio of the maximum to the minimum voltage amplitudes. Since we are given that $V_{max} = 3V_{min}$, we find $s = \underline{3}$.

c) the normalized input impedance of the egg beater as seen looking the right at point $X$: Now we need the chart. From the figure below, $s = 3$ is marked on the positive real axis, which determines the compass radius setting. This point is then transformed, using the compass, to the negative real axis, which corresponds to the location of a voltage minimum. Since the first $V_{min}$ is 5cm in front of $X$, this corresponds to $(5/32)\lambda = 0.1563\lambda$ to the left of $X$. On the chart, we now move this distance from the $V_{min}$ location toward the load, using the WTL scale. A line is drawn from the origin through the $0.1563\lambda$ mark on the WTL scale, and the compass is used to scribe the original radius through this line. The intersection is the normalized input impedance, which is read as $z_{in} = \underline{0.86 - j1.06}$.



**10.31.** In order to compare the relative sharpness of the maxima and minima of a standing wave, assume a load $z_L = 4 + j0$ is located at $z = 0$. Let $|V|_{min} = 1$ and $\lambda = 1$ m. Determine the width of the

a) minimum, where $|V| < 1.1$: We begin with the general phasor voltage in the line:

$$V(z) = V^+(e^{-j\beta z} + \Gamma e^{j\beta z})$$

With $z_L = 4 + j0$, we recognize the real part as the standing wave ratio. Since the load impedance is real, the reflection coefficient is also real, and so we write

$$\Gamma = |\Gamma| = \frac{s-1}{s+1} = \frac{4-1}{4+1} = 0.6$$

The voltage magnitude is then

$$|V(z)| = \sqrt{V(z)V^*(z)} = V^+ \left[(e^{-j\beta z} + \Gamma e^{j\beta z})(e^{j\beta z} + \Gamma e^{-j\beta z})\right]^{1/2}$$
$$= V^+ \left[1 + 2\Gamma\cos(2\beta z) + \Gamma^2\right]^{1/2}$$

Note that with $\cos(2\beta z) = \pm 1$, we obtain $|V| = V^+(1 \pm \Gamma)$ as expected. With $s = 4$ and with $|V|_{min} = 1$, we find $|V|_{max} = 4$. Then with $\Gamma = 0.6$, it follows that $V^+ = 2.5$. The net expression for $|V(z)|$ is then

$$V(z) = 2.5\sqrt{1.36 + 1.2\cos(2\beta z)}$$

To find the width in $z$ of the voltage minimum, defined as $|V| < 1.1$, we set $|V(z)| = 1.1$ and solve for $z$: We find

$$\left(\frac{1.1}{2.5}\right)^2 = 1.36 + 1.2\cos(2\beta z) \quad \Rightarrow \quad 2\beta z = \cos^{-1}(-0.9726)$$

Thus $2\beta z = 2.904$. At this stage, we note the the $|V|_{min}$ point will occur at $2\beta z = \pi$. We therefore compute the range, $\Delta z$, over which $|V| < 1.1$ through the equation:

$$2\beta(\Delta z) = 2(\pi - 2.904) \quad \Rightarrow \quad \Delta z = \frac{\pi - 2.904}{2\pi/1} = 0.0378\,\text{m} = \underline{3.8\,\text{cm}}$$

where $\lambda = 1$ m has been used.

b) Determine the width of the maximum, where $|V| > 4/1.1$: We use the same equation for $|V(z)|$, which in this case reads:

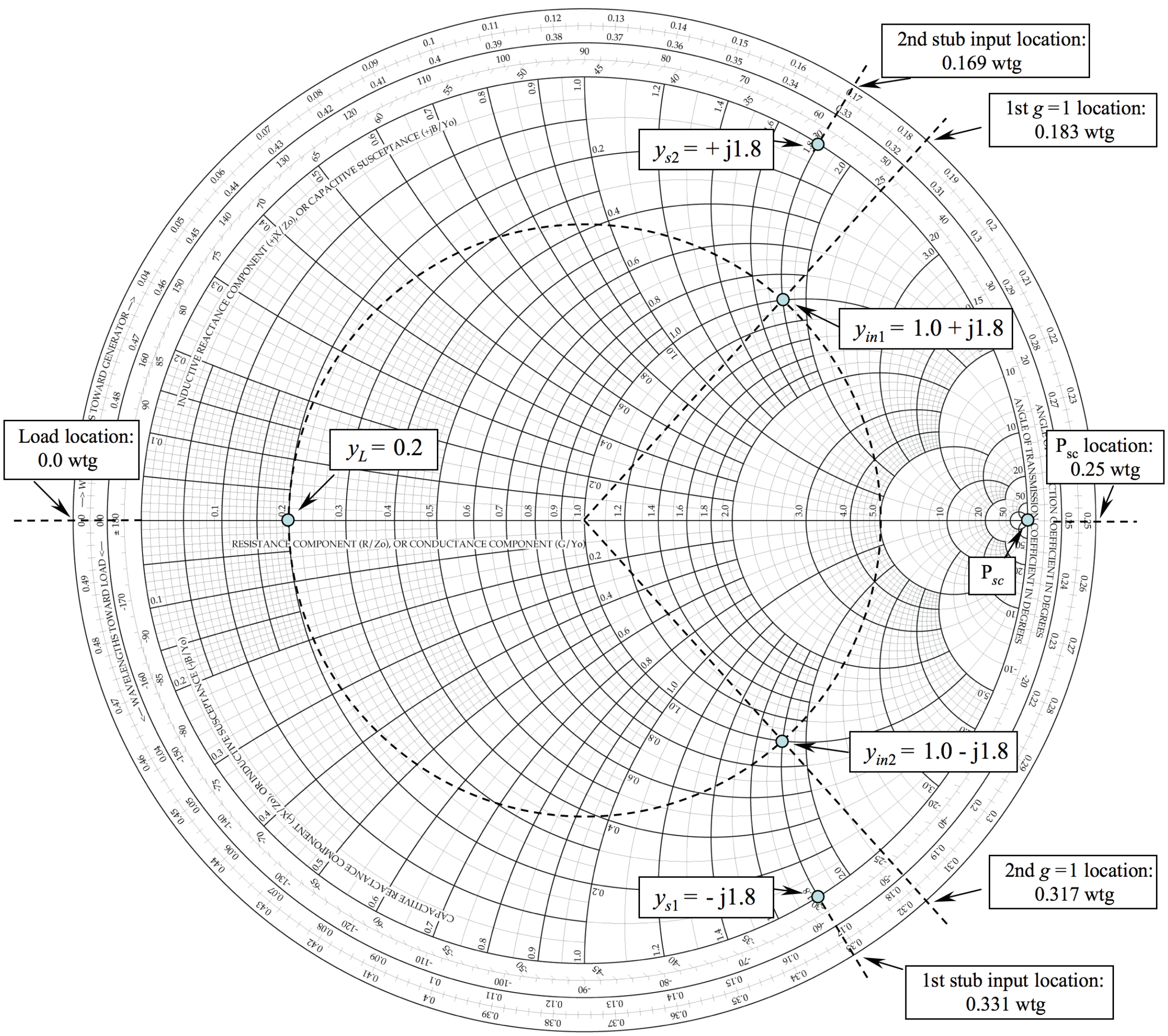
$$4/1.1 = 2.5\sqrt{1.36 + 1.2\cos(2\beta z)} \quad \Rightarrow \quad \cos(2\beta z) = 0.6298$$

Since the maximum corresponds to $2\beta z = 0$, we find the range through

$$2\beta\Delta z = 2\cos^{-1}(0.6298) \quad \Rightarrow \quad \Delta z = \frac{0.8896}{2\pi/1} = 0.142\,\text{m} = \underline{14.2\,\text{cm}}$$

**10.32.** In Fig. 10.7, let $Z_L = 250$ ohms, $Z_0 = 50$ ohms, find the shortest attachment distance $d$ and the shortest length, $d_1$ of a short-circuited stub line that will provide a perfect match on the main line to the left of the stub. Express all answers in wavelengths.

The first step is to mark the normalized load admittance on the chart. This will be $y_L = 1/z_L = 50/250 = 0.20$. Its location is noted as 0.0 on the wavelengths toward generator (WTG) scale. Next, from the load, move clockwise (toward generator) until the admittance real part is unity. The first instance of this is at the point $y_{in1} = 1 + j1.8$, as shown. Moving farther, the second instance is at the point $y_{in2} = 1.0 - j1.8$. The distance in wavelenghs between $y_L$ and $y_{in1}$ is noted on the WTG scale, or $d_a = 0.183\lambda$. The distance in wavelenghs between $y_L$ and $y_{in2}$ is again noted on the WTG scale, or $d_b = 0.317\lambda$. These are the two possible attachment points for the shorted stub. The shortest of these is $\underline{d_a = 0.183\lambda}$. The corresponding stub length is found by transforming from the short circuit (load) position on the stub, $P_{sc}$ toward generator until a normalized admittance of $y_s = b_s = -j1.8$ occurs. This is marked on the chart as the point $y_{s1}$, located at $0.331\lambda$ (WTG). The (shortest) stub length is thus $d_{1a} = (0.331 - 0.250)\lambda = \underline{0.81\lambda}$.



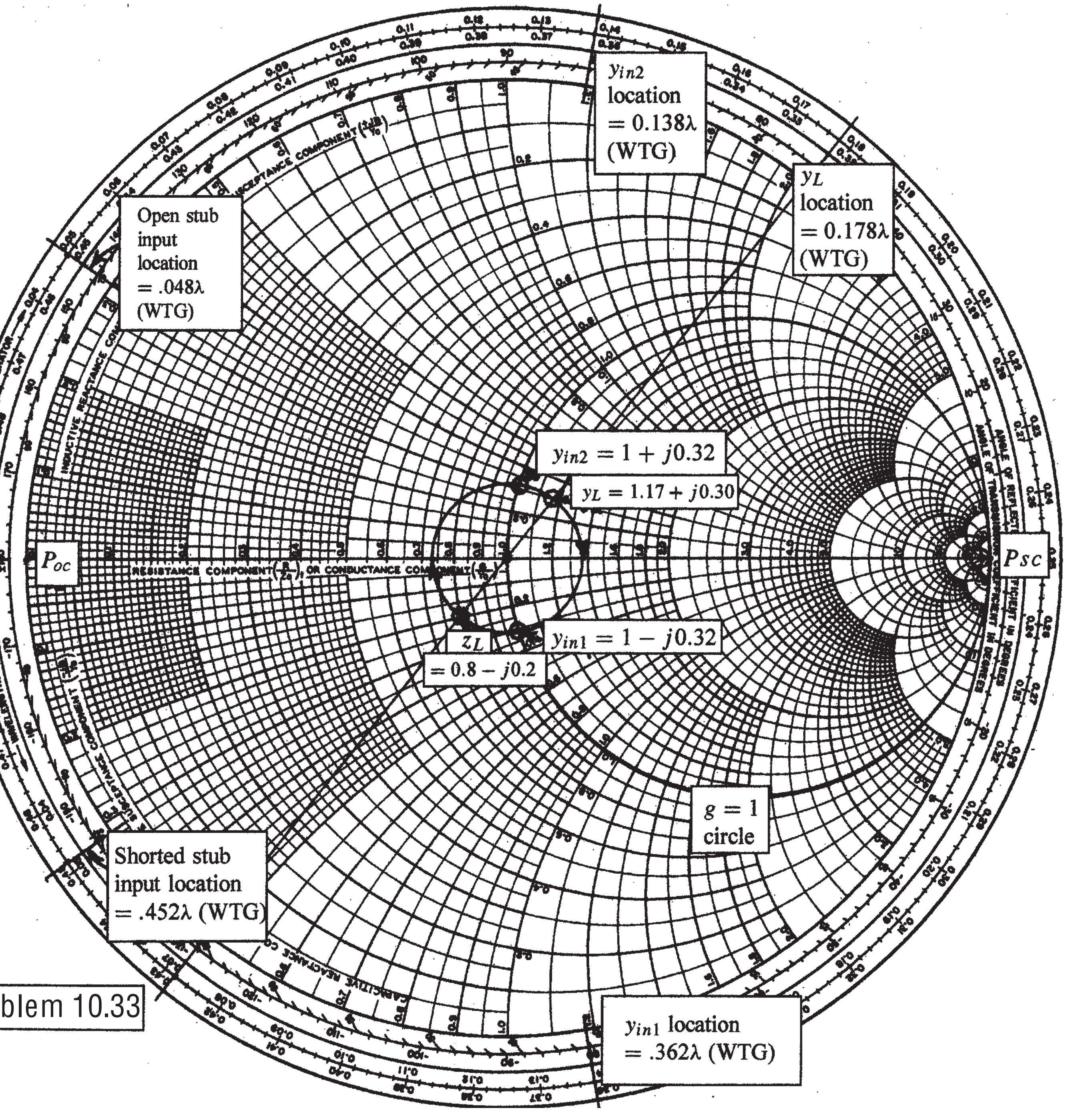
**10.33.** In Fig. 10.17, let $Z_L = 40 - j10\ \Omega$, $Z_0 = 50\ \Omega$, $f = 800$ MHz, and $v = c$.

a) Find the shortest length, $d_1$, of a short-circuited stub, and the shortest distance $d$ that it may be located from the load to provide a perfect match on the main line to the left of the stub:

The Smith chart construction is shown on the next page. First we find $z_L = (40 - j10)/50 = 0.8 - j0.2$ and plot it on the chart. Next, we find $y_L = 1/z_L$ by transforming this point halfway around the chart, where we read $y_L = 1.17 + j0.30$. This point is to be transformed to a location at which the real part of the normalized admittance is unity. The $g = 1$ circle is highlighted on the chart; $y_L$ transforms to two locations on it: $y_{in1} = 1 - j0.32$ and $y_{in2} = 1 + j0.32$. The stub is connected at either of these two points. The stub input admittance must cancel the imaginary part of the line admittance at that point. If $y_{in2}$ is chosen, the stub must have input admittance of $-j0.32$. This point is marked on the outer circle and occurs at $0.452\,\lambda$ on the WTG scale. The length of the stub is found by computing the distance between its input, found above, and the short-circuit position (stub load end), marked as $P_{sc}$. This distance is $d_1 = (0.452 - 0.250)\lambda = 0.202\,\lambda$. With $f = 800$ MHz and $v = c$, the wavelength is $\lambda = (3 \times 10^8)/(8 \times 10^8) = 0.375$ m. The distance is thus $d_1 = (0.202)(0.375) = 0.758\,\text{m} = \underline{7.6\,\text{cm}}$. This is the shortest of the two possible stub lengths, since if we had used $y_{in1}$, we would have needed a stub input admittance of $+j0.32$, which would have required a longer stub length to realize. The length of the main line between its load and the stub attachment point is found on the chart by measuring the distance between $y_L$ and $y_{in2}$, in moving clockwise (toward generator). This distance will be $d = [0.500 - (0.178 - 0.138)]\,\lambda = 0.46\,\lambda$. The actual length is then $d = (0.46)(0.375) = 0.173\text{m} = \underline{17.3\,\text{cm}}$.

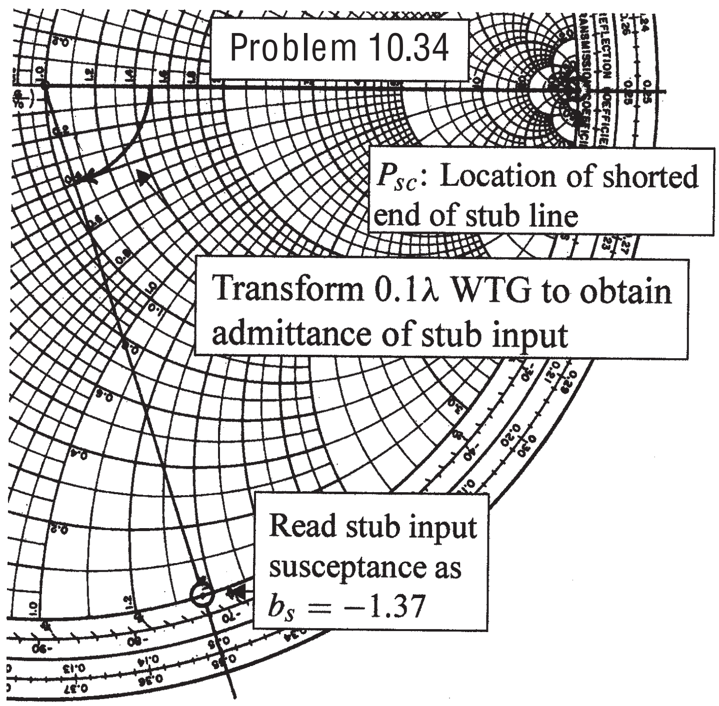
b) Repeat for an open-circuited stub:

In this case, everything is the same, except for the load-end position of the stub, which now occurs at the $P_{oc}$ point on the chart. To use the shortest possible stub, we need to use $y_{in1} = 1 - j0.32$, requiring $y_s = +j0.32$. We find the stub length by moving from $P_{oc}$ to the point at which the admittance is $j0.32$. This occurs at $0.048\,\lambda$ on the WTG scale, which thus determines the required stub length. Now $d_1 = (0.048)(0.375) = 0.18\,\text{m} = \underline{1.8\,\text{cm}}$. The attachment point is found by transforming $y_L$ to $y_{in1}$, where the former point is located at $0.178\,\lambda$ on the WTG scale, and the latter is at $0.362\,\lambda$ on the same scale. The distance is then $d = (0.362 - 0.178)\lambda = 0.184\lambda$. The actual length is $d = (0.184)(0.375) = 0.069\,\text{m} = \underline{6.9\,\text{cm}}$.




Problem 10.33

**10.34.** The lossless line shown in Fig. 10.35 is operating with $\lambda = 100$cm. If $d_1 = 10$cm, $d = 25$cm, and the line is matched to the left of the stub, what is $Z_L$? For the line to be matched, it is required that the sum of the normalized input admittances of the shorted stub and the main line at the point where the stub is connected be unity. So the input susceptances of the two lines must cancel. To find the stub input susceptance, use the Smith chart to transform the short circuit point $0.1\lambda$ toward the generator, and read the input value as $b_s = -1.37$ (note that the stub length is one-tenth of a wavelength). The main line input admittance must now be $y_{in} = 1 + j1.37$. This line is one-quarter wavelength long, so the normalized load impedance is equal to the normalized input admittance. Thus $z_L = 1 + j1.37$, so that $Z_L = 300z_L = \underline{300 + j411\ \Omega}$.

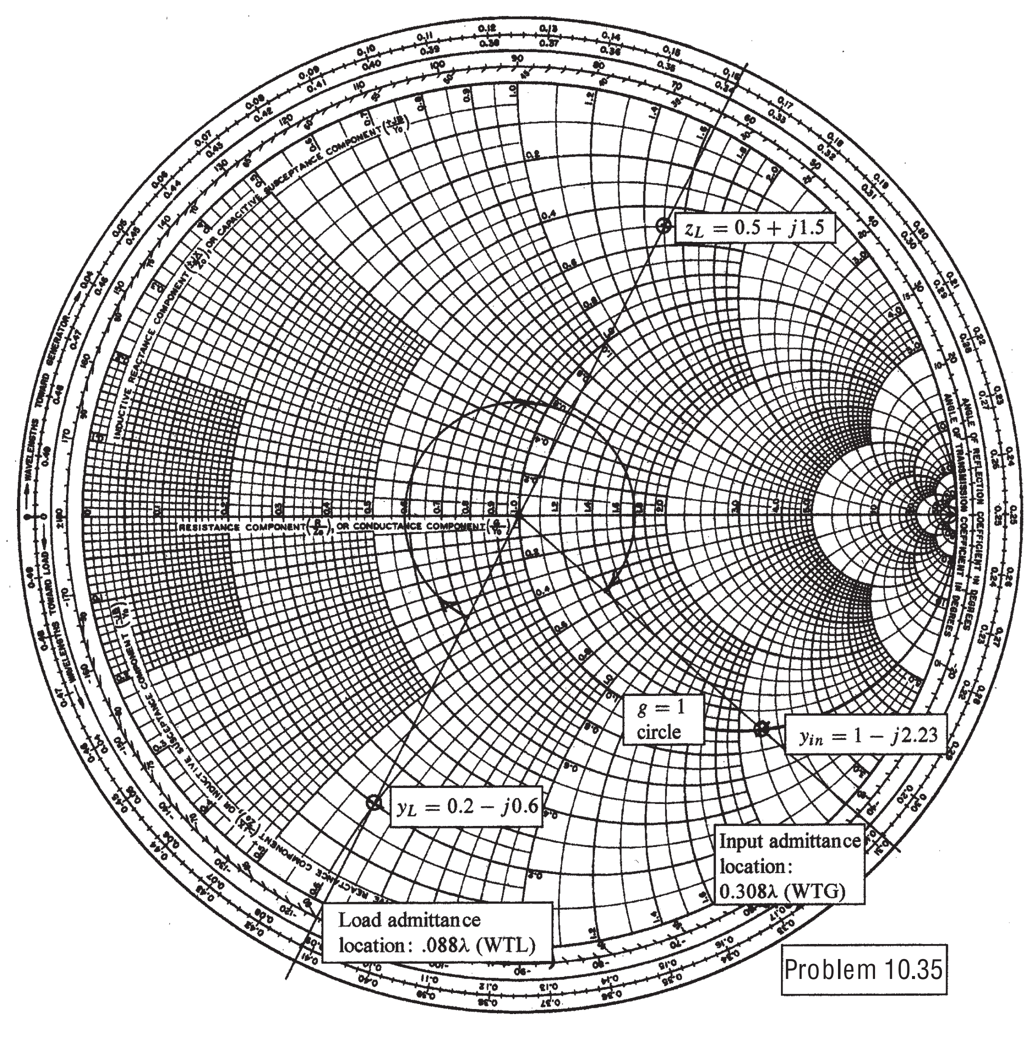


**10.35.** A load, $Z_L = 25 + j75\ \Omega$, is located at $z = 0$ on a lossless two-wire line for which $Z_0 = 50\ \Omega$ and $v = c$.

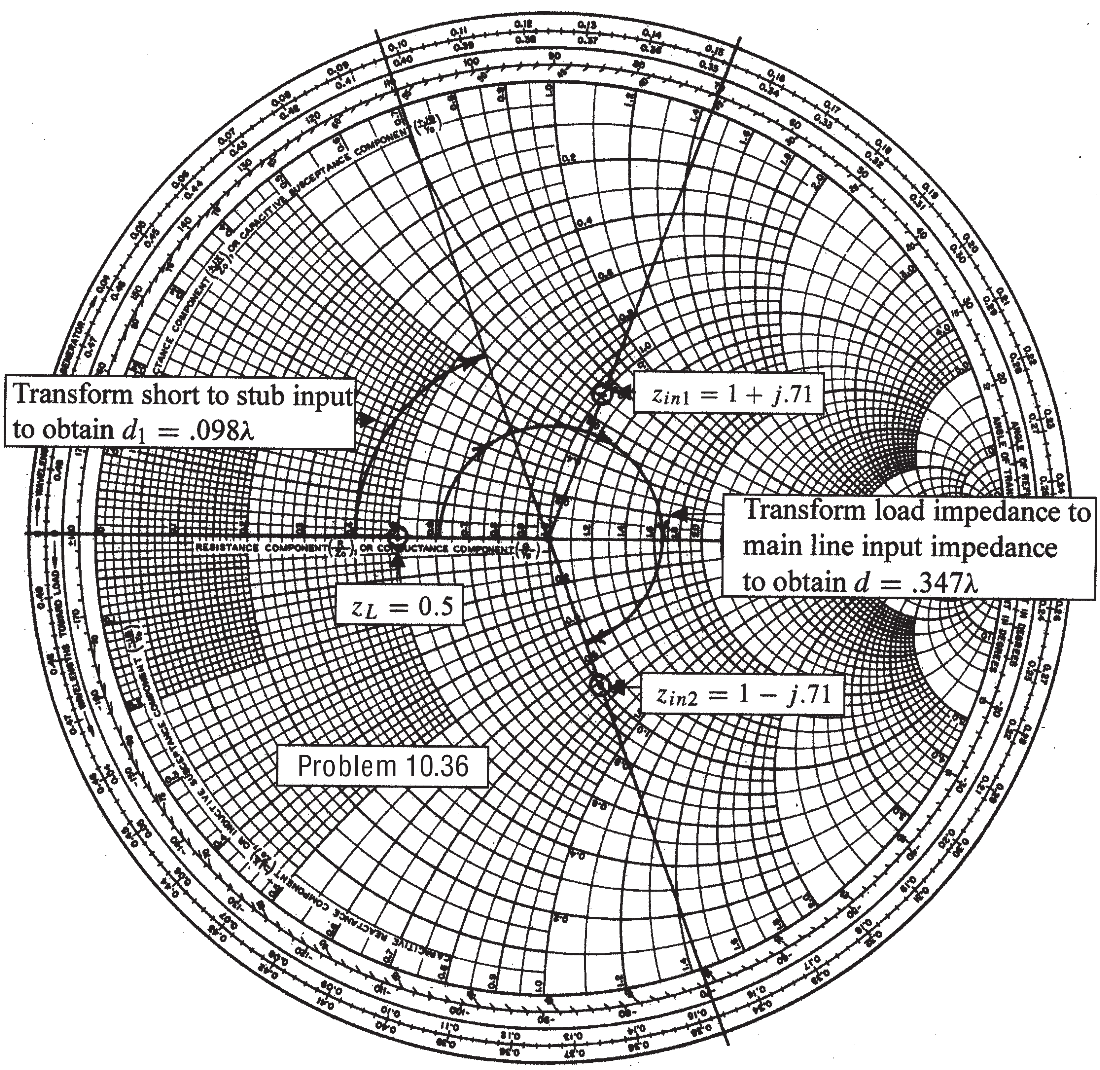
a) If $f = 300$ MHz, find the shortest distance $d$ ($z = -d$) at which the input impedance has a real part equal to $1/Z_0$ and a negative imaginary part: The Smith chart construction is shown below. We begin by calculating $z_L = (25 + j75)/50 = 0.5 + j1.5$, which we then locate on the chart. Next, this point is transformed by rotation halfway around the chart to find $y_L = 1/z_L = 0.20 - j0.60$, which is located at 0.088 $\lambda$ on the WTL scale. This point is then transformed toward the generator until it intersects the $g = 1$ circle (shown highlighted) with a negative imaginary part. This occurs at point $y_{in} = 1.0 - j2.23$, located at 0.308 $\lambda$ on the WTG scale. The total distance between load and input is then $d = (0.088 + 0.308)\lambda = 0.396\lambda$. At 300 MHz, and with $v = c$, the wavelength is $\lambda = 1$ m. Thus the distance is $d = 0.396\,\text{m} = \underline{39.6\,\text{cm}}$.

b) What value of capacitance $C$ should be connected across the line at that point to provide unity standing wave ratio on the remaining portion of the line? To cancel the input normalized susceptance of -2.23, we need a capacitive normalized susceptance of +2.23. We therefore write

$$\omega C = \frac{2.23}{Z_0} \quad \Rightarrow \quad C = \frac{2.23}{(50)(2\pi \times 3 \times 10^8)} = 2.4 \times 10^{-11}\ \text{F} = \underline{24\ \text{pF}}$$



**10.36.** The two-wire lines shown in Fig. 10.36 are all lossless and have $Z_0 = 200\,\Omega$. Find $d$ and the shortest possible value for $d_1$ to provide a matched load if $\lambda = 100$cm. In this case, we have a series combination of the loaded line section and the shorted stub, so we use impedances and the Smith chart as an impedance diagram. The requirement for matching is that the total normalized impedance at the junction (consisting of the sum of the input impedances to the stub and main loaded section) is unity. First, we find $z_L = 100/200 = 0.5$ and mark this on the chart (see below). We then transform this point toward the generator until we reach the $r = 1$ circle. This happens at two possible points, indicated as $z_{in1} = 1 + j.71$ and $z_{in2} = 1 - j.71$. The stub input impedance must cancel the imaginary part of the loaded section input impedance, or $z_{ins} = \pm j.71$. The shortest stub length that accomplishes this is found by transforming the short circuit point on the chart to the point $z_{ins} = +j0.71$, which yields a stub length of $d_1 = .098\lambda = \underline{9.8\,\text{cm}}$. The length of the loaded section is then found by transforming $z_L = 0.5$ to the point $z_{in2} = 1 - j.71$, so that $z_{ins} + z_{in2} = 1$, as required. This transformation distance is $d = 0.347\lambda = \underline{37.7\,\text{cm}}$.

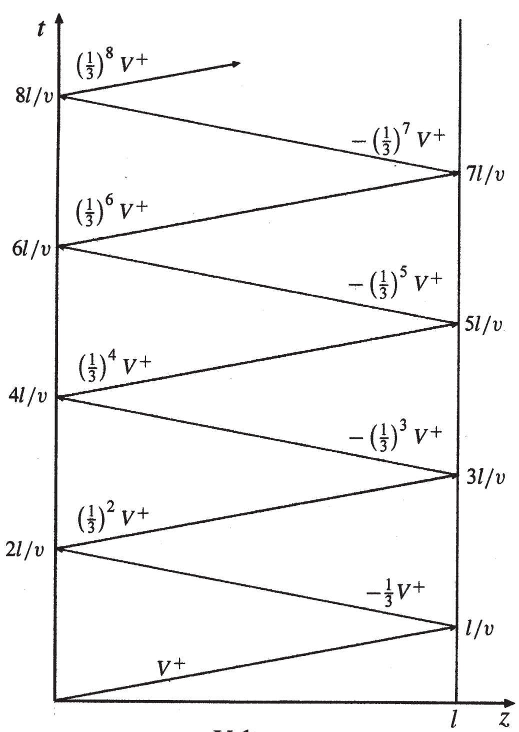


**10.37.** In the transmission line of Fig. 10.20, $R_g = Z_0 = 50\ \Omega$, and $R_L = 25\ \Omega$. Determine and plot the voltage at the load resistor and the current in the battery as functions of time by constructing appropriate voltage and current reflection diagrams: Referring to the figure, closing the switch launches a voltage wave whose value is given by Eq. (119):

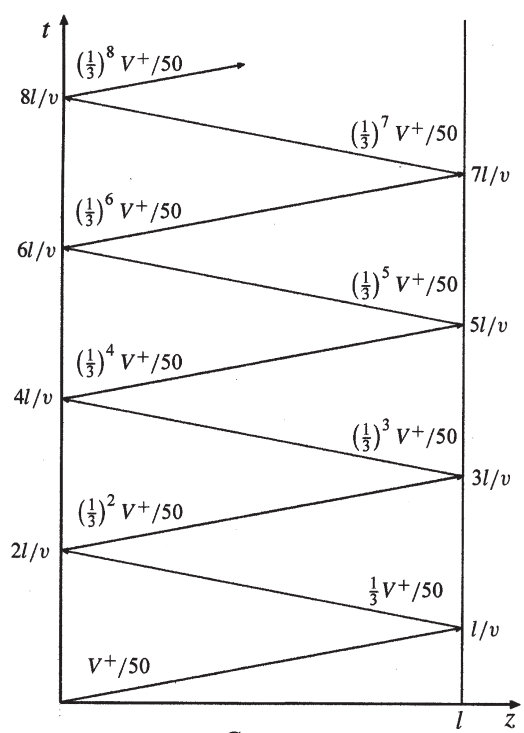
$$V_1^+ = \frac{V_0 Z_0}{R_g + Z_0} = \frac{50}{100} V_0 = \frac{1}{2} V_0$$

Now, $\Gamma_L = (25 - 50)/(25 + 50) = -1/3$. So on reflection from the load, the reflected wave is of value $V_1^- = -V_0/6$. On returning to the input end, the reflection coefficient there is zero, and so all is still. The voltage reflection diagram would be that shown in Fig. 10.21a, except that no waves are present after time $t = 2l/v$. Likewise, the current reflection diagram is that of Fig. 10.22a, except, again, no waves exist after $t = l/v$. The voltage at the load will be $V_L = V_1^+(1 + \Gamma_L) = V_0/3$ for times beyond $l/v$. The current through the battery is initially $I_B = V_1^+/Z_0 = V_0/100$ for times $(0 < t < 2l/v)$. When the reflected wave from the load returns to the input end (at time $t = 2l/v$), the reflected wave current, $I_1^- = V_0/300$, adds to the original current to give $I_B = V_0/75$ A for $(t > 2l/v)$.

**10.38.** Repeat Problem 37, with $Z_0 = 50\Omega$, and $R_L = R_g = 25\Omega$. Carry out the analysis for the time period $0 < t < 8l/v$. At the generator end, we have $\Gamma_g = -1/3$. At the load end, we have $\Gamma_L = -1/3$ as before. The initial wave is of magnitude $V^+ = (2/3)V_0$. Using these values, voltage and current reflection diagrams are constructed, and are shown below:

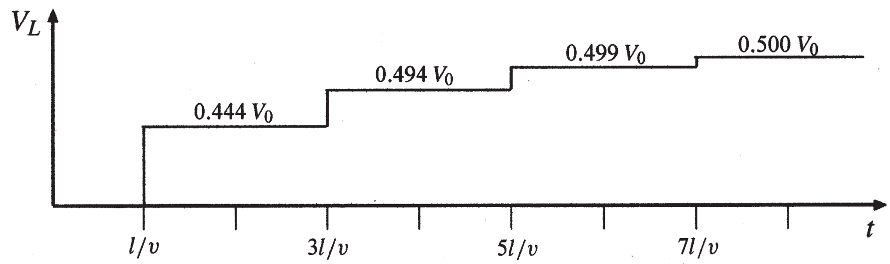


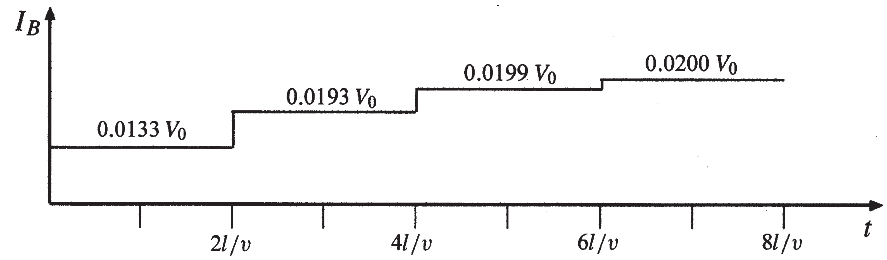
Voltage



Current

**10.38.** (continued) From the diagrams, voltage and current plots are constructed. First, the load voltage is found by adding voltages along the right side of the voltage diagram at the indicated times. Second, the current through the battery is found by adding currents along the left side of the current reflection diagram. Both plots are shown below, where currents and voltages are expressed to three significant figures. The steady state values, $V_L = 0.5$V and $I_B = 0.02$A, are expected as $t \to \infty$.



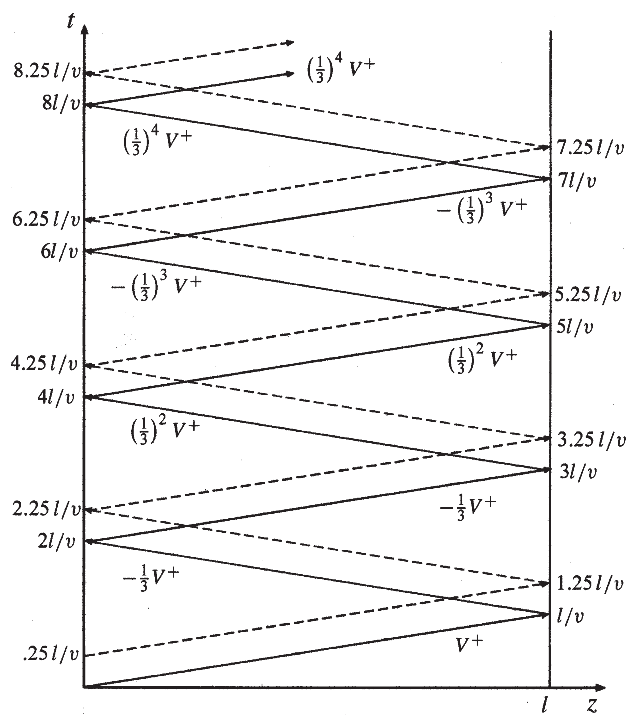

**10.39.** In the transmission line of Fig. 10.20, $Z_0 = 50\,\Omega$ and $R_L = R_g = 25\,\Omega$. The switch is closed at $t = 0$ and *is opened again* at time $t = l/4v$, thus creating a rectangular voltage *pulse* in the line. Construct an appropriate voltage reflection diagram for this case and use it to make a plot of the voltage at the load resistor as a function of time for $0 < t < 8l/v$ (note that the effect of opening the switch is to initiate a second voltage wave, whose value is such that it leaves a net current of zero in its wake): The value of the initial voltage wave, formed by closing the switch, will be

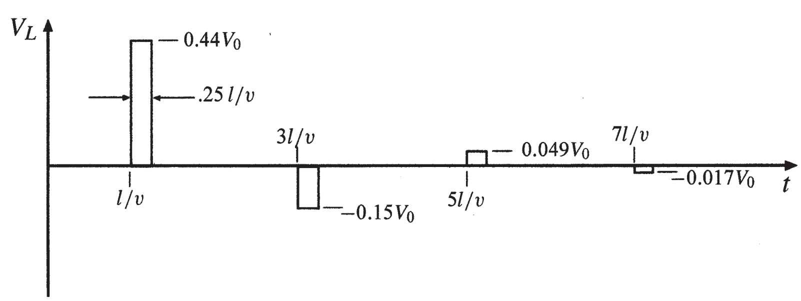
$$V^+ = \frac{Z_0}{R_g + Z_0} V_0 = \frac{50}{25 + 50} V_0 = \frac{2}{3} V_0$$

On opening the switch, a second wave, $V^{+\prime}$, is generated which leaves a net current behind it of zero. This means that $V^{+\prime} = -V^+ = -(2/3)V_0$. Note also that when the switch is opened, the reflection coefficient at the generator end of the line becomes unity. The reflection coefficient at the load end is $\Gamma_L = (25 - 50)/(25 + 50) = -(1/3)$. The reflection diagram is now constructed in the usual manner, and is shown on the next page. The path of the second wave as it reflects from either end is shown in dashed lines, and is a replica of the first wave path, displaced later in time by $l/(4v)$.a All values for the second wave after each reflection are equal but of opposite sign to the immediately preceding first wave values. The load voltage as a function of time is found by accumulating voltage values as they are read moving up along the right hand boundary of the chart. The resulting function, plotted just below the reflection diagram, is found to be a sequence of pulses that alternate signs. The pulse amplitudes are calculated as follows:

**10.39.** (continued)

$$\frac{l}{v} < t < \frac{5l}{4v}: \quad V_1 = \left(1 - \frac{1}{3}\right) V^+ = 0.44\, V_0$$
$$\frac{3l}{v} < t < \frac{13l}{4v}: \quad V_2 = -\frac{1}{3}\left(1 - \frac{1}{3}\right) V^+ = -0.15\, V_0$$
$$\frac{5l}{v} < t < \frac{21l}{4v}: \quad V_3 = \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{3}\right) V^+ = 0.049\, V_0$$
$$\frac{7l}{v} < t < \frac{29l}{4v}: \quad V_4 = -\left(\frac{1}{3}\right)^3 \left(1 - \frac{1}{3}\right) V^+ = -0.017\, V_0$$

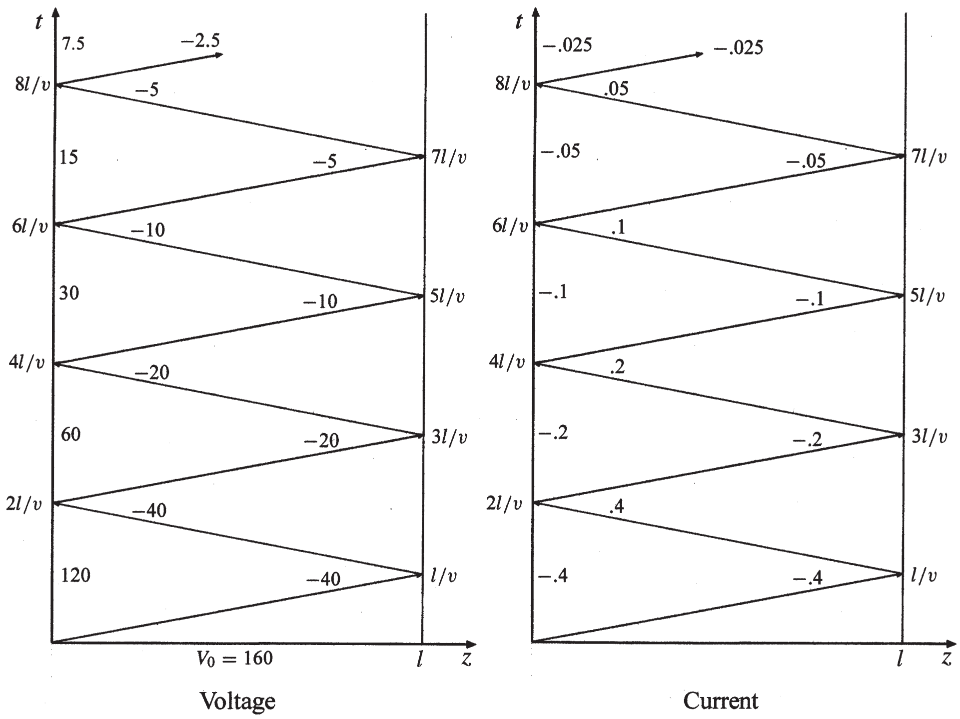


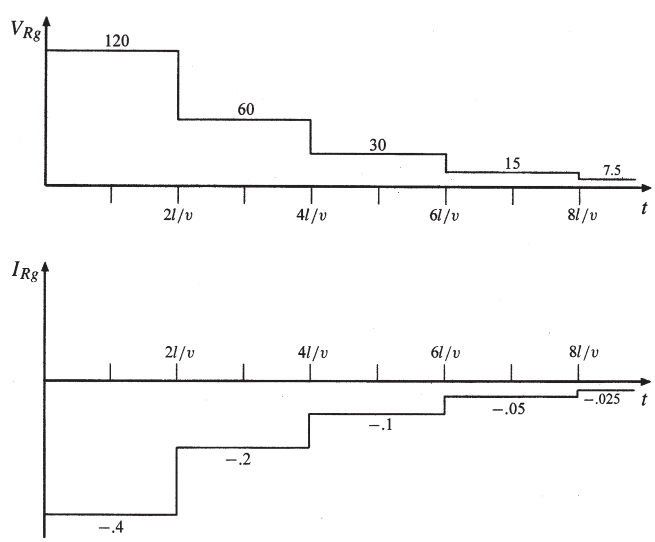



**10.40.** In the charged line of Fig. 10.25, the characteristic impedance is $Z_0 = 100\Omega$, and $R_g = 300\Omega$. The line is charged to initial voltage $V_0 = 160$ V, and the switch is closed at $t = 0$. Determine and plot the voltage and current through the resistor for time $0 < t < 8l/v$ (four round trips). This problem accompanies Example 11.12 as the other special case of the basic charged line problem, in which now $R_g > Z_0$. On closing the switch, the initial voltage wave is

$$V^+ = -V_0 \frac{Z_0}{R_g + Z_0} = -160 \frac{100}{400} = -40\,\text{V}$$

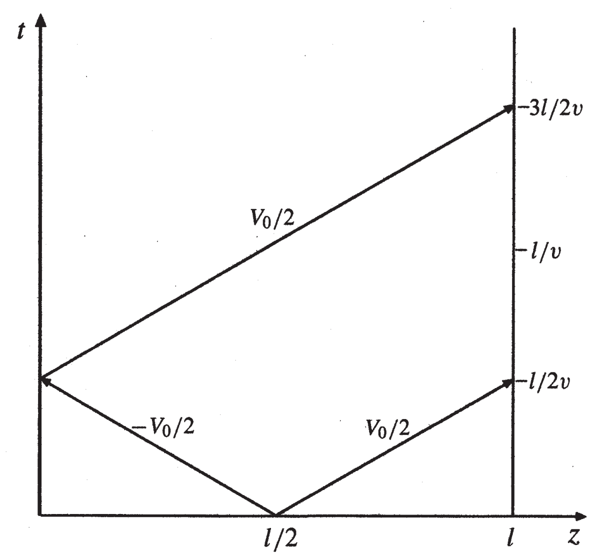
Now, with $\Gamma_g = 1/2$ and $\Gamma_L = 1$, the voltage and current reflection diagrams are constructed as shown below. Plots of the voltage and current at the resistor are then found by accumulating values from the left sides of the two charts, producing the plots as shown.

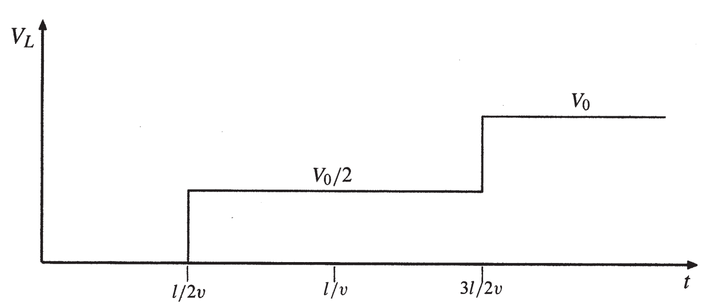


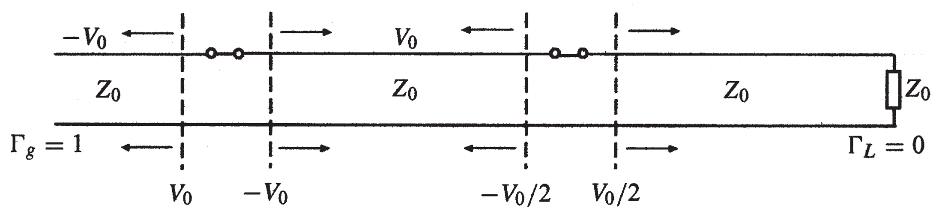
**10.41.** In the transmission line of Fig. 10.37, the switch is located *midway* down the line, and is closed at $t = 0$. Construct a voltage reflection diagram for this case, where $R_L = Z_0$. Plot the load resistor voltage as a function of time: With the left half of the line charged to $V_0$, closing the switch initiates (at the switch location) *two* voltage waves: The first is of value $-V_0/2$ and propagates toward the left; the second is of value $V_0/2$ and propagates toward the right. The backward wave reflects at the battery with $\Gamma_g = -1$. No reflection occurs at the load end, since the load is matched to the line. The reflection diagram and load voltage plot are shown below. The results are summarized as follows:

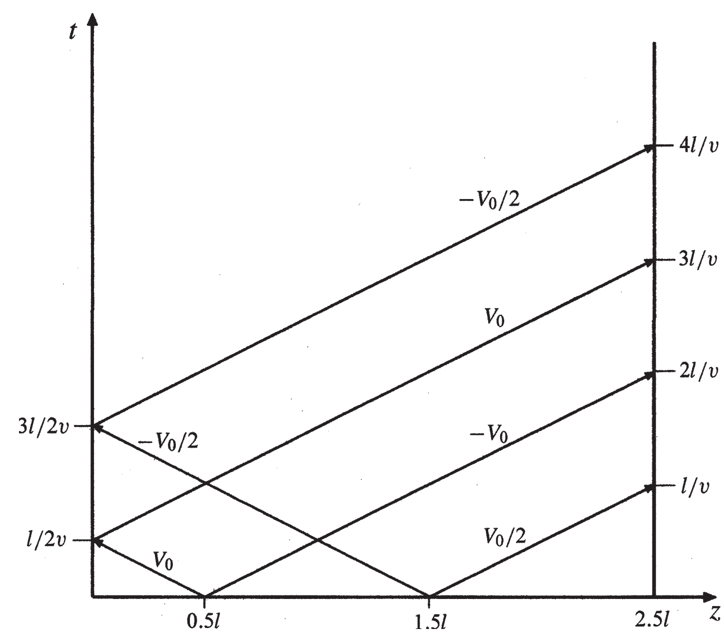
$$0 < t < \frac{l}{2v} : \quad V_L = 0$$
$$\frac{l}{2v} < t < \frac{3l}{2v} : \quad V_L = \frac{V_0}{2}$$
$$t > \frac{3l}{2v} : \quad V_L = V_0$$

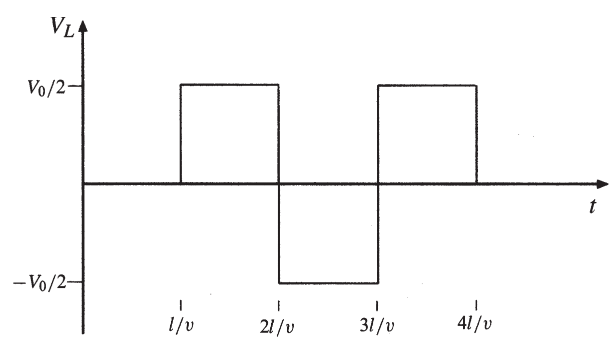


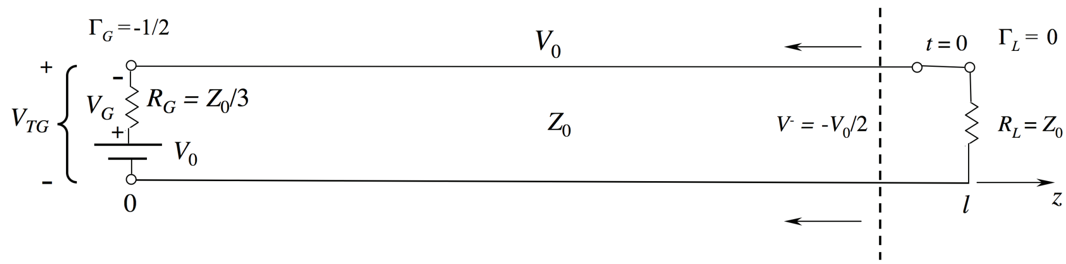
**10.42.** A simple *frozen wave generator* is shown in Fig. 10.38. Both switches are closed simultaneously at $t = 0$. Construct an appropriate voltage reflection diagram for the case in which $R_L = Z_0$. Determine and plot the load voltage as a function of time: Closing the switches sets up a total of four voltage waves as shown in the diagram below. Note that the first and second waves from the left are of magnitude $V_0$, since in fact we are superimposing voltage waves from the $-V_0$ and $+V_0$ charged sections acting alone. The reflection diagram is drawn and is used to construct the load voltage with time by accumulating voltages up the right hand vertical axis.



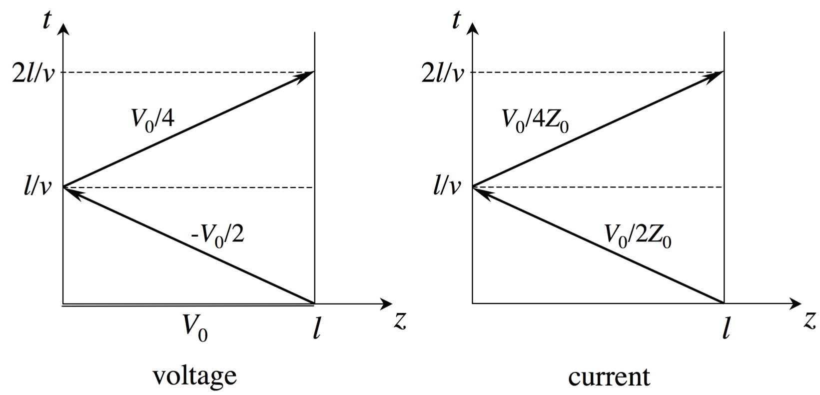



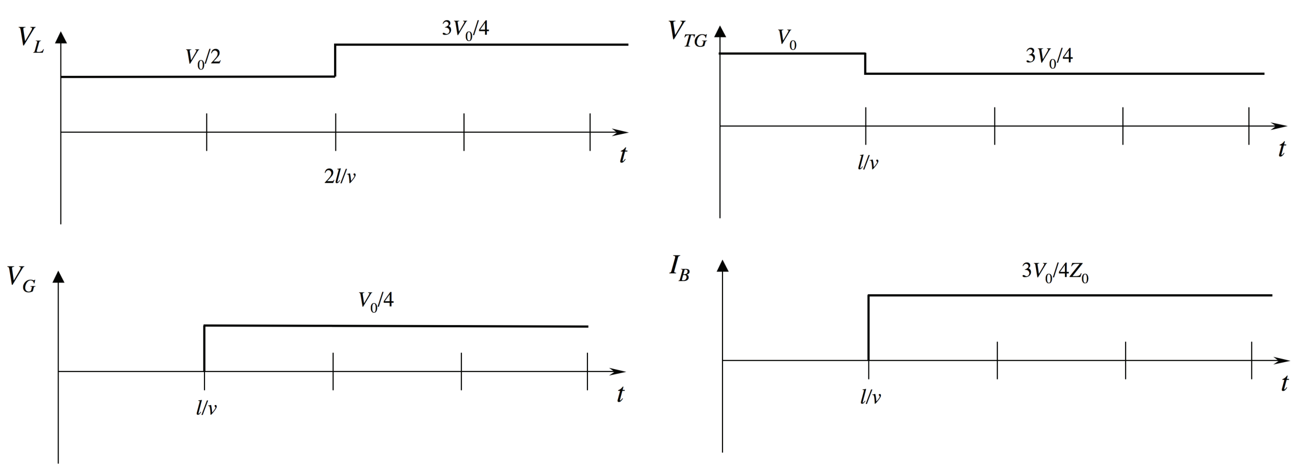
**10.43.** In Fig. 10.39, $R_L = Z_0$ and $R_g = Z_0/3$. The switch is closed at $t = 0$. Determine and plot as functions of time: a) the voltage across $R_L$; b) the voltage across $R_g$; c) the current through the battery.




With the switch at the opposite end from the battery, the entire line is initially charged to $V_0$. So, on closing the switch, the initial wave propagates backward, originating at the switch, and is of value $V^- = -V_0 Z_0/(R_L + Z_0) = -V_0/2$. On reaching the left end, the wave reflects with reflection coefficient $\Gamma_G = (R_G - Z_0)/(R_G + Z_0) = -1/2$. The reflected wave returns to the switch end, sees a matched load there, and there is no further reflection. The resulting voltage and current reflection diagrams are shown below.




The load voltage is read from the right side of the voltage diagram, and is plotted as $V_L$ below. The voltage across the entire left end is read from the left side of the voltage diagram, and is plotted as $V_{TG}$ below. The voltage across the resistor, $R_G$, will be $V_G = V_{TG} - V_0$ (choosing the resistor voltage polarity as positive), which leads to the $V_G$ plot below. Finally, the battery current is read from the left side of the current diagram, and is plotted as $I_B$ below.



# CHAPTER 11

**11.1.** Show that $E_{xs} = Ae^{jk_0z+\phi}$ is a solution to the vector Helmholtz equation, Sec. 11.1, Eq. (30), for $k_0 = \omega\sqrt{\mu_0\epsilon_0}$ and any $\phi$ and $A$: We take

$$\frac{d^2}{dz^2}Ae^{jk_0z+\phi} = (jk_0)^2 Ae^{jk_0z+\phi} = -k_0^2 E_{xs}$$

**11.2.** A 100-MHz uniform plane wave propagates in a lossless medium for which $\epsilon_r = 5$ and $\mu_r = 1$. Find:

a) $v_p$: $v_p = c/\sqrt{\epsilon_r} = 3 \times 10^8/\sqrt{5} = \underline{1.34 \times 10^8 \text{ m/s}}$.

b) $\beta$: $\beta = \omega/v_p = (2\pi \times 10^8)/(1.34 \times 10^8) = \underline{4.69 \text{ m}^{-1}}$.

c) $\lambda$: $\lambda = 2\pi/\beta = \underline{1.34 \text{ m}}$.

d) $\mathbf{E}_s$: Assume real amplitude $E_0$, forward $z$ travel, and $x$ polarization, and write $\mathbf{E}_s = E_0 \exp(-j\beta z)\mathbf{a}_x = \underline{E_0 \exp(-j4.69z)\,\mathbf{a}_x \text{ V/m}}$.

e) $\mathbf{H}_s$: First, the intrinsic impedance of the medium is $\eta = \eta_0/\sqrt{\epsilon_r} = 377/\sqrt{5} = 169\ \Omega$. Then $\mathbf{H}_s = (E_0/\eta)\exp(-j\beta z)\,\mathbf{a}_y = \underline{(E_0/169)\exp(-j4.69z)\,\mathbf{a}_y \text{ A/m}}$.

f) $< \mathbf{S} >= (1/2)\mathcal{R}e\{\mathbf{E}_s \times \mathbf{H}_s^*\} = \underline{(E_0^2/337)\,\mathbf{a}_z \text{ W/m}^2}$

**12.3.** An $\mathbf{H}$ field in free space is given as $\mathcal{H}(x,t) = 10\cos(10^8 t - \beta x)\mathbf{a}_y$ A/m. Find

a) $\beta$: Since we have a uniform plane wave, $\beta = \omega/c$, where we identify $\omega = 10^8$ sec$^{-1}$. Thus $\beta = 10^8/(3 \times 10^8) = \underline{0.33 \text{ rad/m}}$.

b) $\lambda$: We know $\lambda = 2\pi/\beta = \underline{18.9 \text{ m}}$.

c) $\mathcal{E}(x,t)$ at $P(0.1, 0.2, 0.3)$ at $t = 1$ ns: Use $E(x,t) = -\eta_0 H(x,t) = -(377)(10)\cos(10^8 t - \beta x) = -3.77 \times 10^3 \cos(10^8 t - \beta x)$. The vector direction of $\mathbf{E}$ will be $-\mathbf{a}_z$, since we require that $\mathbf{S} = \mathbf{E} \times \mathbf{H}$, where $\mathbf{S}$ is $x$-directed. At the given point, the relevant coordinate is $x = 0.1$. Using this, along with $t = 10^{-9}$ sec, we finally obtain

$$\mathbf{E}(x,t) = -3.77 \times 10^3 \cos[(10^8)(10^{-9}) - (0.33)(0.1)]\mathbf{a}_z = -3.77 \times 10^3 \cos(6.7 \times 10^{-2})\mathbf{a}_z$$
$$= \underline{-3.76 \times 10^3 \mathbf{a}_z \text{ V/m}}$$

**11.4.** Small antennas have low efficiencies (as will be seen in Chapter 14) and the efficiency increases with size up to the point at which a critical dimension of the antenna is an appreciable fraction of a wavelength, say $\lambda/8$.

a) An antenna is that is 12cm long is operated in air at 1 MHz. What fraction of a wavelength long is it? The free space wavelength will be

$$\lambda_{air} = \frac{c}{f} = \frac{3.0 \times 10^8 \text{ m/s}}{10^6 \text{ s}^{-1}} = 300 \text{ m}, \text{ so that the fraction} = \frac{1.2}{300} = \underline{4.0 \times 10^{-3}}$$

b) The same antenna is embedded in a ferrite material for which $\epsilon_r = 20$ and $\mu_r = 2{,}000$. What fraction of a wavelength is it now?

$$\lambda_{ferrite} = \frac{\lambda_{air}}{\sqrt{\mu_r \epsilon_r}} = \frac{300}{\sqrt{(20)(2000)}} = 1.5\text{m} \;\Rightarrow\; \text{fraction} = \frac{1.2}{1.5} = \underline{0.8}$$

**11.5.** A 150-MHz uniform plane wave in free space is described by $\mathbf{H}_s = (4 + j10)(2\mathbf{a}_x + j\mathbf{a}_y)e^{-j\beta z}$ A/m.

a) Find numerical values for $\omega$, $\lambda$, and $\beta$: First, $\omega = 2\pi \times 150 \times 10^6 = \underline{3\pi \times 10^8 \text{ sec}^{-1}}$. Second, for a uniform plane wave in free space, $\lambda = 2\pi c/\omega = c/f = (3 \times 10^8)/(1.5 \times 10^8) = \underline{2\,\text{m}}$. Third, $\beta = 2\pi/\lambda = \underline{\pi\,\text{rad/m}}$.

b) Find $\mathcal{H}(z,t)$ at $t = 1.5$ ns, $z = 20$ cm: Use

$$\begin{aligned}\mathbf{H}(z,t) &= \text{Re}\{\mathbf{H}_s e^{j\omega t}\} = \text{Re}\{(4 + j10)(2\mathbf{a}_x + j\mathbf{a}_y)(\cos(\omega t - \beta z) + j\sin(\omega t - \beta z)\} \\ &= [8\cos(\omega t - \beta z) - 20\sin(\omega t - \beta z)]\,\mathbf{a}_x - [10\cos(\omega t - \beta z) + 4\sin(\omega t - \beta z)]\,\mathbf{a}_y\end{aligned}$$

. Now at the given position and time, $\omega t - \beta z = (3\pi \times 10^8)(1.5 \times 10^{-9}) - \pi(0.20) = \pi/4$. And $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$. So finally,

$$\mathbf{H}(z = 20\text{cm}, t = 1.5\text{ns}) = -\frac{1}{\sqrt{2}}(12\mathbf{a}_x + 14\mathbf{a}_y) = \underline{-8.5\mathbf{a}_x - 9.9\mathbf{a}_y \text{ A/m}}$$

c) What is $|E|_{max}$? Have $|E|_{max} = \eta_0 |H|_{max}$, where

$$|H|_{max} = \sqrt{\mathbf{H}_s \cdot \mathbf{H}_s^*} = [4(4 + j10)(4 - j10) + (j)(-j)(4 + j10)(4 - j10)]^{1/2} = 24.1\,\text{A/m}$$

Then $|E|_{max} = 377(24.1) = \underline{9.08 \text{ kV/m}}$.

**11.6.** A uniform plane wave has electric field $\mathbf{E}_s = (E_{y0}\,\mathbf{a}_y - E_{z0}\,\mathbf{a}_z)\,e^{-\alpha x}\,e^{-j\beta x}$ V/m. The intrinsic impedance of the medium is given as $\eta = |\eta|\,e^{j\phi}$, where $\phi$ is a constant phase.

a) Describe the wave polarization and state the direction of propagation: The wave is linearly polarized in the $y$-$z$ plane, and propagates in the forward $x$ direction (from the $e^{-j\beta x}$ factor).

b) Find $\mathbf{H}_s$: Each component of $\mathbf{E}_s$, when crossed into its companion component of $\mathbf{H}_s^*$, must give a vector in the positive-$x$ direction of travel. Using this rule, we find

$$\mathbf{H}_s = \left[\frac{E_y}{\eta}\,\mathbf{a}_z + \frac{E_z}{\eta}\,\mathbf{a}_y\right] = \left[\frac{E_{y0}}{|\eta|}\,\mathbf{a}_z + \frac{E_{z0}}{|\eta|}\,\mathbf{a}_y\right] e^{-\alpha x}e^{-j\phi}e^{-j\beta x} \text{ A/m}$$

c) Find $\mathcal{E}(x,t)$ and $\mathcal{H}(x,t)$: $\mathcal{E}(x,t) = \mathcal{R}e\left\{\mathbf{E}_s e^{j\omega t}\right\} = [E_{y0}\,\mathbf{a}_y - E_{z0}\,\mathbf{a}_z]\,e^{-\alpha x}\cos(\omega t - \beta x)$

$$\mathcal{H}(x,t) = \mathcal{R}e\left\{\mathbf{H}_s e^{j\omega t}\right\} = [E_{y0}\,\mathbf{a}_z + E_{z0}\,\mathbf{a}_y]\,e^{-\alpha x}\cos(\omega t - \beta x - \phi)$$

where all amplitudes are assumed real.

d) Find $< \mathbf{S} >$ in W/m$^2$:

$$< \mathbf{S} >= \frac{1}{2}\mathcal{R}e\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\} = \frac{1}{2}\left(E_{y0}^2 + E_{z0}^2\right)e^{-2\alpha x}\cos\phi\,\mathbf{a}_x \text{ W/m}^2$$

e) Find the time-average power in watts that is intercepted by an antenna of rectangular cross-section, having width $w$ and height $h$, suspended parallel to the $yz$ plane, and at a distance $d$ from the wave source. This will be

$$P = \int\int_{plate} < \mathbf{S} > \cdot\, d\mathbf{S} = | < \mathbf{S} > |_{x=d} \times area = \frac{1}{2}(wh)\left(E_{y0}^2 + E_{z0}^2\right)e^{-2\alpha d}\cos\phi \text{ W}$$

**11.7.** The phasor magnetic field intensity for a 400-MHz uniform plane wave propagating in a certain lossless material is $(2\mathbf{a}_y - j5\mathbf{a}_z)e^{-j25x}$ A/m. Knowing that the maximum amplitude of $\mathbf{E}$ is 1500 V/m, find $\beta$, $\eta$, $\lambda$, $v_p$, $\epsilon_r$, $\mu_r$, and $\mathcal{H}(x,y,z,t)$: First, from the phasor expression, we identify $\beta = \underline{25 \text{ m}^{-1}}$ from the argument of the exponential function. Next, we evaluate $H_0 = |\mathbf{H}| = \sqrt{\mathbf{H}\cdot\mathbf{H}^*} = \sqrt{2^2+5^2} = \sqrt{29}$. Then $\eta = E_0/H_0 = 1500/\sqrt{29} = \underline{278.5\ \Omega}$. Then $\lambda = 2\pi/\beta = 2\pi/25 = .25 \text{ m} = \underline{25 \text{ cm}}$. Next,

$$v_p = \frac{\omega}{\beta} = \frac{2\pi \times 400 \times 10^6}{25} = \underline{1.01 \times 10^8 \text{ m/s}}$$

Now we note that

$$\eta = 278.5 = 377\sqrt{\frac{\mu_r}{\epsilon_r}} \quad \Rightarrow \quad \frac{\mu_r}{\epsilon_r} = 0.546$$

And

$$v_p = 1.01 \times 10^8 = \frac{c}{\sqrt{\mu_r \epsilon_r}} \quad \Rightarrow \quad \mu_r\epsilon_r = 8.79$$

We solve the above two equations simultaneously to find $\epsilon_r = \underline{4.01}$ and $\mu_r = \underline{2.19}$. Finally,

$$\begin{aligned}\mathbf{H}(x,y,z,t) &= Re\left\{(2\mathbf{a}_y - j5\mathbf{a}_z)e^{-j25x}e^{j\omega t}\right\} \\ &= 2\cos(2\pi \times 400 \times 10^6 t - 25x)\mathbf{a}_y + 5\sin(2\pi \times 400 \times 10^6 t - 25x)\mathbf{a}_z \\ &= \underline{2\cos(8\pi \times 10^8 t - 25x)\mathbf{a}_y + 5\sin(8\pi \times 10^8 t - 25x)\mathbf{a}_z \text{ A/m}}\end{aligned}$$

**11.8.** An electric field in free space is given in spherical coordinates as $\mathbf{E}_s(r) = E_0(r)e^{-jkr}\,\mathbf{a}_\theta$ V/m.

a) find $\mathbf{H}_s(r)$ assuming uniform plane wave behavior: Knowing that the cross product of $\mathbf{E}_s$ with the complex conjugate of the phasor $\mathbf{H}_s$ field must give a vector in the direction of propagation, we obtain,

$$\mathbf{H}_s(r) = \frac{E_0(r)}{\eta_0}e^{-jkr}\,\mathbf{a}_\phi \text{ A/m}$$

b) Find $< \mathbf{S} >$: This will be

$$< \mathbf{S} > = \frac{1}{2}\mathcal{R}e\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\} = \frac{E_0^2(r)}{2\eta_0}\,\mathbf{a}_r \text{ W/m}^2$$

c) Express the average outward power in watts through a closed spherical shell of radius $r$, centered at the origin: The power will be (in this case) just the product of the power density magnitude in part $b$ with the sphere area, or

$$P = 4\pi r^2\,\frac{E_0^2(r)}{2\eta_0} \text{ W}$$

where $E_0(r)$ is assumed real.

d) Establish the required functional form of $E_0(r)$ that will enable the power flow in part $c$ to be independent of radius: Evidently this condition is met when $\underline{E_0(r) \propto 1/r}$

**11.9.** A certain lossless material has $\mu_r = 4$ and $\epsilon_r = 9$. A 10-MHz uniform plane wave is propagating in the $\mathbf{a}_y$ direction with $E_{x0} = 400$ V/m and $E_{y0} = E_{z0} = 0$ at $P(0.6, 0.6, 0.6)$ at $t = 60$ ns.

a) Find $\beta$, $\lambda$, $v_p$, and $\eta$: For a uniform plane wave,

$$\beta = \omega\sqrt{\mu\epsilon} = \frac{\omega}{c}\sqrt{\mu_r\epsilon_r} = \frac{2\pi \times 10^7}{3 \times 10^8}\sqrt{(4)(9)} = \underline{0.4\pi \text{ rad/m}}$$

Then $\lambda = (2\pi)/\beta = (2\pi)/(0.4\pi) = \underline{5\,\text{m}}$. Next,

$$v_p = \frac{\omega}{\beta} = \frac{2\pi \times 10^7}{4\pi \times 10^{-1}} = \underline{5 \times 10^7 \text{ m/s}}$$

Finally,

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \eta_0\sqrt{\frac{\mu_r}{\epsilon_r}} = 377\sqrt{\frac{4}{9}} = \underline{251\ \Omega}$$

b) Find $E(t)$ (at $P$): We are given the amplitude at $t = 60$ ns and at $y = 0.6$ m. Let the maximum amplitude be $E_{max}$, so that in general, $E_x = E_{max}\cos(\omega t - \beta y)$. At the given position and time,

$$E_x = 400 = E_{max}\cos[(2\pi \times 10^7)(60 \times 10^{-9}) - (4\pi \times 10^{-1})(0.6)] = E_{max}\cos(0.96\pi)$$
$$= -0.99E_{max}$$

So $E_{max} = (400)/(-0.99) = -403\,\text{V/m}$. Thus at P, $E(t) = \underline{-403\cos(2\pi \times 10^7 t) \text{ V/m}}$.

c) Find $H(t)$: First, we note that if $E$ at a given instant points in the negative $x$ direction, while the wave propagates in the forward $y$ direction, then $H$ at that same position and time must point in the positive $z$ direction. Since we have a lossless homogeneous medium, $\eta$ is real, and we are allowed to write $H(t) = E(t)/\eta$, where $\eta$ is treated as negative and real. Thus

$$H(t) = H_z(t) = \frac{E_x(t)}{\eta} = \frac{-403}{-251}\cos(2\pi \times 10^{-7}t) = \underline{1.61\cos(2\pi \times 10^{-7}t) \text{ A/m}}$$

**11.10.** In a medium characterized by intrinsic impedance $\eta = |\eta|e^{j\phi}$, a linearly-polarized plane wave propagates, with magnetic field given as $\mathbf{H}_s = (H_{0y}\mathbf{a}_y + H_{0z}\mathbf{a}_z)\,e^{-\alpha x}e^{-j\beta x}$. Find:

a) $\mathbf{E}_s$: Requiring orthogonal components of $\mathbf{E}_s$ for each component of $\mathbf{H}_s$, we find

$$\mathbf{E}_s = |\eta|\,[H_{0z}\,\mathbf{a}_y - H_{0y}\,\mathbf{a}_z]\;e^{-\alpha x}\;e^{-j\beta x}\;e^{j\phi}$$

b) $\mathcal{E}(x,t) = \mathcal{R}e\,\{\mathbf{E}_s e^{j\omega t}\} = |\eta|\,[H_{0z}\,\mathbf{a}_y - H_{0y}\,\mathbf{a}_z]\,e^{-\alpha x}\;\cos(\omega t - \beta x + \phi)$.

c) $\mathcal{H}(x,t) = \mathcal{R}e\,\{\mathbf{H}_s e^{j\omega t}\} = [H_{0y}\,\mathbf{a}_y + H_{0z}\,\mathbf{a}_z]\,e^{-\alpha x}\;\cos(\omega t - \beta x)$.

d) $$< \mathbf{S} >= \frac{1}{2}\,\mathcal{R}e\{\mathbf{E}_s \times \mathbf{H}_s^*\} = \frac{1}{2}|\eta|\left[H_{0y}^2 + H_{0z}^2\right]e^{-2\alpha x}\;\cos\phi\,\mathbf{a}_x \text{ W/m}^2$$

**11.11.** A 2-GHz uniform plane wave has an amplitude of $E_{y0} = 1.4$ kV/m at $(0, 0, 0, t = 0)$ and is propagating in the $\mathbf{a}_z$ direction in a medium where $\epsilon'' = 1.6 \times 10^{-11}$ F/m, $\epsilon' = 3.0 \times 10^{-11}$ F/m, and $\mu = 2.5\,\mu$H/m. Find:

a) $E_y$ at $P(0, 0, 1.8\text{cm})$ at 0.2 ns: To begin, we have the ratio, $\epsilon''/\epsilon' = 1.6/3.0 = 0.533$. So

$$\alpha = \omega\sqrt{\frac{\mu\epsilon'}{2}}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2} - 1\right]^{1/2}$$

$$= (2\pi \times 2 \times 10^9)\sqrt{\frac{(2.5 \times 10^{-6})(3.0 \times 10^{-11})}{2}}\left[\sqrt{1 + (.533)^2} - 1\right]^{1/2} = 28.1\,\text{Np/m}$$

Then

$$\beta = \omega\sqrt{\frac{\mu\epsilon'}{2}}\left[\sqrt{1+\left(\frac{\epsilon''}{\epsilon'}\right)^2} + 1\right]^{1/2} = 112\,\text{rad/m}$$

Thus in general,

$$E_y(z,t) = 1.4e^{-28.1z}\cos(4\pi \times 10^9 t - 112z) \text{ kV/m}$$

Evaluating this at $t = 0.2$ ns and $z = 1.8$ cm, find

$$E_y(1.8\,\text{cm}, 0.2\,\text{ns}) = \underline{0.74 \text{ kV/m}}$$

b) $H_x$ at $P$ at 0.2 ns: We use the phasor relation, $H_{xs} = -E_{ys}/\eta$ where

$$\eta = \sqrt{\frac{\mu}{\epsilon'}}\frac{1}{\sqrt{1 - j(\epsilon''/\epsilon')}} = \sqrt{\frac{2.5 \times 10^{-6}}{3.0 \times 10^{-11}}}\frac{1}{\sqrt{1 - j(.533)}} = 263 + j65.7 = 271\angle\, 14^\circ\ \Omega$$

So now

$$H_{xs} = -\frac{E_{ys}}{\eta} = -\frac{(1.4 \times 10^3)e^{-28.1z}e^{-j112z}}{271e^{j14^\circ}} = -5.16e^{-28.1z}e^{-j112z}e^{-j14^\circ}\ \text{A/m}$$

Then

$$H_x(z,t) = -5.16e^{-28.1z}\cos(4\pi \times 10^{-9}t - 112z - 14^\circ)$$

This, when evaluated at $t = 0.2$ ns and $z = 1.8$ cm, yields

$$H_x(1.8\,\text{cm}, 0.2\,\text{ns}) = \underline{-3.0\,\text{A/m}}$$

**11.12.** Describe how the attenuation coefficient of a liquid medium, assumed to be a good conductor, could be determined through measurement of wavelength in the liquid at a known frequency. What restrictions apply? Could this method be used to find the conductivity as well? In a good conductor, we may use the approximation:

$$\alpha \doteq \beta \doteq \sqrt{\frac{\omega\mu\sigma}{2}} \text{ where } \beta = \frac{2\pi}{\lambda}$$

Therefore, in the good conductor approximation, $\alpha \doteq 2\pi/\lambda$. From the above formula, we could also find

$$\sigma \doteq \frac{4\pi}{\lambda^2 f\mu}$$

which would work provided that again, we are certain that we have a good conductor, and that the permeability is known.

**11.13.** Let $jk = 0.2 + j1.5\,\text{m}^{-1}$ and $\eta = 450 + j60\ \Omega$ for a uniform plane wave propagating in the $\mathbf{a}_z$ direction. If $\omega = 300$ Mrad/s, find $\mu$, $\epsilon'$, and $\epsilon''$: We begin with

$$\eta = \sqrt{\frac{\mu}{\epsilon'}}\frac{1}{\sqrt{1 - j(\epsilon''/\epsilon')}} = 450 + j60$$

and

$$jk = j\omega\sqrt{\mu\epsilon'}\sqrt{1 - j(\epsilon''/\epsilon')} = 0.2 + j1.5$$

Then

$$\eta\eta^* = \frac{\mu}{\epsilon'}\frac{1}{\sqrt{1 + (\epsilon''/\epsilon')^2}} = (450 + j60)(450 - j60) = 2.06 \times 10^5 \qquad (1)$$

and

$$(jk)(jk)^* = \omega^2\mu\epsilon'\sqrt{1 + (\epsilon''/\epsilon')^2} = (0.2 + j1.5)(0.2 - j1.5) = 2.29 \qquad (2)$$

Taking the ratio of (2) to (1),

$$\frac{(jk)(jk)^*}{\eta\eta^*} = \omega^2(\epsilon')^2\left(1 + (\epsilon''/\epsilon')^2\right) = \frac{2.29}{2.06 \times 10^5} = 1.11 \times 10^{-5}$$

Then with $\omega = 3 \times 10^8$,

$$(\epsilon')^2 = \frac{1.11 \times 10^{-5}}{(3 \times 10^8)^2\,(1 + (\epsilon''/\epsilon')^2)} = \frac{1.23 \times 10^{-22}}{(1 + (\epsilon''/\epsilon')^2)} \qquad (3)$$

Now, we use Eqs. (35) and (36). Squaring these and taking their ratio gives

$$\frac{\alpha^2}{\beta^2} = \frac{\sqrt{1 + (\epsilon''/\epsilon')^2}}{\sqrt{1 + (\epsilon''/\epsilon')^2}} = \frac{(0.2)^2}{(1.5)^2}$$

We solve this to find $\epsilon''/\epsilon' = 0.271$. Substituting this result into (3) gives $\epsilon' = 1.07 \times 10^{-11}$ F/m. Since $\epsilon''/\epsilon' = 0.271$, we then find $\epsilon'' = 2.90 \times 10^{-12}$ F/m. Finally, using these results in either (1) or (2) we find $\mu = 2.28 \times 10^{-6}$ H/m. Summary: $\mu = \underline{2.28 \times 10^{-6}\,\text{H/m}}$, $\epsilon' = \underline{1.07 \times 10^{-11}\,\text{F/m}}$, and $\epsilon'' = \underline{2.90 \times 10^{-12}\,\text{F/m}}$.

**11.14.** A certain nonmagnetic material has the material constants $\epsilon_r' = 2$ and $\epsilon''/\epsilon' = 4 \times 10^{-4}$ at $\omega = 1.5$ Grad/s. Find the distance a uniform plane wave can propagate through the material before:

a) it is attenuated by 1 Np: First, $\epsilon'' = (4 \times 10^4)(2)(8.854 \times 10^{-12}) = 7.1 \times 10^{-15}$ F/m. Then, since $\epsilon''/\epsilon' << 1$, we use the approximate form for $\alpha$, given by Eq. (51) (written in terms of $\epsilon''$):

$$\alpha \doteq \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} = \frac{(1.5 \times 10^9)(7.1 \times 10^{-15})}{2}\frac{377}{\sqrt{2}} = 1.42 \times 10^{-3} \text{ Np/m}$$

The required distance is now $z_1 = (1.42 \times 10^{-3})^{-1} = \underline{706 \text{ m}}$

b) the power level is reduced by one-half: The governing relation is $e^{-2\alpha z_{1/2}} = 1/2$, or $z_{1/2} = \ln 2/2\alpha = \ln 2/2(1.42 \times 10{-}3) = \underline{244 \text{ m}}$.

c) the phase shifts 360°: This distance is defined as one wavelength, where $\lambda = 2\pi/\beta = (2\pi c)/(\omega\sqrt{\epsilon_r'}) = [2\pi(3 \times 10^8)]/[(1.5 \times 10^9)\sqrt{2}] = \underline{0.89 \text{ m}}$.

**11.15.** A 10 GHz radar signal may be represented as a uniform plane wave in a sufficiently small region. Calculate the wavelength in centimeters and the attenuation in nepers per meter if the wave is propagating in a non-magnetic material for which

a) $\epsilon_r' = 1$ and $\epsilon_r'' = 0$: In a non-magnetic material, we would have:

$$\alpha = \omega\sqrt{\frac{\mu_0\epsilon_0\epsilon_r'}{2}}\left[\sqrt{1 + \left(\frac{\epsilon_r''}{\epsilon_r'}\right)^2} - 1\right]^{1/2}$$

and

$$\beta = \omega\sqrt{\frac{\mu_0\epsilon_0\epsilon_r'}{2}}\left[\sqrt{1 + \left(\frac{\epsilon_r''}{\epsilon_r'}\right)^2} + 1\right]^{1/2}$$

With the given values of $\epsilon_r'$ and $\epsilon_r''$, it is clear that $\beta = \omega\sqrt{\mu_0\epsilon_0} = \omega/c$, and so $\lambda = 2\pi/\beta = 2\pi c/\omega = 3 \times 10^{10}/10^{10} = \underline{3 \text{ cm}}$. It is also clear that $\alpha = \underline{0}$.

b) $\epsilon_r' = 1.04$ and $\epsilon_r'' = 9.00 \times 10^{-4}$: In this case $\epsilon_r''/\epsilon_r' << 1$, and so $\beta \doteq \omega\sqrt{\epsilon_r'}/c = 2.13 \text{ cm}^{-1}$. Thus $\lambda = 2\pi/\beta = \underline{2.95 \text{ cm}}$. Then

$$\alpha \doteq \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} = \frac{\omega\epsilon_r''}{2}\frac{\sqrt{\mu_0\epsilon_0}}{\sqrt{\epsilon_r'}} = \frac{\omega}{2c}\frac{\epsilon_r''}{\sqrt{\epsilon_r'}} = \frac{2\pi \times 10^{10}}{2 \times 3 \times 10^8}\frac{(9.00 \times 10^{-4})}{\sqrt{1.04}}$$
$$= \underline{9.24 \times 10^{-2} \text{ Np/m}}$$

c) $\epsilon_r' = 2.5$ and $\epsilon_r'' = 7.2$: Using the above formulas, we obtain

$$\beta = \frac{2\pi \times 10^{10}\sqrt{2.5}}{(3 \times 10^{10})\sqrt{2}}\left[\sqrt{1 + \left(\frac{7.2}{2.5}\right)^2} + 1\right]^{1/2} = 4.71 \text{ cm}^{-1}$$

and so $\lambda = 2\pi/\beta = \underline{1.33 \text{ cm}}$. Then

$$\alpha = \frac{2\pi \times 10^{10}\sqrt{2.5}}{(3 \times 10^8)\sqrt{2}}\left[\sqrt{1 + \left(\frac{7.2}{2.5}\right)^2} - 1\right]^{1/2} = \underline{335 \text{ Np/m}}$$

**11.16.** Consider the power dissipation term, $\int \mathbf{E} \cdot \mathbf{J} dv$ in Poynting's theorem (Eq. (70)). This gives the power lost to heat within a volume into which electromagnetic waves enter. The term $p_d = \mathbf{E} \cdot \mathbf{J}$ is thus the power dissipation per unit volume in W/m$^3$. Following the same reasoning that resulted in Eq. (77), the time-average power dissipation per volume will be $< p_d >= (1/2)\mathcal{R}e\{\mathbf{E}_s \cdot \mathbf{J}_s^*\}$.

a) Show that in a conducting medium, through which a uniform plane wave of amplitude $E_0$ propagates in the forward $z$ direction, $< p_d >= (\sigma/2)|E_0|^2 e^{-2\alpha z}$: Begin with the phasor expression for the electric field, assuming complex amplitude $E_0$, and $x$-polarization:

$$\mathbf{E}_s = E_0 e^{-\alpha z} e^{-j\beta z} \, \mathbf{a}_x \text{ V/m}^2$$

Then

$$\mathbf{J}_s = \sigma \mathbf{E}_s = \sigma E_0 e^{-\alpha z} e^{-j\beta z} \, \mathbf{a}_x \text{ A/m}^2$$

So that

$$< p_d >= (1/2)\mathcal{R}e\left\{E_0 e^{-\alpha z} e^{-j\beta z} \, \mathbf{a}_x \cdot \sigma E_0^* e^{-\alpha z} e^{+j\beta z} \, \mathbf{a}_x\right\} = (\sigma/2)|E_0|^2 e^{-2\alpha z}$$

b) Confirm this result for the special case of a good conductor by using the left hand side of Eq. (70), and consider a very small volume. In a good conductor, the intrinsic impedance is, from Eq. (85), $\eta_c = (1+j)/(\sigma\delta)$, where the skin depth, $\delta = 1/\alpha$. The magnetic field phasor is then

$$\mathbf{H}_s = \frac{E_s}{\eta_c} \, \mathbf{a}_y = \frac{\sigma}{(1+j)\alpha} E_0 e^{-\alpha z} e^{-j\beta z} \, \mathbf{a}_y \text{ A/m}$$

The time-average Poynting vector is then

$$< \mathbf{S} >= \frac{1}{2}\mathcal{R}e\{\mathbf{E}_s \times \mathbf{H}_s^*\} = \frac{\alpha}{4\sigma}|E_0|^2 e^{-2\alpha z} \, \mathbf{a}_z \text{ W/m}^2$$

Now, consider a rectangular volume of side lengths, $\Delta x$, $\Delta y$, and $\Delta z$, all of which are very small. As the wave passes through this volume in the forward $z$ direction, the power dissipated will be the difference between the power at entry (at $z = 0$), and the power that exits the volume (at $z = \Delta z$). With small $z$, we may approximate $e^{-2\alpha z} \doteq 1 - 2\alpha z$, and the dissipated power in the volume becomes

$$P_d = P_{in} - P_{out} = \left[\frac{\alpha}{4\sigma}|E_0|^2\right] \Delta x \Delta y - \left[\frac{\alpha}{4\sigma}|E_0|^2(1 - 2\alpha\Delta z)\right] \Delta x \Delta y = \frac{\alpha}{2\sigma}|E_0|^2(\Delta x \Delta y \Delta z)$$

This is just the result of part $a$, evaluated at $z = 0$ and multiplied by the volume. The relation becomes exact as $\Delta z \rightarrow 0$, in which case $< p_d > \rightarrow (\sigma/2)|E_0|^2$.

It is also possible to show the relation by using Eq. (69) (which involves taking the divergence of $< \mathbf{S} >$), or by removing the restriction of a small volume and evaluating the integrals in Eq. (70) without approximations. Either method is straightforward.

**11.17.** Let $\eta = 250 + j30\,\Omega$ and $jk = 0.2 + j2\,\mathrm{m}^{-1}$ for a uniform plane wave propagating in the $\mathbf{a}_z$ direction in a dielectric having some finite conductivity. If $|E_s| = 400$ V/m at $z = 0$, find:

a) $<\mathbf{S}>$ at $z = 0$ and $z = 60$ cm: Assume $x$-polarization for the electric field. Then

$$<\mathbf{S}> = \frac{1}{2}\mathrm{Re}\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\} = \frac{1}{2}\mathrm{Re}\left\{400e^{-\alpha z}e^{-j\beta z}\mathbf{a}_x \times \frac{400}{\eta^*}e^{-\alpha z}e^{j\beta z}\mathbf{a}_y\right\}$$
$$= \frac{1}{2}(400)^2 e^{-2\alpha z}\mathrm{Re}\left\{\frac{1}{\eta^*}\right\}\mathbf{a}_z = 8.0 \times 10^4 e^{-2(0.2)z}\mathrm{Re}\left\{\frac{1}{250 - j30}\right\}\mathbf{a}_z$$
$$= 315\, e^{-2(0.2)z}\,\mathbf{a}_z \ \mathrm{W/m^2}$$

Evaluating at $z = 0$, obtain $<\mathbf{S}>(z = 0) = \underline{315\,\mathbf{a}_z \ \mathrm{W/m^2}}$,
and at $z = 60$ cm, $\mathbf{P}_{z,av}(z = 0.6) = 315e^{-2(0.2)(0.6)}\mathbf{a}_z = \underline{248\,\mathbf{a}_z \ \mathrm{W/m^2}}$.

b) the average ohmic power dissipation in watts per cubic meter at $z = 60$ cm: At this point a flaw becomes evident in the problem statement, since solving this part in two different ways gives results that are not the same. I will demonstrate: In the first method, we use Poynting's theorem in point form Eq.(69), which we modify for the case of time-independent fields to read: $-\nabla\cdot <\mathbf{S}> = <\mathbf{J}\cdot\mathbf{E}>$, where the right hand side is the average power dissipation per volume. Note that the additional right-hand-side terms in Poynting's theorem that describe changes in energy stored in the fields will both be zero in steady state. We apply our equation to the result of part $a$:

$$<\mathbf{J}\cdot\mathbf{E}> = -\nabla\cdot <\mathbf{S}> = -\frac{d}{dz}315\, e^{-2(0.2)z} = (0.4)(315)e^{-2(0.2)z} = 126e^{-0.4z} \ \mathrm{W/m^3}$$

At $z = 60$ cm, this becomes $<\mathbf{J}\cdot\mathbf{E}> = 99.1$ W/m$^3$. In the second method, we solve for the conductivity and evaluate $<\mathbf{J}\cdot\mathbf{E}> = \sigma <E^2>$. We use $jk = j\omega\sqrt{\mu\epsilon'}\sqrt{1 - j(\epsilon''/\epsilon')}$ and

$$\eta = \sqrt{\frac{\mu}{\epsilon'}}\frac{1}{\sqrt{1 - j(\epsilon''/\epsilon')}}$$

We take the ratio,

$$\frac{jk}{\eta} = j\omega\epsilon'\left[1 - j\left(\frac{\epsilon''}{\epsilon'}\right)\right] = j\omega\epsilon' + \omega\epsilon''$$

Identifying $\sigma = \omega\epsilon''$, we find

$$\sigma = \mathrm{Re}\left\{\frac{jk}{\eta}\right\} = \mathrm{Re}\left\{\frac{0.2 + j2}{250 + j30}\right\} = 1.74 \times 10^{-3} \ \mathrm{S/m}$$

Now we find the dissipated power per volume:

$$\sigma <E^2> = 1.74 \times 10^{-3}\left(\frac{1}{2}\right)\left(400e^{-0.2z}\right)^2$$

At $z = 60$ cm, this evaluates as 109 W/m$^3$. One can show that consistency between the two methods requires that

$$\mathrm{Re}\left\{\frac{1}{\eta^*}\right\} = \frac{\sigma}{2\alpha}$$

This relation does not hold using the numbers as given in the problem statement and the value of $\sigma$ found above. Note that in Problem 11.13, where all values are worked out, the relation does hold and consistent results are obtained using both methods.

**11.18.** Given, a 100MHz uniform plane wave in a medium known to be a good dielectric. The phasor electric field is $\mathbf{E}_s = 4e^{-0.5z}e^{-j20z}\mathbf{a}_x$ V/m. Not stated in the problem is the permeability, which we take to be $\mu_0$. Determine:

a) $\epsilon'$: As a first step, it is useful to see just how much of a good dielectric we have. We use the good dielectric approximations, Eqs. (60a) and (60b), with $\sigma = \omega\epsilon''$. Using these, we take the ratio, $\beta/\alpha$, to find

$$\frac{\beta}{\alpha} = \frac{20}{0.5} = \frac{\omega\sqrt{\mu\epsilon'}\left[1+(1/8)(\epsilon''/\epsilon')^2\right]}{(\omega\epsilon''/2)\sqrt{\mu/\epsilon'}} = 2\left(\frac{\epsilon'}{\epsilon''}\right) + \frac{1}{4}\left(\frac{\epsilon''}{\epsilon'}\right)$$

This becomes the quadratic equation:

$$\left(\frac{\epsilon''}{\epsilon'}\right)^2 - 160\left(\frac{\epsilon''}{\epsilon'}\right) + 8 = 0$$

The solution to the quadratic is $(\epsilon''/\epsilon') = 0.05$, which means that we can neglect the second term in Eq. (60b), so that $\beta \doteq \omega\sqrt{\mu\epsilon'} = (\omega/c)\sqrt{\epsilon'_r}$. With the given frequency of 100 MHz, and with $\mu = \mu_0$, we find $\sqrt{\epsilon'_r} = 20(3/2\pi) = 9.55$, so that $\epsilon'_r = 91.3$, and finally $\epsilon' = \epsilon'_r\epsilon_0 = \underline{8.1 \times 10^{-10}\ \text{F/m}}$.

b) $\epsilon''$: Using Eq. (60a), the set up is

$$\alpha = 0.5 = \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}} \quad\Rightarrow\quad \epsilon'' = \frac{2(0.5)}{2\pi\times 10^8}\sqrt{\frac{\epsilon'}{\mu}} = \frac{10^{-8}}{2\pi(377)}\sqrt{91.3} = \underline{4.0\times 10^{-11}\ \text{F/m}}$$

c) $\eta$: Using Eq. (62b), we find

$$\eta \doteq \sqrt{\frac{\mu}{\epsilon'}}\left[1 + j\frac{1}{2}\left(\frac{\epsilon''}{\epsilon'}\right)\right] = \frac{377}{\sqrt{91.3}}(1 + j.025) = \underline{(39.5 + j0.99)\ \text{ohms}}$$

d) $\mathbf{H}_s$: This will be a $y$-directed field, and will be

$$\mathbf{H}_s = \frac{E_s}{\eta}\,\mathbf{a}_y = \frac{4}{(39.5+j0.99)}e^{-0.5z}e^{-j20z}\,\mathbf{a}_y = \underline{0.101e^{-0.5z}e^{-j20z}e^{-j0.025}\,\mathbf{a}_y\ \text{A/m}}$$

e) $<\mathbf{S}>$: Using the given field and the result of part $d$, obtain

$$<\mathbf{S}> = \frac{1}{2}\mathcal{R}e\{\mathbf{E}_s \times \mathbf{H}_s^*\} = \frac{(0.101)(4)}{2}e^{-2(0.5)z}\cos(0.025)\,\mathbf{a}_z = \underline{0.202e^{-z}\,\mathbf{a}_z\ \text{W/m}^2}$$

f) the power in watts that is incident on a rectangular surface measuring 20m x 30m at $z = 10$m: At 10m, the power density is $<\mathbf{S}> = 0.202e^{-10} = 9.2\times 10^{-6}$ W/m$^2$. The incident power on the given area is then $P = 9.2\times 10^{-6} \times (20)(30) = \underline{5.5\ \text{mW}}$.

**11.19.** Perfectly-conducting cylinders with radii of 8 mm and 20 mm are coaxial. The region between the cylinders is filled with a perfect dielectric for which $\epsilon = 10^{-9}/4\pi$ F/m and $\mu_r = 1$. If **E** in this region is $(500/\rho)\cos(\omega t - 4z)\mathbf{a}_\rho$ V/m, find:

a) $\omega$, with the help of Maxwell's equations in cylindrical coordinates: We use the two curl equations, beginning with $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$, where in this case,

$$\nabla \times \mathbf{E} = \frac{\partial E_\rho}{\partial z}\mathbf{a}_\phi = \frac{2000}{\rho}\sin(\omega t - 4z)\mathbf{a}_\phi = -\frac{\partial B_\phi}{\partial t}\mathbf{a}_\phi$$

So

$$B_\phi = \int \frac{2000}{\rho}\sin(\omega t - 4z)dt = \frac{2000}{\omega\rho}\cos(\omega t - 4z)\ \mathrm{T}$$

Then

$$H_\phi = \frac{B_\phi}{\mu_0} = \frac{2000}{(4\pi \times 10^{-7})\omega\rho}\cos(\omega t - 4z)\ \mathrm{A/m}$$

We next use $\nabla \times \mathbf{H} = \partial \mathbf{D}/\partial t$, where in this case

$$\nabla \times \mathbf{H} = -\frac{\partial H_\phi}{\partial z}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho}\mathbf{a}_z$$

where the second term on the right hand side becomes zero when substituting our $H_\phi$. So

$$\nabla \times \mathbf{H} = -\frac{\partial H_\phi}{\partial z}\mathbf{a}_\rho = -\frac{8000}{(4\pi \times 10^{-7})\omega\rho}\sin(\omega t - 4z)\mathbf{a}_\rho = \frac{\partial D_\rho}{\partial t}\mathbf{a}_\rho$$

And

$$D_\rho = \int -\frac{8000}{(4\pi \times 10^{-7})\omega\rho}\sin(\omega t - 4z)dt = \frac{8000}{(4\pi \times 10^{-7})\omega^2\rho}\cos(\omega t - 4z)\ \mathrm{C/m^2}$$

Finally, using the given $\epsilon$,

$$E_\rho = \frac{D_\rho}{\epsilon} = \frac{8000}{(10^{-16})\omega^2\rho}\cos(\omega t - 4z)\ \mathrm{V/m}$$

This must be the same as the given field, so we require

$$\frac{8000}{(10^{-16})\omega^2\rho} = \frac{500}{\rho} \quad \Rightarrow \quad \omega = \underline{4 \times 10^8\ \mathrm{rad/s}}$$

b) $\mathbf{H}(\rho, z, t)$: From part $a$, we have

$$\mathbf{H}(\rho, z, t) = \frac{2000}{(4\pi \times 10^{-7})\omega\rho}\cos(\omega t - 4z)\mathbf{a}_\phi = \underline{\frac{4.0}{\rho}\cos(4 \times 10^8 t - 4z)\mathbf{a}_\phi\ \mathrm{A/m}}$$

c) $\mathbf{S}(\rho, \phi, z)$: This will be

$$\mathbf{S}(\rho, \phi, z) = \mathbf{E} \times \mathbf{H} = \frac{500}{\rho}\cos(4 \times 10^8 t - 4z)\mathbf{a}_\rho \times \frac{4.0}{\rho}\cos(4 \times 10^8 t - 4z)\mathbf{a}_\phi$$
$$= \underline{\frac{2.0 \times 10^{-3}}{\rho^2}\cos^2(4 \times 10^8 t - 4z)\mathbf{a}_z\ \mathrm{W/m^2}}$$

**11.19**d) the average power passing through every cross-section $8 < \rho < 20$ mm, $0 < \phi < 2\pi$. Using the result of part $c$, we find $< \mathbf{S} >= (1.0 \times 10^3)/\rho^2 \mathbf{a}_z$ W/m$^2$. The power through the given cross-section is now

$$\mathrm{P} = \int_0^{2\pi} \int_{.008}^{.020} \frac{1.0 \times 10^3}{\rho^2} \rho \, d\rho \, d\phi = 2\pi \times 10^3 \ln\left(\frac{20}{8}\right) = \underline{5.7 \, \mathrm{kW}}$$

**11.20.** Voltage breakdown in air at standard temperature and pressure occurs at an electric field strength of approximately $3 \times 10^6$ V/m. This becomes an issue in some high-power optical experiments, in which tight focusing of light may be necessary. Estimate the lightwave power in watts that can be focused into a cylindrical beam of $10\mu$m radius before breakdown occurs. Assume uniform plane wave behavior (although this assumption will produce an answer that is higher than the actual number by as much as a factor of 2, depending on the actual beam shape).

The power density in the beam in free space can be found as a special case of Eq. (76) (with $\eta = \eta_0$, $\theta_\eta = \alpha = 0$):

$$| < \mathbf{S} > | = \frac{E_0^2}{2\eta_0} = \frac{(3 \times 10^6)^2}{2(377)} = 1.2 \times 10^{10} \ \mathrm{W/m^2}$$

To avoid breakdown, the power in a 10-$\mu$m radius cylinder is then bounded by

$$P < (1.2 \times 10^{10})(\pi \times (10^{-5})^2) = \underline{3.75 \ \mathrm{W}}$$

**11.21.** The cylindrical shell, 1 cm $< \rho <$ 1.2 cm, is composed of a conducting material for which $\sigma = 10^6$ S/m. The external and internal regions are non-conducting. Let $H_\phi = 2000$ A/m at $\rho = 1.2$ cm.

a) Find **H** everywhere: Use Ampere's circuital law, which states:

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho(2000) = 2\pi(1.2 \times 10^{-2})(2000) = 48\pi \text{ A} = I_{encl}$$

Then in this case

$$\mathbf{J} = \frac{I}{Area}\mathbf{a}_z = \frac{48}{(1.44 - 1.00) \times 10^{-4}}\mathbf{a}_z = 1.09 \times 10^6\,\mathbf{a}_z \quad \text{A/m}^2$$

With this result we again use Ampere's circuital law to find **H** everywhere within the shell as a function of $\rho$ (in meters):

$$H_{\phi 1}(\rho) = \frac{1}{2\pi\rho}\int_0^{2\pi}\int_{.01}^{\rho} 1.09 \times 10^6 \;\rho\, d\rho\, d\phi = \underline{\frac{54.5}{\rho}(10^4\rho^2 - 1) \text{ A/m} \quad (.01 < \rho < .012)}$$

Outside the shell, we would have

$$H_{\phi 2}(\rho) = \frac{48\pi}{2\pi\rho} = \underline{24/\rho \;\text{ A/m} \quad (\rho > .012)}$$

Inside the shell ($\rho < .01$ m), $H_\phi = 0$ since there is no enclosed current.

b) Find **E** everywhere: We use

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} = \frac{1.09 \times 10^6}{10^6}\mathbf{a}_z = \underline{1.09\,\mathbf{a}_z \quad \text{V/m}}$$

which is valid, presumeably, outside as well as inside the shell.

c) Find **S** everywhere: Use

$$\mathbf{P} = \mathbf{E} \times \mathbf{H} = 1.09\,\mathbf{a}_z \times \frac{54.5}{\rho}(10^4\rho^2 - 1)\,\mathbf{a}_\phi$$

$$= \underline{-\frac{59.4}{\rho}(10^4\rho^2 - 1)\,\mathbf{a}_\rho \quad \text{W/m}^2 \quad (.01 < \rho < .012\,\text{m})}$$

Outside the shell,

$$\mathbf{S} = 1.09\,\mathbf{a}_z \times \frac{24}{\rho}\,\mathbf{a}_\phi = \underline{-\frac{26}{\rho}\,\mathbf{a}_\rho \quad \text{W/m}^2 \quad (\rho > .012\,\text{m})}$$

**11.22.** The inner and outer dimensions of a copper coaxial transmission line are 2 and 7 mm, respectively. Both conductors have thicknesses much greater than $\delta$. The dielectric is lossless and the operating frequency is 400 MHz. Calculate the resistance per meter length of the:

a) inner conductor: First

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi(4 \times 10^8)(4\pi \times 10^{-7})(5.8 \times 10^7)}} = 3.3 \times 10^{-6}\text{m} = 3.3\mu\text{m}$$

Now, using (90) with a unit length, we find

$$R_{in} = \frac{1}{2\pi a \sigma \delta} = \frac{1}{2\pi(2 \times 10^{-3})(5.8 \times 10^7)(3.3 \times 10^{-6})} = \underline{0.42 \text{ ohms/m}}$$

b) outer conductor: Again, (90) applies but with a different conductor radius. Thus

$$R_{out} = \frac{a}{b} R_{in} = \frac{2}{7}(0.42) = \underline{0.12 \text{ ohms/m}}$$

c) transmission line: Since the two resistances found above are in series, the line resistance is their sum, or $R = R_{in} + R_{out} = \underline{0.54 \text{ ohms/m}}$.

**11.23.** A hollow tubular conductor is constructed from a type of brass having a conductivity of $1.2 \times 10^7$ S/m. The inner and outer radii are 9 mm and 10 mm respectively. Calculate the resistance per meter length at a frequency of

a) dc: In this case the current density is uniform over the entire tube cross-section. We write:

$$R(\text{dc}) = \frac{L}{\sigma A} = \frac{1}{(1.2 \times 10^7)\pi(.01^2 - .009^2)} = \underline{1.4 \times 10^{-3} \ \Omega/\text{m}}$$

b) 20 MHz: Now the skin effect will limit the effective cross-section. At 20 MHz, the skin depth is

$$\delta(20\text{MHz}) = [\pi f \mu_0 \sigma]^{-1/2} = [\pi(20 \times 10^6)(4\pi \times 10^{-7})(1.2 \times 10^7)]^{-1/2} = 3.25 \times 10^{-5} \text{ m}$$

This is much less than the outer radius of the tube. Therefore we can approximate the resistance using the formula:

$$R(20\text{MHz}) = \frac{L}{\sigma A} = \frac{1}{2\pi b \delta} = \frac{1}{(1.2 \times 10^7)(2\pi(.01))(3.25 \times 10^{-5})} = \underline{4.1 \times 10^{-2} \ \Omega/\text{m}}$$

c) 2 GHz: Using the same formula as in part $b$, we find the skin depth at 2 GHz to be $\delta = 3.25 \times 10^{-6}$ m. The resistance (using the other formula) is $R(2\text{GHz}) = \underline{4.1 \times 10^{-1} \ \Omega/\text{m}}$.

**11.24** a) Most microwave ovens operate at 2.45 GHz. Assume that $\sigma = 1.2 \times 10^6$ S/m and $\mu_r = 500$ for the stainless steel interior, and find the depth of penetration:

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi(2.45 \times 10^9)(4\pi \times 10^{-7})(1.2 \times 10^6)}} = 9.28 \times 10^{-6}\text{m} = 9.28\mu\text{m}$$

b) Let $E_s = 50\angle\, 0^\circ$ V/m at the surface of the conductor, and plot a curve of the amplitude of $E_s$ vs. the angle of $E_s$ as the field propagates into the stainless steel: Since the conductivity is high, we use (82) to write $\alpha \doteq \beta \doteq \sqrt{\pi f \mu \sigma} = 1/\delta$. So, assuming that the direction into the conductor is $z$, the depth-dependent field is written as

$$E_s(z) = 50e^{-\alpha z}e^{-j\beta z} = 50e^{-z/\delta}e^{-jz/\delta} = \underbrace{50\exp(-z/9.28)}_{\text{amplitude}}\exp(-j\underbrace{z/9.28}_{\text{angle}})$$

where $z$ is in microns. Therefore, the plot of amplitude versus angle is simply a plot of $e^{-x}$ versus $x$, where $x = z/9.28$; the starting amplitude is 50 and the $1/e$ amplitude (at $z = 9.28$ $\mu$m) is 18.4.

**11.25.** A good conductor is planar in form and carries a uniform plane wave that has a wavelength of 0.3 mm and a velocity of $3 \times 10^5$ m/s. Assuming the conductor is non-magnetic, determine the frequency and the conductivity: First, we use

$$f = \frac{v}{\lambda} = \frac{3 \times 10^5}{3 \times 10^{-4}} = 10^9 \text{ Hz} = \underline{1 \text{ GHz}}$$

Next, for a good conductor,

$$\delta = \frac{\lambda}{2\pi} = \frac{1}{\sqrt{\pi f \mu \sigma}} \quad \Rightarrow \quad \sigma = \frac{4\pi}{\lambda^2 f \mu} = \frac{4\pi}{(9 \times 10^{-8})(10^9)(4\pi \times 10^{-7})} = \underline{1.1 \times 10^5 \text{ S/m}}$$

**11.26.** The dimensions of a certain coaxial transmission line are $a = 0.8$mm and $b = 4$mm. The outer conductor thickness is 0.6mm, and all conductors have $\sigma = 1.6 \times 10^7$ S/m.

a) Find $R$, the resistance per unit length, at an operating frequency of 2.4 GHz: First

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi (2.4 \times 10^8)(4\pi \times 10^{-7})(1.6 \times 10^7)}} = 2.57 \times 10^{-6}\text{m} = 2.57\mu\text{m}$$

Then, using (90) with a unit length, we find

$$R_{in} = \frac{1}{2\pi a \sigma \delta} = \frac{1}{2\pi (0.8 \times 10^{-3})(1.6 \times 10^7)(2.57 \times 10^{-6})} = 4.84 \text{ ohms/m}$$

The outer conductor resistance is then found from the inner through

$$R_{out} = \frac{a}{b} R_{in} = \frac{0.8}{4}(4.84) = 0.97 \text{ ohms/m}$$

The net resistance per length is then the sum, $R = R_{in} + R_{out} = \underline{5.81 \text{ ohms/m}}$.

b) Use information from Secs. 6.3 and 8.10 to find $C$ and $L$, the capacitance and inductance per unit length, respectively. The coax is air-filled. From those sections, we find (in free space)

$$C = \frac{2\pi\epsilon_0}{\ln(b/a)} = \frac{2\pi(8.854 \times 10^{-12})}{\ln(4/.8)} = \underline{3.46 \times 10^{-11} \text{ F/m}}$$

$$L = \frac{\mu_0}{2\pi} \ln(b/a) = \frac{4\pi \times 10^{-7}}{2\pi} \ln(4/.8) = \underline{3.22 \times 10^{-7} \text{ H/m}}$$

c) Find $\alpha$ and $\beta$ if $\alpha + j\beta = \sqrt{j\omega C(R + j\omega L)}$: Taking real and imaginary parts of the given expression, we find

$$\alpha = \text{Re}\left\{\sqrt{j\omega C(R + j\omega L)}\right\} = \frac{\omega\sqrt{LC}}{\sqrt{2}} \left[\sqrt{1 + \left(\frac{R}{\omega L}\right)^2} - 1\right]^{1/2}$$

and

$$\beta = \text{Im}\left\{\sqrt{j\omega C(R + j\omega L)}\right\} = \frac{\omega\sqrt{LC}}{\sqrt{2}} \left[\sqrt{1 + \left(\frac{R}{\omega L}\right)^2} + 1\right]^{1/2}$$

These can be found by writing out

$$\alpha = \text{Re}\left\{\sqrt{j\omega C(R + j\omega L)}\right\} = (1/2)\sqrt{j\omega C(R + j\omega L)} + c.c.$$

where $c.c$ denotes the complex conjugate. The result is squared, terms collected, and the square root taken. Now, using the values of $R$, $C$, and $L$ found in parts $a$ and $b$, we find $\alpha = \underline{3.0 \times 10^{-2} \text{ Np/m}}$ and $\beta = \underline{50.3 \text{ rad/m}}$.

**11.27.** The planar surface at $z = 0$ is a brass-Teflon interface. Use data available in Appendix C to evaluate the following ratios for a uniform plane wave having $\omega = 4 \times 10^{10}$ rad/s:

a) $\alpha_{\text{Tef}}/\alpha_{\text{brass}}$: From the appendix we find $\epsilon''/\epsilon' = .0003$ for Teflon, making the material a good dielectric. Also, for Teflon, $\epsilon'_r = 2.1$. For brass, we find $\sigma = 1.5 \times 10^7$ S/m, making brass a good conductor at the stated frequency. For a good dielectric (Teflon) we use the approximations:

$$\alpha \doteq \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon'}} = \left(\frac{\epsilon''}{\epsilon'}\right)\left(\frac{1}{2}\right)\omega\sqrt{\mu\epsilon'} = \frac{1}{2}\left(\frac{\epsilon''}{\epsilon'}\right)\frac{\omega}{c}\sqrt{\epsilon'_r}$$

$$\beta \doteq \omega\sqrt{\mu\epsilon'}\left[1 + \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)\right] \doteq \omega\sqrt{\mu\epsilon'} = \frac{\omega}{c}\sqrt{\epsilon'_r}$$

For brass (good conductor) we have

$$\alpha \doteq \beta \doteq \sqrt{\pi f \mu \sigma_{\text{brass}}} = \sqrt{\pi\left(\frac{1}{2\pi}\right)(4 \times 10^{10})(4\pi \times 10^{-7})(1.5 \times 10^7)} = 6.14 \times 10^5 \text{ m}^{-1}$$

Now

$$\frac{\alpha_{\text{Tef}}}{\alpha_{\text{brass}}} = \frac{1/2\,(\epsilon''/\epsilon')\,(\omega/c)\sqrt{\epsilon'_r}}{\sqrt{\pi f \mu \sigma_{\text{brass}}}} = \frac{(1/2)(.0003)(4 \times 10^{10}/3 \times 10^8)\sqrt{2.1}}{6.14 \times 10^5} = \underline{4.7 \times 10^{-8}}$$

b)

$$\frac{\lambda_{\text{Tef}}}{\lambda_{\text{brass}}} = \frac{(2\pi/\beta_{\text{Tef}})}{(2\pi/\beta_{\text{brass}})} = \frac{\beta_{\text{brass}}}{\beta_{\text{Tef}}} = \frac{c\sqrt{\pi f \mu \sigma_{\text{brass}}}}{\omega\sqrt{\epsilon'_{r\,\text{Tef}}}} = \frac{(3 \times 10^8)(6.14 \times 10^5)}{(4 \times 10^{10})\sqrt{2.1}} = \underline{3.2 \times 10^3}$$

c)

$$\frac{v_{\text{Tef}}}{v_{\text{brass}}} = \frac{(\omega/\beta_{\text{Tef}})}{(\omega/\beta_{\text{brass}})} = \frac{\beta_{\text{brass}}}{\beta_{\text{Tef}}} = \underline{3.2 \times 10^3} \text{ as before}$$

**11.28.** A uniform plane wave in free space has electric field given by $\mathbf{E}_s = 10e^{-j\beta x}\mathbf{a}_z + 15e^{-j\beta x}\mathbf{a}_y$ V/m.

a) Describe the wave polarization: Since the two components have a fixed phase difference (in this case zero) with respect to time and position, the wave has linear polarization, with the field vector in the $yz$ plane at angle $\phi = \tan^{-1}(10/15) = 33.7°$ to the $y$ axis.

b) Find $\mathbf{H}_s$: With propagation in forward $x$, we would have

$$\mathbf{H}_s = \frac{-10}{377}e^{-j\beta x}\mathbf{a}_y + \frac{15}{377}e^{-j\beta x}\mathbf{a}_z \text{ A/m} = \underline{-26.5e^{-j\beta x}\mathbf{a}_y + 39.8e^{-j\beta x}\mathbf{a}_z \text{ mA/m}}$$

c) determine the average power density in the wave in W/m$^2$: Use

$$\mathbf{P}_{avg} = \frac{1}{2}\text{Re}\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\} = \frac{1}{2}\left[\frac{(10)^2}{377}\mathbf{a}_x + \frac{(15)^2}{377}\mathbf{a}_x\right] = 0.43\mathbf{a}_x \text{ W/m}^2 \text{ or } P_{avg} = \underline{0.43 \text{ W/m}^2}$$

**11.29.** Consider a left-circularly polarized wave in free space that propagates in the forward $z$ direction. The electric field is given by the appropriate form of Eq. (100).

a) Determine the magnetic field phasor, $\mathbf{H}_s$:
We begin, using (100), with $\mathbf{E}_s = E_0(\mathbf{a}_x + j\mathbf{a}_y)e^{-j\beta z}$. We find the two components of $\mathbf{H}_s$ separately, using the two components of $\mathbf{E}_s$. Specifically, the $x$ component of $\mathbf{E}_s$ is associated with a $y$ component of $\mathbf{H}_s$, and the $y$ component of $\mathbf{E}_s$ is associated with a negative $x$ component of $\mathbf{H}_s$. The result is

$$\mathbf{H}_s = \underline{\frac{E_0}{\eta_0}\left(\mathbf{a}_y - j\mathbf{a}_x\right) e^{-j\beta z}}$$

b) Determine an expression for the average power density in the wave in W/m$^2$ by direct application of Eq. (77): We have

$$\mathbf{P}_{z,avg} = \frac{1}{2}Re(\mathbf{E}_s \times \mathbf{H}_s^*) = \frac{1}{2}Re\left(E_0(\mathbf{a}_x + j\mathbf{a}_y)e^{-j\beta z} \times \frac{E_0}{\eta_0}(\mathbf{a}_y - j\mathbf{a}_x)e^{+j\beta z}\right)$$
$$= \underline{\frac{E_0^2}{\eta_0}\mathbf{a}_z \text{ W/m}^2} \quad \text{(assuming } E_0 \text{ is real)}$$

**11.30.** In an *anisotropic* medium, permittivity varies with electric field *direction*, and is a property seen in most crystals. Consider a uniform plane wave propagating in the $z$ direction in such a medium, and which enters the material with equal field components along the $x$ and $y$ axes. The field phasor will take the form:

$$\mathbf{E}_s(z) = E_0(\mathbf{a}_x + \mathbf{a}_y\, e^{j\Delta\beta z})\, e^{-j\beta z}$$

where $\Delta\beta = \beta_x - \beta_y$ is the difference in phase constants for waves that are linearly-polarized in the $x$ and $y$ directions. Find distances into the material (in terms of $\Delta\beta$) at which the field is:

a) Linearly-polarized: We want the $x$ and $y$ components to be in phase, so therefore

$$\Delta\beta z_{lin} = m\pi \;\Rightarrow\; z_{lin} = \underline{\frac{m\pi}{\Delta\beta}\ ,\ (m = 1, 2, 3...)}$$

b) Circularly-polarized: In this case, we want the two field components to be in quadrature phase, such that the total field is of the form, $\mathbf{E}_s = E_0(\mathbf{a}_x \pm j\mathbf{a}_y)e^{-j\beta z}$. Therefore,

$$\Delta\beta z_{circ} = \frac{(2n+1)\pi}{2} \;\Rightarrow\; z_{circ} = \underline{\frac{(2n+1)\pi}{2\Delta\beta}\ ,\quad (n = 0, 1, 2, 3...)}$$

c) Assume intrinsic impedance $\eta$ that is approximately constant with field orientation and find $\mathbf{H}_s$ and $< \mathbf{S} >$: Magnetic field is found by looking at the individual components:

$$\mathbf{H}_s(z) = \underline{\frac{E_0}{\eta}\left(\mathbf{a}_y - \mathbf{a}_x e^{j\Delta\beta z}\right) e^{-j\beta z}} \text{ and}$$

$$< \mathbf{S} >= \frac{1}{2}\mathcal{R}e\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\} = \underline{\frac{E_0^2}{\eta}\,\mathbf{a}_z \text{ W/m}^2}$$

where it is assumed that $E_0$ is real. $\eta$ is real because the medium is evidently lossless.

**11.31.** A linearly-polarized uniform plane wave, propagating in the forward $z$ direction, is input to a lossless *anisotropic* material, in which the dielectric constant encountered by waves polarized along $y$ ($\epsilon_{ry}$) differs from that seen by waves polarized along $x$ ($\epsilon_{rx}$). Suppose $\epsilon_{rx} = 2.15$, $\epsilon_{ry} = 2.10$, and the wave electric field at input is polarized at 45° to the positive $x$ and $y$ axes. *Assume free space wavelength* $\lambda$.

a) Determine the shortest length of the material such that the wave as it emerges from the output end is circularly polarized: With the input field at 45°, the $x$ and $y$ components are of equal magnitude, and circular polarization will result if the phase difference between the components is $\pi/2$. Our requirement over length $L$ is thus $\beta_x L - \beta_y L = \pi/2$, or

$$L = \frac{\pi}{2(\beta_x - \beta_y)} = \frac{\pi c}{2\omega(\sqrt{\epsilon_{rx}} - \sqrt{\epsilon_{ry}})}$$

With the given values, we find,

$$L = \frac{(58.3)\pi c}{2\omega} = 58.3\,\frac{\lambda}{4} = \underline{14.6\,\lambda}$$

b) Will the output wave be right- or left-circularly-polarized? With the dielectric constant greater for $x$-polarized waves, the $x$ component will lag the $y$ component in time at the output. The field can thus be written as $\mathbf{E} = E_0(\mathbf{a}_y - j\mathbf{a}_x)$, which is left circular polarization.

**11.32.** Suppose that the length of the medium of Problem 11.31 is made to be *twice* that as determined in the problem. Describe the polarization of the output wave in this case: With the length doubled, a phase shift of $\pi$ radians develops between the two components. At the input, we can write the field as $\mathbf{E}_s(0) = E_0(\mathbf{a}_x + \mathbf{a}_y)$. After propagating through length $L$, we would have,

$$\mathbf{E}_s(L) = E_0[e^{-j\beta_x L}\mathbf{a}_x + e^{-j\beta_y L}\mathbf{a}_y] = E_0 e^{-j\beta_x L}[\mathbf{a}_x + e^{-j(\beta_y - \beta_x)L}\mathbf{a}_y]$$

where $(\beta_y - \beta_x)L = -\pi$ (since $\beta_x > \beta_y$), and so $\mathbf{E}_s(L) = E_0 e^{-j\beta_x L}[\mathbf{a}_x - \mathbf{a}_y]$. With the reversal of the $y$ component, the wave polarization is rotated by 90°, but is still linear polarization.

**11.33.** Given a wave for which $\mathbf{E}_s = 15e^{-j\beta z}\mathbf{a}_x + 18e^{-j\beta z}e^{j\phi}\mathbf{a}_y$ V/m, propagating in a medium characterized by complex intrinsic impedance, $\eta$.

a) Find $\mathbf{H}_s$: With the wave propagating in the forward $z$ direction, we find:

$$\underline{\mathbf{H}_s = \frac{1}{\eta}\left[-18e^{j\phi}\mathbf{a}_x + 15\mathbf{a}_y\right]e^{-j\beta z}\ \text{A/m}}$$

b) Determine the average power density in W/m²: We find

$$P_{z,avg} = \frac{1}{2}\text{Re}\{\mathbf{E}_s \times \mathbf{H}_s^*\} = \frac{1}{2}\text{Re}\left\{\frac{(15)^2}{\eta^*} + \frac{(18)^2}{\eta^*}\right\} = \underline{275\,\text{Re}\left\{\frac{1}{\eta^*}\right\}\ \text{W/m}^2}$$

**11.34.** Given the general elliptically-polarized wave as per Eq. (93):

$$\mathbf{E}_s = [E_{x0}\mathbf{a}_x + E_{y0}e^{j\phi}\mathbf{a}_y]e^{-j\beta z}$$

a) Show, using methods similar to those of Example 11.7, that a linearly polarized wave results when superimposing the given field and a phase-shifted field of the form:

$$\mathbf{E}_s = [E_{x0}\mathbf{a}_x + E_{y0}e^{-j\phi}\mathbf{a}_y]e^{-j\beta z}e^{j\delta}$$

where $\delta$ is a constant: Adding the two fields gives

$$\mathbf{E}_{s,tot} = \left[E_{x0}\left(1+e^{j\delta}\right)\mathbf{a}_x + E_{y0}\left(e^{j\phi}+e^{-j\phi}e^{j\delta}\right)\mathbf{a}_y\right]e^{-j\beta z}$$

$$= \left[E_{x0}e^{j\delta/2}\underbrace{\left(e^{-j\delta/2}+e^{j\delta/2}\right)}_{2\cos(\delta/2)}\mathbf{a}_x + E_{y0}e^{j\delta/2}\underbrace{\left(e^{-j\delta/2}e^{j\phi}+e^{-j\phi}e^{j\delta/2}\right)}_{2\cos(\phi-\delta/2)}\mathbf{a}_y\right]e^{-j\beta z}$$

This simplifies to $\mathbf{E}_{s,tot} = 2\left[E_{x0}\cos(\delta/2)\mathbf{a}_x + E_{y0}\cos(\phi-\delta/2)\mathbf{a}_y\right]e^{j\delta/2}e^{-j\beta z}$, which is linearly polarized.

b) Find $\delta$ in terms of $\phi$ such that the resultant wave is polarized along $x$: By inspecting the part $a$ result, we achieve a zero $y$ component when $\underline{2\phi - \delta = \pi}$ (or odd multiples of $\pi$).

# CHAPTER 12

**12.1.** A uniform plane wave in air, $E^+_{x1} = E^+_{x10}\cos(10^{10}t - \beta z)$ V/m, is normally-incident on a copper surface at $z = 0$. What percentage of the incident power density is transmitted into the copper? We need to find the reflection coefficient. The intrinsic impedance of copper (a good conductor) is

$$\eta_c = \sqrt{\frac{j\omega\mu}{\sigma}} = (1+j)\sqrt{\frac{\omega\mu}{2\sigma}} = (1+j)\sqrt{\frac{10^{10}(4\pi \times 10^7)}{2(5.8 \times 10^7)}} = (1+j)(.0104)$$

Note that the accuracy here is questionable, since we know the conductivity to only two significant figures. We nevertheless proceed: Using $\eta_0 = 376.7288$ ohms, we write

$$\Gamma = \frac{\eta_c - \eta_0}{\eta_c + \eta_0} = \frac{.0104 - 376.7288 + j.0104}{.0104 + 376.7288 + j.0104} = -.9999 + j.0001$$

Now $|\Gamma|^2 = .9999$, and so the transmitted power fraction is $1 - |\Gamma|^2 = .0001$, or about $\underline{0.01\%}$ is transmitted.

**12.2.** The plane $z = 0$ defines the boundary between two dielectrics. For $z < 0$, $\epsilon_{r1} = 9$, $\epsilon''_{r1} = 0$, and $\mu_1 = \mu_0$. For $z > 0$, $\epsilon'_{r2} = 3$, $\epsilon''_{r2} = 0$, and $\mu_2 = \mu_0$. Let $E^+_{x1} = 10\cos(\omega t - 15z)$ V/m and find

a) $\omega$: We have $\beta = \omega\sqrt{\mu_0\epsilon'_1} = \omega\sqrt{\epsilon'_{r1}}/c = 15$. So $\omega = 15c/\sqrt{\epsilon'_{r1}} = 15 \times (3 \times 10^8)/\sqrt{9} = \underline{1.5 \times 10^9 \text{ s}^{-1}}$.

b) $< \mathbf{S}^+_1 >$: First we need $\eta_1 = \sqrt{\mu_0/\epsilon'_1} = \eta_0/\sqrt{\epsilon'_{r1}} = 377/\sqrt{9} = 126$ ohms. Next we apply Eq. (76), Chapter 11, to evaluate the Poynting vector (with no loss and consequently with no phase difference between electric and magnetic fields). We find $< \mathbf{S}^+_1 >= (1/2)|E_1|^2/\eta_1\,\mathbf{a}_z = (1/2)(10)^2/126\,\mathbf{a}_z = \underline{0.40\,\mathbf{a}_z \text{ W/m}^2}$.

c) $< \mathbf{S}^-_1 >$: First, we need to evaluate the reflection coefficient:

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\eta_0/\sqrt{\epsilon'_{r2}} - \eta_0/\sqrt{\epsilon'_{r1}}}{\eta_0/\sqrt{\epsilon'_{r2}} + \eta_0/\sqrt{\epsilon'_{r1}}} = \frac{\sqrt{\epsilon'_{r1}} - \sqrt{\epsilon'_{r2}}}{\sqrt{\epsilon'_{r1}} + \sqrt{\epsilon'_{r2}}} = \frac{\sqrt{9} - \sqrt{3}}{\sqrt{9} + \sqrt{3}} = 0.27$$

Then $< \mathbf{S}^-_1 >= -|\Gamma|^2 < \mathbf{S}^+_1 >= -(0.27)^2(0.40)\,\mathbf{a}_z = \underline{-0.03\,\mathbf{a}_z \text{ W/m}^2}$.

d) $< \mathbf{S}^+_2 >$: This will be the remaining power, propagating in the forward $z$ direction, or $< \mathbf{S}^+_2 >= \underline{0.37\,\mathbf{a}_z \text{ W/m}^2}$.

**12.3.** A uniform plane wave in region 1 is normally-incident on the planar boundary separating regions 1 and 2. If $\epsilon_1'' = \epsilon_2'' = 0$, while $\epsilon_{r1}' = \mu_{r1}^3$ and $\epsilon_{r2}' = \mu_{r2}^3$, find the ratio $\epsilon_{r2}'/\epsilon_{r1}'$ if 20% of the energy in the incident wave is reflected at the boundary. There are two possible answers. First, since $|\Gamma|^2 = .20$, and since both permittivities and permeabilities are real, $\Gamma = \pm 0.447$. we then set up

$$\Gamma = \pm 0.447 = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\eta_0\sqrt{(\mu_{r2}/\epsilon_{r2}')} - \eta_0\sqrt{(\mu_{r1}/\epsilon_{r1}')}}{\eta_0\sqrt{(\mu_{r2}/\epsilon_{r2}')} + \eta_0\sqrt{(\mu_{r1}/\epsilon_{r1}')}}$$

$$= \frac{\sqrt{(\mu_{r2}/\mu_{r2}^3)} - \sqrt{(\mu_{r1}/\mu_{r1}^3)}}{\sqrt{(\mu_{r2}/\mu_{r2}^3)} + \sqrt{(\mu_{r1}/\mu_{r1}^3)}} = \frac{\mu_{r1} - \mu_{r2}}{\mu_{r1} + \mu_{r2}}$$

Therefore

$$\frac{\mu_{r2}}{\mu_{r1}} = \frac{1 \mp 0.447}{1 \pm 0.447} = (0.382, 2.62) \quad \Rightarrow \quad \frac{\epsilon_{r2}'}{\epsilon_{r1}'} = \left(\frac{\mu_{r2}}{\mu_{r1}}\right)^3 = \underline{(0.056, 17.9)}$$

**12.4.** A 10-MHz uniform plane wave having an initial average power density of 5W/m$^2$ is normally-incident from free space onto the surface of a lossy material in which $\epsilon_2''/\epsilon_2' = 0.05$, $\epsilon_{r2}' = 5$, and $\mu_2 = \mu_0$. Calculate the distance into the lossy medium at which the transmitted wave power density is down by 10dB from the initial 5W/m$^2$:

First, since $\epsilon_2''/\epsilon_2' = 0.05 << 1$, we recognize region 2 as a good dielectric. Its intrinsic impedance is therefore approximated well by Eq. (62b), Chapter 11:

$$\eta_2 = \sqrt{\frac{\mu_0}{\epsilon_2'}}\left[1 + j\frac{1}{2}\frac{\epsilon_2''}{\epsilon_2'}\right] = \frac{377}{\sqrt{5}}\left[1 + j0.025\right]$$

The reflection coefficient encountered by the incident wave from region 1 is therefore

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{(377/\sqrt{5})[1 + j.025] - 377}{(377/\sqrt{5})[1 + j.025] + 377} = \frac{(1 - \sqrt{5}) + j.025}{(1 + \sqrt{5}) + j.025} = -0.383 + j0.011$$

The fraction of the incident power that is reflected is then $|\Gamma|^2 = 0.147$, and thus the fraction of the power that is transmitted into region 2 is $1 - |\Gamma|^2 = 0.853$. Still using the good dielectric approximation, the attenuation coefficient in region 2 is found from Eq. (60a), Chapter 11:

$$\alpha \doteq \frac{\omega\epsilon_2''}{2}\sqrt{\frac{\mu_0}{\epsilon_2'}} = (2\pi \times 10^7)(0.05 \times 5 \times 8.854 \times 10^{-12})\frac{377}{2\sqrt{5}} = 2.34 \times 10^{-2} \text{ Np/m}$$

Now, the power that propagates into region 2 is expressed in terms of the incident power through

$$< S_2 > (z) = 5(1 - |\Gamma|^2)e^{-2\alpha z} = 5(.853)e^{-2(2.34 \times 10^{-2})z} = 0.5 \text{ W/m}^2$$

in which the last equality indicates a factor of ten reduction from the incident power, as occurs for a 10 dB loss. Solve for $z$ to obtain

$$z = \frac{\ln(8.53)}{2(2.34 \times 10^{-2})} = \underline{45.8 \text{ m}}$$

**12.5.** The region $z < 0$ is characterized by $\epsilon'_r = \mu_r = 1$ and $\epsilon''_r = 0$. The total **E** field here is given as the sum of the two uniform plane waves, $\mathbf{E}_s = 150e^{-j10z}\,\mathbf{a}_x + (50\angle 20^\circ)e^{j10z}\,\mathbf{a}_x$ V/m.

a) What is the operating frequency? In free space, $\beta = k_0 = 10 = \omega/c = \omega/3 \times 10^8$. Thus, $\omega = 3 \times 10^9$ s$^{-1}$, or $f = \omega/2\pi = \underline{4.7 \times 10^8 \text{ Hz}}$.

b) Specify the intrinsic impedance of the region $z > 0$ that would provide the appropriate reflected wave: Use

$$\Gamma = \frac{E_r}{E_{inc}} = \frac{50e^{j20^\circ}}{150} = \frac{1}{3}e^{j20^\circ} = 0.31 + j0.11 = \frac{\eta - \eta_0}{\eta + \eta_0}$$

Now

$$\eta = \eta_0 \left(\frac{1+\Gamma}{1-\Gamma}\right) = 377 \left(\frac{1 + 0.31 + j0.11}{1 - 0.31 - j0.31}\right) = \underline{691 + j177\ \Omega}$$

c) At what value of $z$ ($-10\,\text{cm} < z < 0$) is the total electric field intensity a maximum amplitude? We found the phase of the reflection coefficient to be $\phi = 20^\circ = .349$rad, and we use

$$z_{max} = \frac{-\phi}{2\beta} = \frac{-.349}{20} = -0.017\,\text{m} = \underline{-1.7\,\text{cm}}$$

**12.6.** In the beam-steering prism of Example 12.8, suppose the anti-reflective coatings are removed, leaving bare glass-to-air interfaces. Calcluate the ratio of the prism output power to the input power, assuming a single transit.

In making the transit, the light encounters two interfaces at normal incidence, at which loss will occur. The reflection coefficient at the front surface (air to glass) is

$$\Gamma_f = \frac{\eta_g - \eta_0}{\eta_g + \eta_0} = \frac{\eta_0/n_g - \eta_0}{\eta_0/n_g + \eta_0} = \frac{1 - n_g}{1 + n_g}$$

Taking the glass index, $n_g$, as 1.45, we find $\Gamma_f = -0.18$. The interface on exit from the prism is glass to air, and so the reflection coefficient there will be equal and opposite to $\Gamma_f$; i.e., $\Gamma_b = -\Gamma_f$.

Now, the wave power that makes it through (assuming total reflection at the 45° interface) will be

$$P_{out} = P_{in}(1 - |\Gamma_f|^2)(1 - |\Gamma_b|^2) = P_{in}(1 - |0.18|^2)^2 = \underline{0.93}P_{in}$$

So we have 93% net transmission.

**12.7.** The semi-infinite regions $z < 0$ and $z > 1$ m are free space. For $0 < z < 1$ m, $\epsilon'_r = 4$, $\mu_r = 1$, and $\epsilon''_r = 0$. A uniform plane wave with $\omega = 4 \times 10^8$ rad/s is travelling in the $\mathbf{a}_z$ direction toward the interface at $z = 0$.

a) Find the standing wave ratio in each of the three regions: First we find the phase constant in the middle region,

$$\beta_2 = \frac{\omega\sqrt{\epsilon'_r}}{c} = \frac{2(4 \times 10^8)}{3 \times 10^8} = 2.67\,\text{rad/m}$$

Then, with the middle layer thickness of 1 m, $\beta_2 d = 2.67\,\text{rad}$. Also, the intrinsic impedance of the middle layer is $\eta_2 = \eta_0/\sqrt{\epsilon'_r} = \eta_0/2$. We now find the input impedance:

$$\eta_{in} = \eta_2 \left[\frac{\eta_0 \cos(\beta_2 d) + j\eta_2 \sin(\beta_2 d)}{\eta_2 \cos(\beta_2 d) + j\eta_0 \sin(\beta_2 d)}\right] = \frac{377}{2}\left[\frac{2\cos(2.67) + j\sin(2.67)}{\cos(2.67) + j2\sin(2.67)}\right] = 231 + j141$$

Now, at the first interface,

$$\Gamma_{12} = \frac{\eta_{in} - \eta_0}{\eta_{in} + \eta_0} = \frac{231 + j141 - 377}{231 + j141 + 377} = -.176 + j.273 = .325\angle 123^\circ$$

The standing wave ratio measured in region 1 is thus

$$s_1 = \frac{1 + |\Gamma_{12}|}{1 - |\Gamma_{12}|} = \frac{1 + 0.325}{1 - 0.325} = \underline{1.96}$$

In region 2 the standing wave ratio is found by considering the reflection coefficient for waves incident from region 2 on the second interface:

$$\Gamma_{23} = \frac{\eta_0 - \eta_0/2}{\eta_0 + \eta_0/2} = \frac{1 - 1/2}{1 + 1/2} = \frac{1}{3}$$

Then

$$s_2 = \frac{1 + 1/3}{1 - 1/3} = \underline{2}$$

Finally, $s_3 = \underline{1}$, since no reflected waves exist in region 3.

b) Find the location of the maximum $|\mathbf{E}|$ for $z < 0$ that is nearest to $z = 0$. We note that the phase of $\Gamma_{12}$ is $\phi = 123^\circ = 2.15$ rad. Thus

$$z_{max} = \frac{-\phi}{2\beta} = \frac{-2.15}{2(4/3)} = \underline{-.81\,\text{m}}$$

**12.8.** A wave starts at point $a$, propagates 1m through a lossy dielectric rated at $\alpha_{dB} = 0.1$dB/cm, reflects at normal incidence at a boundary at which $\Gamma = 0.3 + j0.4$, and then returns to point $a$. Calculate the ratio of the final power to the incident power after this round trip: Final power, $P_f$, and incident power, $P_i$, are related through

$$P_f = P_i 10^{-0.1\alpha_{dB}L}|\Gamma|^2 10^{-0.1\alpha_{dB}L} \quad \Rightarrow \quad \frac{P_f}{P_i} = |0.3 + j0.4|^2 10^{-0.2(0.1)100} = \underline{2.5 \times 10^{-3}}$$

**12.9.** Region 1, $z < 0$, and region 2, $z > 0$, are both perfect dielectrics ($\mu = \mu_0$, $\epsilon'' = 0$). A uniform plane wave traveling in the $\mathbf{a}_z$ direction has a radian frequency of $3 \times 10^{10}$ rad/s. Its wavelengths in the two regions are $\lambda_1 = 5$ cm and $\lambda_2 = 3$ cm. What percentage of the energy incident on the boundary is

a) reflected; We first note that

$$\epsilon'_{r1} = \left(\frac{2\pi c}{\lambda_1 \omega}\right)^2 \quad \text{and} \quad \epsilon'_{r2} = \left(\frac{2\pi c}{\lambda_2 \omega}\right)^2$$

Therefore $\epsilon'_{r1}/\epsilon'_{r2} = (\lambda_2/\lambda_1)^2$. Then with $\mu = \mu_0$ in both regions, we find

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\eta_0\sqrt{1/\epsilon'_{r2}} - \eta_0\sqrt{1/\epsilon'_{r1}}}{\eta_0\sqrt{1/\epsilon'_{r2}} + \eta_0\sqrt{1/\epsilon'_{r1}}} = \frac{\sqrt{\epsilon'_{r1}/\epsilon'_{r2}} - 1}{\sqrt{\epsilon'_{r1}/\epsilon'_{r2}} + 1} = \frac{(\lambda_2/\lambda_1) - 1}{(\lambda_2/\lambda_1) + 1}$$
$$= \frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1} = \frac{3-5}{3+5} = -\frac{1}{4}$$

The fraction of the incident energy that is reflected is then $|\Gamma|^2 = 1/16 = \underline{6.25 \times 10^{-2}}$.

b) transmitted? We use part $a$ and find the transmitted fraction to be $1 - |\Gamma|^2 = 15/16 = \underline{0.938}$.

c) What is the standing wave ratio in region 1? Use

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 1/4}{1 - 1/4} = \frac{5}{3} = \underline{1.67}$$

**12.10.** In Fig. 12.1, let region 2 be free space, while $\mu_{r1} = 1$, $\epsilon''_{r1} = 0$, and $\epsilon'_{r1}$ is unknown. Find $\epsilon'_{r1}$ if

a) the amplitude of $\mathbf{E}_1^-$ is one-half that of $\mathbf{E}_1^+$: Since region 2 is free space, the reflection coefficient is

$$\Gamma = \frac{|\mathbf{E}_1^-|}{|\mathbf{E}_1^+|} = \frac{\eta_0 - \eta_1}{\eta_0 + \eta_1} = \frac{\eta_0 - \eta_0/\sqrt{\epsilon'_{r1}}}{\eta_0 + \eta_0/\sqrt{\epsilon'_{r1}}} = \frac{\sqrt{\epsilon'_{r1}} - 1}{\sqrt{\epsilon'_{r1}} + 1} = \frac{1}{2} \quad \Rightarrow \quad \epsilon'_{r1} = \underline{9}$$

.

b) $< \mathbf{S}_1^- >$ is one-half of $< \mathbf{S}_1^+ >$: This time

$$|\Gamma|^2 = \left|\frac{\sqrt{\epsilon'_{r1}} - 1}{\sqrt{\epsilon'_{r1}} + 1}\right|^2 = \frac{1}{2} \quad \Rightarrow \quad \epsilon'_{r1} = \underline{34}$$

c) $|\mathbf{E}_1|_{min}$ is one-half $|\mathbf{E}_1|_{max}$: Use

$$\frac{|\mathbf{E}_1|_{max}}{|\mathbf{E}_1|_{min}} = s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = 2 \quad \Rightarrow \quad |\Gamma| = \Gamma = \frac{1}{3} = \frac{\sqrt{\epsilon'_{r1}} - 1}{\sqrt{\epsilon'_{r1}} + 1} \quad \Rightarrow \quad \epsilon'_{r1} = \underline{4}$$

**12.11.** A 150 MHz uniform plane wave in normally-incident from air onto a material whose intrinsic impedance is unknown. Measurements yield a standing wave ratio of 3 and the appearance of an electric field minimum at 0.3 wavelengths in front of the interface. Determine the impedance of the unknown material: First, the field minimum is used to find the phase of the reflection coefficient, where

$$z_{min} = -\frac{1}{2\beta}(\phi + \pi) = -0.3\lambda \quad \Rightarrow \quad \phi = 0.2\pi$$

where $\beta = 2\pi/\lambda$ has been used. Next,

$$|\Gamma| = \frac{s-1}{s+1} = \frac{3-1}{3+1} = \frac{1}{2}$$

So we now have

$$\Gamma = 0.5e^{j0.2\pi} = \frac{\eta_u - \eta_0}{\eta_u + \eta_0}$$

We solve for $\eta_u$ to find

$$\eta_u = \eta_0(1.70 + j1.33) = \underline{641 + j501\ \Omega}$$

**12.12.** A 50MHz uniform plane wave is normally incident from air onto the surface of a calm ocean. For seawater, $\sigma = 4$ S/m, and $\epsilon'_r = 78$.

a) Determine the fractions of the incident power that are reflected and transmitted: First we find the loss tangent:

$$\frac{\sigma}{\omega\epsilon'} = \frac{4}{2\pi(50 \times 10^6)(78)(8.854 \times 10^{-12})} = 18.4$$

This value is sufficiently greater than 1 to enable seawater to be considered a good conductor at 50MHz. Then, using the approximation (Eq. 85, Chapter 11), the intrinsic impedance is $\eta_s = \sqrt{\pi f \mu/\sigma}(1+j)$, and the reflection coefficient becomes

$$\Gamma = \frac{\sqrt{\pi f \mu/\sigma}\ (1+j) - \eta_0}{\sqrt{\pi f \mu/\sigma}\ (1+j) + \eta_0}$$

where $\sqrt{\pi f \mu/\sigma} = \sqrt{\pi(50 \times 10^6)(4\pi \times 10^{-7})/4} = 7.0$. The fraction of the power reflected is

$$\frac{P_r}{P_i} = |\Gamma|^2 = \frac{[\sqrt{\pi f \mu/\sigma} - \eta_0]^2 + \pi f \mu/\sigma}{[\sqrt{\pi f \mu/\sigma} + \eta_0]^2 + \pi f \mu/\sigma} = \frac{[7.0 - 377]^2 + 49.0}{[7.0 + 377]^2 + 49.0} = \underline{0.93}$$

The transmitted fraction is then

$$\frac{P_t}{P_i} = 1 - |\Gamma|^2 = 1 - 0.93 = \underline{0.07}$$

b) Qualitatively, how will these answers change (if at all) as the frequency is increased? Within the limits of our good conductor approximation (loss tangent greater than about ten), the reflected power fraction, using the formula derived in part *a*, is found to decrease with increasing frequency. The transmitted power fraction thus increases.

**12.13.** A right-circularly-polarized plane wave is normally incident from air onto a semi-infinite slab of plexiglas ($\epsilon'_r = 3.45$, $\epsilon''_r = 0$). Calculate the fractions of the incident power that are reflected and transmitted. Also, describe the polarizations of the reflected and transmitted waves. First, the impedance of the plexiglas will be $\eta = \eta_0/\sqrt{3.45} = 203\,\Omega$. Then

$$\Gamma = \frac{203 - 377}{203 + 377} = -0.30$$

The reflected power fraction is thus $|\Gamma|^2 = \underline{0.09}$. The total electric field in the plane of the interface must rotate in the same direction as the incident field, in order to continually satisfy the boundary condition of tangential electric field continuity across the interface. Therefore, the reflected wave will have to be left circularly polarized in order to make this happen. The transmitted power fraction is now $1 - |\Gamma|^2 = \underline{0.91}$. The transmitted field will be right circularly polarized (as the incident field) for the same reasons.

**12.14.** A left-circularly-polarized plane wave is normally-incident onto the surface of a perfect conductor.

a) Construct the superposition of the incident and reflected waves in phasor form: Assume positive $z$ travel for the incident electric field. Then, with reflection coefficient, $\Gamma = -1$, the incident and reflected fields will add to give the total field:

$$\mathbf{E}_{tot} = \mathbf{E}_i + \mathbf{E}_r = E_0(\mathbf{a}_x + j\mathbf{a}_y)e^{-j\beta z} - E_0(\mathbf{a}_x + j\mathbf{a}_y)e^{+j\beta z}$$

$$= E_0\left[\underbrace{\left(e^{-j\beta z} - e^{j\beta z}\right)}_{-2j\sin(\beta z)}\mathbf{a}_x + j\underbrace{\left(e^{-j\beta z} - e^{j\beta z}\right)}_{-2j\sin(\beta z)}\mathbf{a}_y\right] = \underline{2E_0\sin(\beta z)\left[\mathbf{a}_y - j\mathbf{a}_x\right]}$$

b) Determine the real instantaneous form of the result of part $a$:

$$\mathbf{E}(z,t) = \text{Re}\left\{\mathbf{E}_{tot}e^{j\omega t}\right\} = \underline{2E_0\sin(\beta z)\left[\cos(\omega t)\mathbf{a}_y + \sin(\omega t)\mathbf{a}_x\right]}$$

c) Describe the wave that is formed: This is a standing wave exhibiting circular polarization in time. At each location along the $z$ axis, the field vector rotates clockwise in the $xy$ plane, and has amplitude (constant with time) given by $2E_0\sin(\beta z)$.

**12.15.** Sulfur hexafluoride ($SF_6$) is a high-density gas that has refractive index, $n_s = 1.8$ at a specified pressure, temperature, and wavelength. Consider the retro-reflecting prism shown in Fig. 12.16, that is immersed in $SF_6$. Light enters through a quarter-wave antireflective coating and then totally reflects from the back surfaces of the glass. In principle, the beam should experience zero loss at the design wavelength ($P_{out} = P_{in}$).

a) Determine the minimum required value of the glass refractive index, $n_g$, so that the interior beam will totally reflect: We set the critical angle of total reflection equal to 45°, which gives

$$\sin\theta_c = \frac{n_s}{n_g} = \sin(45^\circ) = \frac{1}{\sqrt{2}} \;\Rightarrow\; n_g = n_s\sqrt{2} = \underline{2.55}$$

b) Knowing $n_g$, find the required refractive index of the quarter-wave film, $n_f$: For a quarter-wave section, we know that the film intrinsic impedance will be

$$\eta_f = \sqrt{\eta_s \eta_g} \;\Rightarrow\; n_f = \sqrt{n_s n_g} = \sqrt{(1.80)(2.55)} = \underline{2.14}$$

c) With the $SF_6$ gas evacuated from the chamber, and with the glass and film values as previously found, find the ratio, $P_{out}/P_{in}$. Assume very slight misalignment, so that the long beam path through the prism is not re-traced by reflected waves. The beam loses power at the two normal-incidence boundaries, whereas the back reflections at 45° will still be lossless, as that angle is now greater than $\theta_c$ with the reduced surrounding index. At the first normal incidence boundary (from air to film to glass), the input intrinsic impedance is

$$\eta_{in1} = \frac{\eta_f^2}{\eta_g} = \frac{\eta_0^2/n_f^2}{\eta_0/n_g} = \eta_0\left(\frac{n_g}{n_f^2}\right)$$

At the second normal incidence boundary at the prism exit (glass to film to air), the input intrinsic impedance is

$$\eta_{in2} = \frac{\eta_f^2}{\eta_0} = \frac{\eta_0^2/n_f^2}{\eta_0} = \eta_0\left(\frac{1}{n_f^2}\right)$$

The reflection coefficients at the two boundaries will be

$$\Gamma_1 = \frac{\eta_{in1} - \eta_0}{\eta_{in1} + \eta_0} = \frac{n_g - n_f^2}{n_g + n_f^2} = \frac{2.55 - (2.14)^2}{2.55 + (2.14)^2} = -0.285$$

$$\Gamma_2 = \frac{\eta_{in2} - \eta_g}{\eta_{in2} + \eta_g} = \frac{n_g - n_f^2}{n_g + n_f^2} = \Gamma_1$$

The power ratio will be:

$$\frac{P_{out}}{P_{in}} = \left(1 - |\Gamma_1|^2\right)\left(1 - |\Gamma_2|^2\right) = \left(1 - (0.285)^2\right)^2 = \underline{0.845}$$

**12.16.** In Fig. 12.5, let regions 2 and 3 both be of quarter-wave thickness. Region 4 is glass, having refractive index, $n_4 = 1.45$; region 1 is air.

a) Find $\eta_{in,b}$: Since region 3 is a quarter-wave layer, $\beta_3 l_b = \pi/2$, and (47) reduces to

$$\eta_{in,b} = \frac{\eta_3^2}{\eta_4}$$

b) Find $\eta_{in,a}$: Again, with region 2 of quarter-wave thickness, $\beta_2 l_a = \pi/2$ and (48) becomes

$$\eta_{in,a} = \frac{\eta_2^2}{\eta_{in,b}} = \frac{\eta_2^2 \eta_4}{\eta_3^2}$$

c) Specify a relation between the four intrinsic impedances that will enable total transmission of waves incident from the left into region 4: At the front surface, we need to have a zero reflection coefficient, so the input impedance there must match that of free space:

$$\eta_{in,a} = \eta_0 \;\Rightarrow\; \underline{\eta_2^2 \eta_4 = \eta_3^2 \eta_0}$$

d) Specify refractive index values for regions 2 and 3 that will accomplish the condition of part c: We can rewrite the part *c* result as

$$\left(\frac{\eta_0^2}{n_2^2}\right)\left(\frac{\eta_0}{\eta_4}\right) = \left(\frac{\eta_0^2}{n_3^2}\right)\eta_0 \;\Rightarrow\; n_4 = \frac{n_3^2}{n_2^2}$$

So any combination of indices that satisfy this result will work. One combination, for example, would be $n_2 = 1.10$ and $n_3 = 1.33$. It is better to have the indices ascending (or descending) monotonically in value from layer to layer because the high transmission feature is then less sensitive to changes in wavelength (as an exercise for fun, show this).

e) Find the fraction of incident power transmitted if the two layers were of half-wave thickness instead of quarter-wave: For any half-wave layer, we know that the input impedance is equal to that of the load. Therefore, $\eta_{in,b} = \eta_{in,a} = \eta_4$. The reflection coefficient at the front surface is therefore

$$\Gamma_{in} = \frac{\eta_{in,a} - \eta_0}{\eta_{in,a} + \eta_0} = \frac{\eta_4 - \eta_0}{\eta_4 + \eta_0} = \frac{1 - n_4}{1 + n_4} = \frac{1 - 1.45}{1 + 1.45} = -0.184$$

The transmitted power fraction is then

$$\frac{P_t}{P_{in}} = 1 - |\Gamma_{in}|^2 = 1 - (0.184)^2 = \underline{0.97}$$

**12.17.** A uniform plane wave in free space is normally-incident onto a dense dielectric plate of thickness $\lambda/4$, having refractive index $n$. Find the required value of $n$ such that exactly half the incident power is reflected (and half transmitted). Remember that $n > 1$.

In this problem, $\eta_1 = \eta_3 = \eta_0$, and the quarter-wave section input impedance is therefore

$$\eta_{in} = \frac{\eta_2^2}{\eta_3} = \frac{\eta_0^2/n^2}{\eta_0} = \frac{\eta_0}{n^2}$$

The reflection coefficient at the front surface is then

$$\Gamma_{in} = \frac{\eta_{in} - \eta_0}{\eta_{in} + \eta_0} = \frac{1-n^2}{1+n^2}$$

For half-power reflection, we require that $|\Gamma_{in}|^2 = 0.5$, or $\Gamma_{in} = \pm 1/\sqrt{2}$. Since $n$ must be greater than 1, we choose the minus sign option and write:

$$\frac{1-n^2}{1+n^2} = -\frac{1}{\sqrt{2}} \;\Rightarrow\; n = \left[\frac{\sqrt{2}+1}{\sqrt{2}-1}\right]^{1/2} = \underline{2.41}$$

**12.18.** A uniform plane wave is normally-incident onto a slab of glass ($n = 1.45$) whose back surface is in contact with a perfect conductor. Determine the reflective phase shift at the front surface of the glass if the glass thickness is: (a) $\lambda/2$; (b) $\lambda/4$; (c) $\lambda/8$.

With region 3 being a perfect conductor, $\eta_3 = 0$, and Eq. (36) gives the input impedance to the structure as $\eta_{in} = j\eta_2 \tan\beta\ell$. The reflection coefficient is then

$$\Gamma = \frac{\eta_{in} - \eta_0}{\eta_{in} + \eta_0} = \frac{j\eta_2 \tan\beta\ell - \eta_0}{j\eta_2 \tan\beta\ell + \eta_0} = \frac{\eta_2^2 \tan^2\beta\ell - \eta_0^2 + j2\eta_0\eta_2 \tan\beta\ell}{\eta_2^2 \tan^2\beta\ell + \eta_0^2} = \Gamma_r + j\Gamma_i$$

where the last equality occurs by multiplying the numerator and denominator of the middle term by the complex conjugate of its denominator. The reflective phase is now

$$\phi = \tan^{-1}\left(\frac{\Gamma_i}{\Gamma_r}\right) = \tan^{-1}\left[\frac{2\eta_2\eta_0 \tan\beta\ell}{\eta_2^2 \tan^2\beta\ell - \eta_0^2}\right] = \tan^{-1}\left[\frac{(2.90)\tan\beta\ell}{\tan\beta\ell - 2.10}\right]$$

where $\eta_2 = \eta_0/1.45$ has been used. We can now evaluate the phase shift for the three given cases. First, when $\ell = \lambda/2$, $\beta\ell = \pi$, and thus $\underline{\phi(\lambda/2) = 0}$. Next, when $\ell = \lambda/4$, $\beta\ell = \pi/2$, and

$$\phi(\lambda/4) \to \tan^{-1}[2.90] = \underline{71^\circ}$$

as $\ell \to \lambda/4$. Finally, when $\ell = \lambda/8$, $\beta\ell = \pi/4$, and

$$\phi(\lambda/8) = \tan^{-1}\left[\frac{2.90}{1-2.10}\right] = \underline{-69.2^\circ} \text{ (or } 291^\circ\text{)}$$

**12.19.** You are given four slabs of lossless dielectric, all with the same intrinsic impedance, $\eta$, known to be different from that of free space. The thickness of each slab is $\lambda/4$, where $\lambda$ is the wavelength as measured in the slab material. The slabs are to be positioned parallel to one another, and the combination lies in the path of a uniform plane wave, normally-incident. The slabs are to be arranged such that the air spaces between them are either zero, one-quarter wavelength, or one-half wavelength in thickness. Specify an arrangement of slabs and air spaces such that

a) the wave is totally transmitted through the stack: In this case, we look for a combination of half-wave sections. Let the inter-slab distances be $d_1$, $d_2$, and $d_3$ (from left to right). Two possibilities are i.) $\underline{d_1 = d_2 = d_3 = 0}$, thus creating a single section of thickness $\lambda$, or ii.) $\underline{d_1 = d_3 = 0,\ d_2 = \lambda/2}$, thus yielding two half-wave sections separated by a half-wavelength.

b) the stack presents the highest reflectivity to the incident wave: The best choice here is to make $\underline{d_1 = d_2 = d_3 = \lambda/4}$. Thus every thickness is one-quarter wavelength. The impedances transform as follows: First, the input impedance at the front surface of the last slab (slab 4) is $\eta_{in,1} = \eta^2/\eta_0$. We transform this back to the back surface of slab 3, moving through a distance of $\lambda/4$ in free space: $\eta_{in,2} = \eta_0^2/\eta_{in,1} = \eta_0^3/\eta^2$. We next transform this impedance to the front surface of slab 3, producing $\eta_{in,3} = \eta^2/\eta_{in,2} = \eta^4/\eta_0^3$. We continue in this manner until reaching the front surface of slab 1, where we find $\eta_{in,7} = \eta^8/\eta_0^7$. Assuming $\eta < \eta_0$, the ratio $\eta^n/\eta_0^{n-1}$ becomes smaller as $n$ increases (as the number of slabs increases). The reflection coefficient for waves incident on the front slab thus gets close to unity, and approaches 1 as the number of slabs approaches infinity.

**12.20.** The 50MHz plane wave of Problem 12.12 is incident onto the ocean surface at an angle to the normal of 60°. Determine the fractions of the incident power that are reflected and transmitted for

a) s polarization: To review Problem 12, we first we find the loss tangent:

$$\frac{\sigma}{\omega\epsilon'} = \frac{4}{2\pi(50\times 10^6)(78)(8.854\times 10^{-12})} = 18.4$$

This value is sufficiently greater than 1 to enable seawater to be considered a good conductor at 50MHz. Then, using the approximation (Eq. 85, Chapter 11), and with $\mu = \mu_0$, the intrinsic impedance is $\eta_s = \sqrt{\pi f \mu/\sigma}(1+j) = 7.0(1+j)$. Next we need the angle of refraction, which means that we need to know the refractive index of seawater at 50MHz. For a uniform plane wave in a good conductor, the phase constant is

$$\beta = \frac{n_{sea}\,\omega}{c} \doteq \sqrt{\pi f \mu \sigma} \quad \Rightarrow \quad n_{sea} \doteq c\sqrt{\frac{\mu\sigma}{4\pi f}} = 26.8$$

Then, using Snell's law, the angle of refraction is found:

$$\sin\theta_2 = \frac{n_{sea}}{n_1}\sin\theta_1 = 26.8\sin(60^\circ) \quad \Rightarrow \quad \theta_2 = 1.9^\circ$$

This angle is small enough so that $\cos\theta_2 \doteq 1$. Therefore, for s polarization,

$$\Gamma_s \doteq \frac{\eta_{s2}-\eta_{s1}}{\eta_{s2}+\eta_{s1}} = \frac{7.0(1+j) - 377/\cos 60^\circ}{7.0(1+j) + 377/\cos 60^\circ} = -0.98 + j0.018 = 0.98\angle 179^\circ$$

The fraction of the power reflected is now $|\Gamma_s|^2 = \underline{0.96}$. The fraction transmitted is then $\underline{0.04}$.

b) p polarization: Again, with the refracted angle close to zero, the relection coefficient for p polarization is

$$\Gamma_p \doteq \frac{\eta_{p2}-\eta_{p1}}{\eta_{p2}+\eta_{p1}} = \frac{7.0(1+j) - 377\cos 60^\circ}{7.0(1+j) + 377\cos 60^\circ} = -0.93 + j0.069 = 0.93\angle 176^\circ$$

The fraction of the power reflected is now $|\Gamma_p|^2 = \underline{0.86}$. The fraction transmitted is then $\underline{0.14}$.

**12.21.** A right-circularly polarized plane wave in air is incident at Brewster's angle onto a semi-infinite slab of plexiglas ($\epsilon'_r = 3.45$, $\epsilon''_r = 0$, $\mu = \mu_0$).

a) Determine the fractions of the incident power that are reflected and transmitted: In plexiglas, Brewster's angle is $\theta_B = \theta_1 = \tan^{-1}(\epsilon'_{r2}/\epsilon'_{r1}) = \tan^{-1}(\sqrt{3.45}) = 61.7°$. Then the angle of refraction is $\theta_2 = 90° - \theta_B$ (see Example 12.9), or $\theta_2 = 28.3°$. With incidence at Brewster's angle, all $p$-polarized power will be transmitted — only $s$-polarized power will be reflected. This is found through

$$\Gamma_s = \frac{\eta_{2s} - \eta_{1s}}{\eta_{2s} + \eta_{1s}} = \frac{.614\eta_0 - 2.11\eta_0}{.614\eta_0 + 2.11\eta_0} = -0.549$$

where $\eta_{1s} = \eta_1 \sec\theta_1 = \eta_0 \sec(61.7°) = 2.11\eta_0$,
and $\eta_{2s} = \eta_2 \sec\theta_2 = (\eta_0/\sqrt{3.45}) \sec(28.3°) = 0.614\eta_0$. Now, the reflected power fraction is $|\Gamma|^2 = (-.549)^2 = .302$. Since the wave is circularly-polarized, the $s$-polarized component represents one-half the total incident wave power, and so the fraction of the *total* power that is reflected is $.302/2 = 0.15$, or 15%. The fraction of the incident power that is transmitted is then the remainder, or 85%.

b) Describe the polarizations of the reflected and transmitted waves: Since all the $p$-polarized component is transmitted, the reflected wave will be entirely $s$-polarized (linear). The transmitted wave, while having all the incident $p$-polarized power, will have a reduced $s$-component, and so this wave will be right-elliptically polarized.

**12.22.** A dielectric waveguide is shown in Fig. 12.16 with refractive indices as labeled. Incident light enters the guide at angle $\phi$ from the front surface normal as shown. Once inside, the light totally reflects at the upper $n_1 - n_2$ interface, where $n_1 > n_2$. All subsequent reflections from the upper an lower boundaries will be total as well, and so the light is confined to the guide. Express, in terms of $n_1$ and $n_2$, the maximum value of $\phi$ such that total confinement will occur, with $n_0 = 1$. The quantity $\sin\phi$ is known as the *numerical aperture* of the guide.

From the illustration we see that $\phi_1$ maximizes when $\theta_1$ is at its minimum value. This minimum will be the critical angle for the $n_1 - n_2$ interface, where $\sin\theta_c = \sin\theta_1 = n_2/n_1$. Let the refracted angle to the right of the vertical interface (not shown) be $\phi_2$, where $n_0 \sin\phi_1 = n_1 \sin\phi_2$. Then we see that $\phi_2 + \theta_1 = 90°$, and so $\sin\theta_1 = \cos\phi_2$. Now, the numerical aperture becomes

$$\sin\phi_{1max} = \frac{n_1}{n_0}\sin\phi_2 = n_1\cos\theta_1 = n_1\sqrt{1-\sin^2\theta_1} = n_1\sqrt{1-(n_2/n_1)^2} = \sqrt{n_1^2 - n_2^2}$$

Finally, $\phi_{1max} = \underline{\sin^{-1}\left(\sqrt{n_1^2 - n_2^2}\right)}$ is the numerical aperture angle.

**12.23.** Suppose that $\phi_1$ in Fig. 12.16 is Brewster's angle, and that $\theta_1$ is the critical angle. Find $n_0$ in terms of $n_1$ and $n_2$: With the incoming ray at Brewster's angle, the refracted angle of this ray (measured from the inside normal to the front surface) will be $90° - \phi_1$. Therefore, $\phi_1 = \theta_1$, and thus $\sin\phi_1 = \sin\theta_1$. Thus

$$\sin\phi_1 = \frac{n_1}{\sqrt{n_0^2 + n_1^2}} = \sin\theta_1 = \frac{n_2}{n_1} \quad\Rightarrow\quad n_0 = \underline{(n_1/n_2)\sqrt{n_1^2 - n_2^2}}$$

Alternatively, we could have used the result of Problem 12.22, in which it was found that $\sin\phi_1 = (1/n_0)\sqrt{n_1^2 - n_2^2}$, which we then set equal to $\sin\theta_1 = n_2/n_1$ to get the same result.

**12.24.** A *Brewster prism* is designed to pass $p$-polarized light without any reflective loss. The prism of Fig. 12.17 is made of glass ($n = 1.45$), and is in air. Considering the light path shown, determine the vertex angle, $\alpha$: With entrance and exit rays at Brewster's angle (to eliminate reflective loss), the interior ray must be horizontal, or parallel to the bottom surface of the prism. From the geometry, the angle between the interior ray and the normal to the prism surfaces that it intersects is $\alpha/2$. Since this angle is also Brewster's angle, we may write:

$$\alpha = 2\sin^{-1}\left(\frac{1}{\sqrt{1+n^2}}\right) = 2\sin^{-1}\left(\frac{1}{\sqrt{1+(1.45)^2}}\right) = 1.21\,\text{rad} = \underline{69.2^\circ}$$

**12.25.** In the Brewster prism of Fig. 12.17, determine for $s$-polarized light the fraction of the incident power that is transmitted through the prism, and from this specify the dB *insertion loss*, defined as $10\log_{10}$ of that number:

We use $\Gamma_s = (\eta_{s2} - \eta_{s1})/(\eta_{s2} + \eta_{s1})$, where

$$\eta_{s2} = \frac{\eta_2}{\cos(\theta_{B2})} = \frac{\eta_2}{n/\sqrt{1+n^2}} = \frac{\eta_0}{n^2}\sqrt{1+n^2}$$

and

$$\eta_{s1} = \frac{\eta_1}{\cos(\theta_{B1})} = \frac{\eta_1}{1/\sqrt{1+n^2}} = \eta_0\sqrt{1+n^2}$$

Thus, at the first interface, $\Gamma = (1-n^2)/(1+n^2)$. At the second interface, $\Gamma$ will be equal but of opposite sign to the above value. The power transmission coefficient through each interface is $1 - |\Gamma|^2$, so that for both interfaces, we have, with $n = 1.45$:

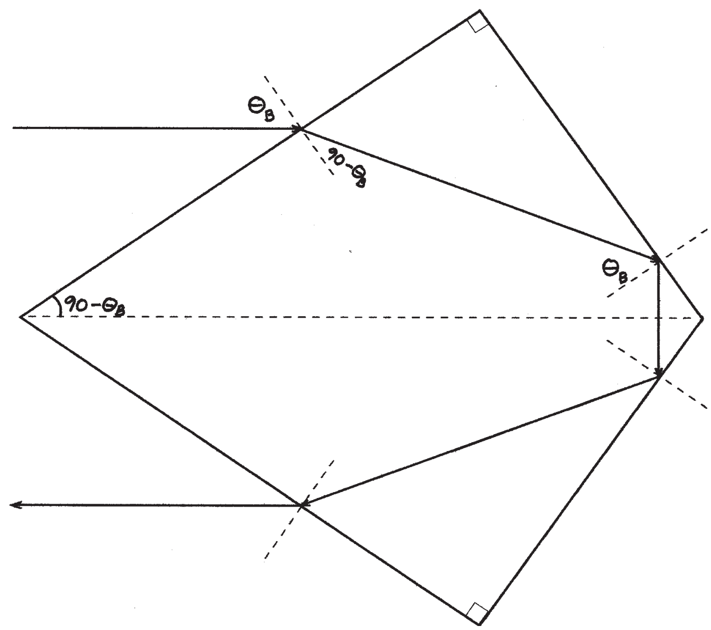
$$\frac{P_{tr}}{P_{inc}} = \left(1 - |\Gamma|^2\right)^2 = \left[1 - \left(\frac{n^2-1}{n^2+1}\right)^2\right]^2 = \underline{0.76}$$

The insertion loss is now

$$\ell_i(\text{dB}) = 10\log_{10}(0.76) = \underline{-1.19}\,\text{dB}$$

**12.26.** Show how a single block of glass can be used to turn a p-polarized beam of iight through 180°, with the light suffering, in principle, zero reflective loss. The light is incident from air, and the returning beam (also in air) may be displaced sideways from the incident beam. Specify all pertinent angles and use $n = 1.45$ for glass. More than one design is possible here.

The prism below is designed such that light enters at Brewster's angle, and once inside, is turned around using total reflection. Using the result of Example 12.9, we find that with glass, $\theta_B = 55.4°$, which, by the geometry, is also the incident angle for total reflection at the back of the prism. For this to work, the Brewster angle must be greater than or equal to the critical angle. This is in fact the case, since $\theta_c = \sin^{-1}(n_2/n_1) = \sin^{-1}(1/1.45) = 43.6°$.




**12.27.** Using Eq. (79) in Chapter 11 as a starting point, determine the ratio of the group and phase velocities of an electromagnetic wave in a good conductor. Assume conductivity does not vary with frequency: In a good conductor:

$$\beta = \sqrt{\pi f \mu \sigma} = \sqrt{\frac{\omega \mu \sigma}{2}} \qquad \longrightarrow \qquad \frac{d\beta}{d\omega} = \frac{1}{2}\left[\frac{\omega\mu\sigma}{2}\right]^{-1/2} \frac{\mu\sigma}{2}$$

Thus

$$\frac{d\omega}{d\beta} = \left(\frac{d\beta}{d\omega}\right)^{-1} = 2\sqrt{\frac{2\omega}{\mu\sigma}} = v_g \qquad \text{and} \qquad v_p = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\omega\mu\sigma/2}} = \sqrt{\frac{2\omega}{\mu\sigma}}$$

Therefore $v_g/v_p = \underline{2}$.

**12.28.** Over a small wavelength range, the refractive index of a certain material varies approximately linearly with wavelength as $n(\lambda) \doteq n_a + n_b(\lambda - \lambda_a)$, where $n_a$, $n_b$, and $\lambda_a$ are constants, and where $\lambda$ is the free space wavelength.

a) Show that $d/d\omega = -(2\pi c/\omega^2)d/d\lambda$: With $\lambda$ as the free space wavelength, we use $\lambda = 2\pi c/\omega$, from which $d\lambda/d\omega = -2\pi c/\omega^2$. Then $d/d\omega = (d\lambda/d\omega)\, d/d\lambda = -(2\pi c/\omega^2)\, d/d\lambda$.

b) Using $\beta(\lambda) = 2\pi n/\lambda$, determine the wavelength-dependent (or independent) group delay over a unit distance: This will be

$$t_g = \frac{1}{v_g} = \frac{d\beta}{d\omega} = \frac{d}{d\omega}\left[\frac{2\pi n(\lambda)}{\lambda}\right] = -\frac{2\pi c}{\omega^2}\frac{d}{d\lambda}\left[\frac{2\pi}{\lambda}\left[n_a + n_b(\lambda - \lambda_a)\right]\right]$$

$$= -\frac{2\pi c}{\omega^2}\left[-\frac{2\pi}{\lambda^2}\left[n_a + n_b(\lambda - \lambda_a)\right] + \frac{2\pi}{\lambda}n_b\right]$$

$$= -\frac{\lambda^2}{2\pi c}\left[-\frac{2\pi n_a}{\lambda^2} + \frac{2\pi n_b \lambda_a}{\lambda^2}\right] = \underline{\frac{1}{c}(n_a - n_b\lambda_a)\ \text{s/m}}$$

c) Determine $\beta_2$ from your result of part *b*: $\beta_2 = d^2\beta/d\omega^2|_{\omega_0}$. Since the part *b* result is independent of wavelength (and of frequency), it follows that $\underline{\beta_2 = 0}$.

d) Discuss the implications of these results, if any, on pulse broadening: A wavelength-independent group delay (leading to zero $\beta_2$) means that there will simply be no pulse broadening at all. All frequency components arrive simultaneously. This sort of thing happens in most transparent materials – that is, there will be a certain wavelength, known as the *zero dispersion wavelength*, around which the variation of $n$ with $\lambda$ is locally linear. Transmitting pulses at this wavelength will result in no pulse broadening (to first order).

**12.29.** A $T = 5$ ps transform-limited pulse propagates in a dispersive channel for which $\beta_2 = 10\ \text{ps}^2/\text{km}$. Over what distance will the pulse spread to twice its initial width? After propagation, the width is $T' = \sqrt{T^2 + (\Delta\tau)^2} = 2T$. Thus $\Delta\tau = \sqrt{3}T$, where $\Delta\tau = \beta_2 z/T$. Therefore

$$\frac{\beta_2 z}{T} = \sqrt{3}T \text{ or } z = \frac{\sqrt{3}T^2}{\beta_2} = \frac{\sqrt{3}(5\,\text{ps})^2}{10\,\text{ps}^2/\text{km}} = \underline{4.3\ \text{km}}$$

**12.30.** A $T = 20$ ps transform-limited pulse propagates through 10 km of a dispersive channel for which $\beta_2 = 12\,\text{ps}^2/\text{km}$. The pulse then propagates through a second 10 km channel for which $\beta_2 = -12\,\text{ps}^2/\text{km}$. Describe the pulse at the output of the second channel and give a physical explanation for what happened.

Our theory of pulse spreading will allow for changes in $\beta_2$ down the length of the channel. In fact, we may write in general:

$$\Delta\tau = \frac{1}{T}\int_0^L \beta_2(z)\, dz$$

Having $\beta_2$ change sign at the midpoint, yields a zero $\Delta\tau$, and so the pulse emerges from the output unchanged! Physically, the pulse acquires a positive linear chirp (frequency increases with time over the pulse envelope) during the first half of the channel. When $\beta_2$ switches sign, the pulse begins to acquire a negative chirp in the second half, which, over an equal distance, will completely eliminate the chirp acquired during the first half. The pulse, if originally transform-limited at input, will emerge, again transform-limited, at its original width. More generally, complete *dispersion compensation* is achieved using a two-segment channel when $\beta_2 L = -\beta_2' L'$, assuming dispersion terms of higher order than $\beta_2$ do not exist.

# CHAPTER 13

**13.1.** The conductors of a coaxial transmission line are copper ($\sigma_c = 5.8 \times 10^{-7}$ S/m) and the dielectric is polyethylene ($\epsilon'_r = 2.26$, $\sigma/\omega\epsilon' = 0.0002$). If the inner radius of the outer conductor is 4 mm, find the radius of the inner conductor so that (assuming a lossless line):

a) $Z_0 = 50\,\Omega$: Use

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon'}}\ln\left(\frac{b}{a}\right) = 50 \quad \Rightarrow \quad \ln\left(\frac{b}{a}\right) = \frac{2\pi\sqrt{\epsilon'_r}(50)}{377} = 1.25$$

Thus $b/a = e^{1.25} = 3.50$, or $a = 4/3.50 = \underline{1.142\,\text{mm}}$

b) $C = 100$ pF/m: Begin with

$$C = \frac{2\pi\epsilon'}{\ln(b/a)} = 10^{-10} \quad \Rightarrow \quad \ln\left(\frac{b}{a}\right) = 2\pi(2.26)(8.854 \times 10^{-2}) = 1.257$$

So $b/a = e^{1.257} = 3.51$, or $a = 4/3.51 = \underline{1.138\,\text{mm}}$.

c) $L = 0.2\,\mu$H/m: Use

$$L = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right) = 0.2 \times 10^{-6} \quad \Rightarrow \quad \ln\left(\frac{b}{a}\right) = \frac{2\pi(0.2 \times 10^{-6})}{4\pi \times 10^{-7}} = 1$$

Thus $b/a = e^1 = 2.718$, or $a = b/2.718 = \underline{1.472\,\text{mm}}$.

**13.2.** Find $R$, $L$, $C$, and $G$ for a coaxial cable with $a = 0.25$ mm, $b = 2.50$ mm, $c = 3.30$ mm, $\epsilon_r = 2.0$, $\mu_r = 1$, $\sigma_c = 1.0 \times 10^7$ S/m, $\sigma = 1.0 \times 10^{-5}$ S/m, and $f = 300$ MHz.

First, we note that the metal is a good conductor, as confirmed by the loss tangent:

$$\frac{\sigma_c}{\omega\epsilon'} = \frac{1.0 \times 10^7}{2\pi(3.00 \times 10^8)(2)(8.854 \times 10^{-12})} = 3.0 \times 10^8 >> 1$$

So the skin depth into the metal is

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma_c}} = \sqrt{\frac{2}{2\pi(3.00 \times 10^8)(4\pi \times 10^{-7})(1.0 \times 10^7)}} = 9.2\,\mu\text{m}$$

The current can be said to exist in layers of thickness $\delta$ just beneath the inner conductor radius, $a$, and just inside the outer conductor at its inner radius, $b$. The outer conductor far radius, $c$, is of no consequence. Both conductors behave essentially as thin films of thickness $\delta$. We are thus in the high frequency regime, and the equations in Sec. 13.1.1 apply. The calculations for the primary constants are as follows:

$$R = \frac{1}{2\pi\delta\sigma_c}\left(\frac{1}{a} + \frac{1}{b}\right) = \frac{1}{2\pi(9.2 \times 10^{-6})(1.0 \times 10^7)}\left(\frac{1}{0.25 \times 10^{-3}} + \frac{1}{2.5 \times 10^{-3}}\right) = \underline{7.6\,\Omega/\text{m}}$$

using Eq. (12).

$$L = L_{ext} = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right) = \frac{4\pi \times 10^{-7}}{2\pi}\ln\left(\frac{2.5}{0.25}\right) = \underline{0.46\,\mu\text{H/m}}$$

using Eq. (11).

$$C = \frac{2\pi\epsilon'}{\ln(b/a)} = \frac{2\pi(2)(8.854 \times 10^{-12})}{\ln(2.5/0.25)} = \underline{48\,\text{pF/m}}$$

using Eq. (9).

$$G = \frac{2\pi\sigma}{\ln(b/a)} = \frac{2\pi(1.0 \times 10^{-5})}{\ln(2.5/0.25)} = \underline{27\,\mu\text{S/m}}$$

using Eq. (10).

**13.3.** Two aluminum-clad steel conductors are used to construct a two-wire transmission line. Let $\sigma_{Al} = 3.8 \times 10^7$ S/m, $\sigma_{St} = 5 \times 10^6$ S/m, and $\mu_{St} = 100\,\mu$H/m. The radius of the steel wire is 0.5 in., and the aluminum coating is 0.05 in. thick. The dielectric is air, and the center-to-center wire separation is 4 in. Find $C$, $L$, $G$, and $R$ for the line at 10 MHz: The first question is whether we are in the high frequency or low frequency regime. Calculation of the skin depth, $\delta$, will tell us. We have, for aluminum,

$$\delta = \frac{1}{\sqrt{\pi f \mu_0 \sigma_{Al}}} = \frac{1}{\sqrt{\pi(10^7)(4\pi \times 10^{-7})(3.8 \times 10^7)}} = 2.58 \times 10^{-5} \text{ m}$$

so we are clearly in the high frequency regime, where uniform current distributions cannot be assumed. Furthermore, the skin depth is considerably less than the aluminum layer thickness, so the bulk of the current resides in the aluminum, and we may neglect the steel. Assuming solid aluminum wires of radius $a = 0.5 + 0.05 = 0.55$ in $= 0.014$ m, the resistance of the two-wire line is now

$$R = \frac{1}{\pi a \delta \sigma_{Al}} = \frac{1}{\pi(.014)(2.58 \times 10^{-5})(3.8 \times 10^7)} = \underline{0.023\ \Omega/\text{m}}$$

Next, since the dielectric is air, no leakage will occur from wire to wire, and so $G = \underline{0 \text{ S/m}}$. Now the capacitance will be

$$C = \frac{\pi \epsilon_0}{\cosh^{-1}(d/2a)} = \frac{\pi \times 8.85 \times 10^{-12}}{\cosh^{-1}(4/(2 \times 0.55))} = 1.42 \times 10^{-11} \text{ F/m} = \underline{14.2 \text{ pF/m}}$$

Finally, the inductance per unit length will be

$$L = \frac{\mu_0}{\pi} \cosh(d/2a) = \frac{4\pi \times 10^{-7}}{\pi} \cosh(4/(2 \times 0.55)) = 7.86 \times 10^{-7} \text{ H/m} = \underline{0.786\ \mu\text{H/m}}$$

**13.4.** Find $R$, $L$, $C$, and $G$ for a two-wire transmission line in polyethylene at $f = 800$ MHz. Assume copper conductors of radius 0.50 mm and separation 0.80 cm. Use $\epsilon_r = 2.26$ and $\sigma/(\omega\epsilon') = 4.0 \times 10^{-4}$.

From the loss tangent, we find the conductivity of polyethylene:

$$\sigma = (4.0 \times 10^{-4})(2\pi \times 8.00 \times 10^8)(2.26)(8.854 \times 10^{-12}) = 4.0 \times 10^{-5}\,\text{S/m}$$

As polyethylene is a good dielectric, its penetration depth is (using Eq. (60a), Chapter 11):

$$\delta_p = \frac{1}{\alpha_p} = \frac{2}{\sigma}\sqrt{\frac{\epsilon'}{\mu}} = \frac{2\epsilon'_r}{\sigma\eta_0} = \frac{2\sqrt{2.26}}{(4.0 \times 10^{-5})(377)} = 199\,\text{m}$$

Therefore, we can assume field distributions over a any cross-sectional plane to be the same as those of the lossless line. On the other hand, within the copper conductors (for which $\sigma_c = 5.8 \times 10^7$) the skin depth will be (from Eq. (82), Chapter 11):

$$\delta_c = \frac{1}{\pi f \mu \sigma_c} = \frac{1}{\pi(8.00 \times 10^8)(4\pi \times 10^{-7})(5.8 \times 10^7)} = 2.3\,\mu\text{m}$$

As this value is much less than the overall conductor dimensions, the line operates in the high frequency regime. The primary constants are found using Eqs. (20) - (23). We have

$$C = \frac{\pi\epsilon'}{\cosh^{-1}(d/2a)} = \frac{\pi(2.26)(8.854 \times 10^{-12})}{\cosh^{-1}(8)} = \underline{22.7\,\text{pF/m}}$$

$$L = L_{ext} = \frac{\mu}{\pi}\cosh^{-1}(d/2a) = \frac{4\pi \times 10^{-7}}{\pi}\cosh^{-1}(8) = \underline{1.11\,\mu\text{H/m}}$$

$$G = \frac{\pi\sigma}{\cosh^{-1}(d/2a)} = \frac{\pi(4.0 \times 10^{-5})}{\cosh^{-1}(8)} = \underline{46\,\mu\text{S/m}}$$

$$R = \frac{1}{\pi a \delta_c \sigma_c} = \frac{1}{\pi(5.0 \times 10^{-4})(2.3 \times 10^{-6})(5.8 \times 10^7)} = \underline{4.8\,\text{ohms/m}}$$

**13.5.** Each conductor of a two-wire transmission line has a radius of 0.5mm; their center-to-center distance is 0.8cm. Let $f = 150MHz$ and assume $\sigma$ and $\sigma_c$ are zero. Find the dielectric constant of the insulating medium if

a) $Z_0 = 300\,\Omega$: Use

$$300 = \frac{1}{\pi}\sqrt{\frac{\mu_0}{\epsilon_r'\epsilon_0}}\cosh^{-1}\left(\frac{d}{2a}\right) \quad\Rightarrow\quad \sqrt{\epsilon_r'} = \frac{120\pi}{300\pi}\cosh^{-1}\left(\frac{8}{2(.5)}\right) = 1.107 \quad\Rightarrow\quad \epsilon_r' = \underline{1.23}$$

b) $C = 20$ pF/m: Use

$$20\times 10^{-12} = \frac{\pi\epsilon'}{\cosh^{-1}(d/2a)} \quad\Rightarrow\quad \epsilon_r' = \frac{20\times 10^{-12}}{\pi\epsilon_0}\cosh^{-1}(8) = \underline{1.99}$$

c) $v_p = 2.6\times 10^8$ m/s:

$$v_p = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu_0\epsilon_0\epsilon_r'}} = \frac{c}{\sqrt{\epsilon_r'}} \quad\Rightarrow\quad \epsilon_r' = \left(\frac{3.0\times 10^8}{2.6\times 10^8}\right)^2 = \underline{1.33}$$

**13.6.** The transmission line in Fig. 6.8 is filled with polyethylene. If it were filled with air, the capacitance would be 57.6 pF/m. Assuming that the line is lossless, find $C$, $L$, and $Z_0$.

The line cross-section is of little consequence here because we have its capacitance and we know that it is lossless. The capacitance with polyethylene added will be just the air-filled line capacitance multiplied by the dielectric constant of polyethylene, which is $\epsilon_r' = 2.26$, or

$$C_p = \epsilon_r' C_{air} = 2.26(57.6) = \underline{130\,\text{pF/m}}$$

The inductance is found from the capacitance and the wave velocity, which in the air-filled line is just the speed of light in vacuum:

$$v_p = c = 3\times 10^8\,\text{m/s} = \frac{1}{\sqrt{LC_{air}}} \Rightarrow L = \frac{1}{c^2 C_{air}} = \frac{1}{(3.00\times 10^8)^2(57.6\times 10^{-12})} = \underline{0.193\,\mu\text{H/m}}$$

Finally, the characteristic impedance in the dielectric-filled line will be

$$Z_0 = \sqrt{\frac{L}{C_p}} = \sqrt{\frac{1.93\times 10^{-7}}{1.30\times 10^{-10}}} = \underline{38.5\text{ ohms}}$$

where the inductance is unaffected by the incorporation of the dielectric.

**13.7.** Pertinent dimensions for the transmission line shown in Fig. 13.2 are $b = 3$ mm, and $d = 0.2$ mm. The conductors and the dielectric are non-magnetic.

a) If the characteristic impedance of the line is 15 $\Omega$, find $\epsilon_r'$: We use

$$Z_0 = \sqrt{\frac{\mu}{\epsilon'}}\left(\frac{d}{b}\right) = 15 \quad \Rightarrow \quad \epsilon_r' = \left(\frac{377}{15}\right)^2 \frac{.04}{9} = \underline{2.8}$$

b) Assume copper conductors and operation at $2 \times 10^8$ rad/s. If $RC = GL$, determine the loss tangent of the dielectric: For copper, $\sigma_c = 5.8 \times 10^7$ S/m, and the skin depth is

$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma_c}} = \sqrt{\frac{2}{(2 \times 10^8)(4\pi \times 10^{-7})(5.8 \times 10^7)}} = 1.2 \times 10^{-5} \text{ m}$$

Then

$$R = \frac{2}{\sigma_c \delta b} = \frac{2}{(5.8 \times 10^7)(1.2 \times 10^{-5})(.003)} = 0.98 \ \Omega/\text{m}$$

Now

$$C = \frac{\epsilon' b}{d} = \frac{(2.8)(8.85 \times 10^{-12})(3)}{0.2} = 3.7 \times 10^{-10} \text{ F/m}$$

and

$$L = \frac{\mu_0 d}{b} = \frac{(4\pi \times 10^{-7})(0.2)}{3} = 8.4 \times 10^{-8} \text{ H/m}$$

Then, with $RC = GL$,

$$G = \frac{RC}{L} = \frac{(.98)(3.7 \times 10^{-10})}{(8.4 \times 10^{-8})} = 4.4 \times 10^{-3} \text{ S/m} = \frac{\sigma_d b}{d}$$

Thus $\sigma_d = (4.4 \times 10^{-3})(0.2/3) = 2.9 \times 10^{-4}$ S/m. The loss tangent is

$$l.t. = \frac{\sigma_d}{\omega\epsilon'} = \frac{2.9 \times 10^{-4}}{(2 \times 10^8)(2.8)(8.85 \times 10^{-12})} = \underline{5.85 \times 10^{-2}}$$

**13.8.** A transmission line constructed from perfect conductors and an air dielectric is to have a maximum dimension of 8mm for its cross-section. The line is to be used at high frequencies. Specify its dimensions if it is:

a) a two-wire line with $Z_0 = 300\,\Omega$: With the maximum dimension of 8mm, we have, using (24):

$$Z_0 = \frac{1}{\pi}\sqrt{\frac{\mu}{\epsilon'}}\cosh^{-1}\left(\frac{8-2a}{2a}\right) = 300 \quad \Rightarrow \quad \frac{8-2a}{2a} = \cosh\left(\frac{300\pi}{120\pi}\right) = 6.13$$

Solve for $a$ to find $a = \underline{0.56\,\text{mm}}$. Then $d = 8 - 2a = \underline{6.88\,\text{mm}}$.

b) a planar line with $Z_0 = 15\,\Omega$: In this case our maximum dimension dictates that $\sqrt{d^2+b^2} = 8$. So, using (8), we write

$$Z_0 = \sqrt{\frac{\mu}{\epsilon'}}\frac{\sqrt{64-b^2}}{b} = 15 \quad \Rightarrow \quad \sqrt{64-b^2} = \frac{15}{377}b$$

Solving, we find $b = \underline{7.99\,\text{mm}}$ and $d = \underline{0.32\,\text{mm}}$.

c) a $72\,\Omega$ coax having a zero-thickness outer conductor: With a zero-thickness outer conductor, we note that the outer radius is $b = 8/2 = 4$mm. Using (13), we write

$$Z_0 = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon'}}\ln\left(\frac{b}{a}\right) = 72 \quad \Rightarrow \quad \ln\left(\frac{b}{a}\right) = \frac{2\pi(72)}{120\pi} = 1.20 \quad \Rightarrow \quad a = be^{-1.20} = 4e^{-1.20} = 1.2$$

Summarizing, $a = \underline{1.2\,\text{mm}}$ and $b = \underline{4\,\text{mm}}$.

**13.9.** A microstrip line is to be constructed using a lossless dielectric for which $\epsilon'_r = 7.0$. If the line is to have a 50-$\Omega$ characteristic impedance, determine:

a) $\epsilon_{r,eff}$: We use Eq. (34) (under the assumption that $w/d > 1.3$) to find:

$$\epsilon_{r,eff} = 7.0\left[0.96 + 7.0(0.109 - 0.004\times 7.0)\log_{10}(10+50) - 1\right]^{-1} = \underline{5.0}$$

b) $w/d$: Still under the assumption that $w/d > 1.3$, we solve Eq. (33) for $d/w$ to find

$$\frac{d}{w} = (0.1)\left[\frac{\epsilon'_r - 1}{2}\left(\epsilon_{r,eff} - \frac{\epsilon'_r + 1}{2}\right)^{-1}\right]^{1/0.555} - 0.10$$

$$= (0.1)\left[3.0\,(5.0 - 4.0)^{-1}\right]^{1/0.555} - 0.10 = 0.624 \quad \Rightarrow \quad \frac{w}{d} = \underline{1.60}$$

**13.10.** Two microstrip lines are fabricated end-to-end on a 2-mm thick wafer of lithium niobate ($\epsilon_r' = 4.8$). Line 1 is of 4mm width; line 2 (unfortunately) has been fabricated with a 5mm width. Determine the power loss in dB for waves transmitted through the junction.

We first note that $w_1/d_1 = 2.0$ and $w_2/d_2 = 2.5$, so that Eqs. (32) and (33) apply. As the first step, solve for the effective dielectric constants for the two lines, using (33). For line 1:

$$\epsilon_{r1,eff} = \frac{5.8}{2} + \frac{3.8}{2}\left[1 + 10\left(\frac{1}{2}\right)\right]^{-0.555} = 3.60$$

For line 2:

$$\epsilon_{r2,eff} = \frac{5.8}{2} + \frac{3.8}{2}\left[1 + 10\left(\frac{1}{2.5}\right)\right]^{-0.555} = 3.68$$

We next use Eq. (32) to find the characteristic impedances for the air-filled cases. For line 1:

$$Z_{01}^{air} = 60\ln\left[4\left(\frac{1}{2}\right) + \sqrt{16\left(\frac{1}{2}\right)^2 + 2}\right] = 89.6 \text{ ohms}$$

and for line 2:

$$Z_{02}^{air} = 60\ln\left[4\left(\frac{1}{2.5}\right) + \sqrt{16\left(\frac{1}{2.5}\right)^2 + 2}\right] = 79.1 \text{ ohms}$$

The actual line impedances are given by Eq. (31). Using our results, we find

$$Z_{01} = \frac{Z_{01}^{air}}{\sqrt{\epsilon_{r1,eff}}} = \frac{89.6}{\sqrt{3.60}} = \underline{47.2\ \Omega} \quad \text{and} \quad Z_{02} = \frac{Z_{02}^{air}}{\sqrt{\epsilon_{r2,eff}}} = \frac{79.1}{\sqrt{3.68}} = \underline{41.2\ \Omega}$$

The reflection coefficient at the junction is now

$$\Gamma = \frac{Z_{02} - Z_{01}}{Z_{02} + Z_{01}} = \frac{47.2 - 41.2}{47.2 + 41.2} = 0.068$$

The transmission loss in dB is then

$$P_L(\text{dB}) = -10\log_{10}(1 - |\Gamma|^2) = -10\log_{10}(0.995) = \underline{0.02 \text{ dB}}$$

**13.11.** A parallel-plate waveguide is known to have a cutoff wavelength for the $m = 1$ TE and TM modes of $\lambda_{c1} = 0.4$ cm. The guide is operated at wavelength $\lambda = 1$ mm. How many modes propagate? The cutoff wavelength for mode $m$ is $\lambda_{cm} = 2nd/m$, where $n$ is the refractive index of the guide interior. For the first mode, we are given

$$\lambda_{c1} = \frac{2nd}{1} = 0.4 \text{ cm} \quad \Rightarrow \quad d = \frac{0.4}{2n} = \frac{0.2}{n} \text{ cm}$$

Now, for mode $m$ to propagate, we require

$$\lambda \leq \frac{2nd}{m} = \frac{0.4}{m} \quad \Rightarrow \quad m \leq \frac{0.4}{\lambda} = \frac{0.4}{0.1} = 4$$

So, accounting for 2 modes (TE and TM) for each value of $m$, and the single TEM mode, we will have a total of $\underline{9}$ modes.

**13.12.** A parallel-plate guide is to be constructed for operation in the TEM mode only over the frequency range $0 < f < 3$ GHz. The dielectric between plates is to be teflon ($\epsilon'_r = 2.1$). Determine the maximum allowable plate separation, $d$: We require that $f < f_{c1}$, which, using (41), becomes

$$f < \frac{c}{2nd} \quad \Rightarrow \quad d_{max} = \frac{c}{2nf_{max}} = \frac{3 \times 10^8}{2\sqrt{2.1}\,(3 \times 10^9)} = \underline{3.45 \, \text{cm}}$$

**13.13.** A lossless parallel-plate waveguide is known to propagate the $m = 2$ TE and TM modes at frequencies as low as 10GHz. If the plate separation is 1 cm, determine the dielectric constant of the medium between plates: Use

$$f_{c2} = \frac{c}{nd} = \frac{3 \times 10^{10}}{n(1)} = 10^{10} \;\; \Rightarrow \;\; n = 3 \;\; \text{or} \;\; \epsilon_r = \underline{9}$$

**13.14.** A $d = 1$ cm parallel-plate guide is made with glass ($n = 1.45$) between plates. If the operating frequency is 32 GHz, which modes will propagate? For a propagating mode, we require $f > f_{cm}$ Using (41) and the given values, we write

$$f > \frac{mc}{2nd} \quad \Rightarrow \quad m < \frac{2fnd}{c} = \frac{2(32 \times 10^9)(1.45)(.01)}{3 \times 10^8} = 3.09$$

The maximum allowed $m$ in this case is thus 3, and the propagating modes will be $\underline{TM_1}$, $\underline{TE_1}$, $\underline{TM_2}$, $\underline{TE_2}$, $\underline{TM_3}$, and $\underline{TE_3}$.

**13.15.** For the guide of Problem 13.14, and at the 32 GHz frequency, determine the difference between the group delays of the highest order mode (TE or TM) and the TEM mode. Assume a propagation distance of 10 cm: From Problem 13.14, we found $m_{max} = 3$. The group velocity of a TE or TM mode for $m = 3$ is

$$v_{g3} = \frac{c}{n}\sqrt{1 - \left(\frac{f_{c3}}{f}\right)^2} \quad \text{where} \quad f_{c3} = \frac{3(3 \times 10^{10})}{2(1.45)(1)} = 3.1 \times 10^{10} = 31\,\text{GHz}$$

Thus

$$v_{g3} = \frac{3 \times 10^{10}}{1.45}\sqrt{1 - \left(\frac{31}{32}\right)^2} = 5.13 \times 10^9 \text{ cm/s}$$

For the TEM mode (assuming no material dispersion) $v_{g,TEM} = c/n = 3 \times 10^{10}/1.45 = 2.07 \times 10^{10}$ cm/s. The group delay difference is now

$$\Delta t_g = z\left(\frac{1}{v_{g3}} - \frac{1}{v_{g,TEM}}\right) = 10\left(\frac{1}{5.13 \times 10^9} - \frac{1}{2.07 \times 10^{10}}\right) = \underline{1.5 \text{ ns}}$$

**13.16.** The cutoff frequency of the $m = 1$ TE and TM modes in an air-filled parallel-plate guide is known to be $f_{c1} = 7.5$ GHz. The guide is used at wavelength $\lambda = 1.5$ cm. Find the group velocity of the $m = 2$ TE and TM modes. First we know that $f_{c2} = 2f_{c1} = 15$ GHz. Then $f = c/\lambda = 3 \times 10^8/.015 = 20$ GHz. Now, using (57),

$$v_{g2} = \frac{c}{n}\sqrt{1 - \left(\frac{f_{c2}}{f}\right)^2} = \frac{c}{(1)}\sqrt{1 - \left(\frac{15}{20}\right)^2} = \underline{2 \times 10^8 \text{ m/s}}$$

**13.17.** A parallel-plate guide is partially filled with two lossless dielectrics (Fig. 13.25) where $\epsilon'_{r1} = 4.0$, $\epsilon'_{r2} = 2.1$, and $d = 1$ cm. At a certain frequency, it is found that the $TM_1$ mode propagates through the guide without suffering any reflective loss at the dielectric interface.

a) Find this frequency: The ray angle is such that the wave is incident on the interface at Brewster's angle. In this case

$$\theta_B = \tan^{-1}\sqrt{\frac{2.1}{4.0}} = 35.9°$$

The ray angle is thus $\theta = 90 - 35.9 = 54.1°$. The cutoff frequency for the $m = 1$ mode is

$$f_{c1} = \frac{c}{2d\sqrt{\epsilon'_{r1}}} = \frac{3 \times 10^{10}}{2(1)(2)} = 7.5\,\text{GHz}$$

The frequency is thus $f = f_{c1}/\cos\theta = 7.5/\cos(54.1°) = \underline{12.8\,\text{GHz}}$.

b) Is the guide operating at a single TM mode at the frequency found in part *a*? The cutoff frequency for the next higher mode, $TM_2$ is $f_{c2} = 2f_{c1} = 15$ GHz. The 12.8 GHz operating frequency is below this, so $TM_2$ will not propagate. So the answer is <u>yes</u>.

**13.18.** In the guide of Figure 13.25, it is found that $m = 1$ modes propagating from left to right totally reflect at the interface, so that no power is transmitted into the region of dielectric constant $\epsilon'_{r2}$.

a) Determine the range of frequencies over which this will occur: For total reflection, the ray angle measured from the normal to the interface must be greater than or equal to the critical angle, $\theta_c$, where $\sin\theta_c = (\epsilon'_{r2}/\epsilon'_{r1})^{1/2}$. The *maximum* mode ray angle is then $\theta_{1\,max} = 90° - \theta_c$. Now, using (39), we write

$$90° - \theta_c = \cos^{-1}\left(\frac{\pi}{k_{max}d}\right) = \cos^{-1}\left(\frac{\pi c}{2\pi f_{max} d\sqrt{4}}\right) = \cos^{-1}\left(\frac{c}{4df_{max}}\right)$$

Now

$$\cos(90 - \theta_c) = \sin\theta_c = \sqrt{\frac{\epsilon'_{r2}}{\epsilon'_{r1}}} = \frac{c}{4df_{max}}$$

Therefore $f_{max} = c/(2\sqrt{2.1}d) = (3\times10^8)/(2\sqrt{2.1}(.01)) = 10.35\,\text{GHz}$. The frequency range is thus $\underline{f < 10.35\,\text{GHz}}$.

b) Does your part *a* answer in any way relate to the cutoff frequency for $m = 1$ modes in any region? We note that $f_{max} = c/(2\sqrt{2.1}d) = f_{c1}$ in guide 2. To summarize, as frequency is lowered, the ray angle in guide 1 decreases, which leads to the incident angle at the interface increasing to eventually reach and surpass the critical angle. At the critical angle, the refracted angle in guide 2 is 90°, which corresponds to a zero degree ray angle in that guide. This defines the cutoff condition in guide 2. So it would make sense that $f_{max} = f_{c1}$(guide 2).

**13.19.** A rectangular waveguide has dimensions $a = 6$ cm and $b = 4$ cm.

a) Over what range of frequencies will the guide operate single mode? The cutoff frequency for mode $mp$ is, using Eq. (101):

$$f_{c,mn} = \frac{c}{2n}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{p}{b}\right)^2}$$

where $n$ is the refractive index of the guide interior. We require that the frequency lie between the cutoff frequencies of the $TE_{10}$ and $TE_{01}$ modes. These will be:

$$f_{c10} = \frac{c}{2na} = \frac{3\times10^8}{2n(.06)} = \frac{2.5\times10^9}{n} \qquad f_{c01} = \frac{c}{2nb} = \frac{3\times10^8}{2n(.04)} = \frac{3.75\times10^9}{n}$$

Thus, the range of frequencies for single mode operation is $\underline{2.5/n < f < 3.75/n \text{ GHz}}$

b) Over what frequency range will the guide support *both* $TE_{10}$ and $TE_{01}$ modes and no others? We note first that $f$ must be greater than $f_{c01}$ to support both modes, but must be less than the cutoff frequency for the next higher order mode. This will be $f_{c11}$, given by

$$f_{c11} = \frac{c}{2n}\sqrt{\left(\frac{1}{.06}\right)^2 + \left(\frac{1}{.04}\right)^2} = \frac{30c}{2n} = \frac{4.5\times10^9}{n}$$

The allowed frequency range is then

$$\underline{\frac{3.75}{n}\text{ GHz} < f < \frac{4.5}{n}\text{ GHz}}$$

**13.20.** Two rectangular waveguides are joined end-to-end. The guides have identical dimensions, where $a = 2b$. One guide is air-filled; the other is filled with a lossless dielectric characterized by $\epsilon'_r$.

a) Determine the maximum allowable value of $\epsilon'_r$ such that single mode operation can be simultaneously ensured in *both* guides at some frequency: Since $a = 2b$, the cutoff frequency for any mode in either guide is written using (101):

$$f_{cmp} = \sqrt{\left(\frac{mc}{4nb}\right)^2 + \left(\frac{pc}{2nb}\right)^2}$$

where $n = 1$ in guide 1 and $n = \sqrt{\epsilon'_r}$ in guide 2. We see that, with $a = 2b$, the next modes (having the next higher cutoff frequency) above $TE_{10}$ with be $TE_{20}$ and $TE_{01}$. We also see that in general, $f_{cmp}$(guide 2) $< f_{cmp}$(guide 1). To assure single mode operation in both guides, the operating frequency must be above cutoff for $TE_{10}$ in both guides, and below cutoff for the next mode in both guides. The allowed frequency range is therefore $f_{c10}$(guide 1) $< f < f_{c20}$(guide 2). This leads to $c/(2a) < f < c/(a\sqrt{\epsilon'_r})$. For this range to be viable, it is required that $\underline{\epsilon'_r < 4}$.

b) Write an expression for the frequency range over which single mode operation will occur in both guides; your answer should be in terms of $\epsilon'_r$, guide dimensions as needed, and other known constants: This was already found in part *a*:

$$\underline{\frac{c}{2a} < f < \frac{c}{\sqrt{\epsilon'_r}\, a}}$$

where $\epsilon'_r < 4$.

**13.21.** An air-filled rectangular waveguide is to be constructed for single-mode operation at 15 GHz. Specify the guide dimensions, $a$ and $b$, such that the design frequency is 10% higher than the cutoff frequency for the $TE_{10}$ mode, while being 10% lower than the cutoff frequency for the next higher-order mode: For an air-filled guide, we have

$$f_{c,mp} = \sqrt{\left(\frac{mc}{2a}\right)^2 + \left(\frac{pc}{2b}\right)^2}$$

For $TE_{10}$ we have $f_{c10} = c/2a$, while for the next mode ($TE_{01}$), $f_{c01} = c/2b$. Our requirements state that $f = 1.1 f_{c10} = 0.9 f_{c01}$. So $f_{c10} = 15/1.1 = 13.6$ GHz and $f_{c01} = 15/0.9 = 16.7$ GHz. The guide dimensions will be

$$a = \frac{c}{2f_{c10}} = \frac{3 \times 10^{10}}{2(13.6 \times 10^9)} = \underline{1.1\,\text{cm}} \quad \text{and} \quad b = \frac{c}{2f_{c01}} = \frac{3 \times 10^{10}}{2(16.7 \times 10^9)} = \underline{0.90\,\text{cm}}$$

**13.22.** Using the relation $P_{av} = (1/2)\text{Re}\{\mathbf{E}_s \times \mathbf{H}_s^*\}$, and Eqs. (106) through (108), show that the average power density in the $TE_{10}$ mode in a rectangular waveguide is given by

$$P_{av} = \frac{\beta_{10}}{2\omega\mu} E_0^2 \sin^2(\kappa_{10} x)\,\mathbf{a}_z \qquad \text{W/m}^2$$

Inspecting (106) through (108), we see that (108) includes a factor of $j$, and so would lead to an imaginary part of the total power when the cross product with $E_y$ is taken. Therefore, the real power in this case is found through the cross product of (106) with the complex conjugate of (107), or

$$P_{av} = \frac{1}{2}\text{Re}\left\{\mathbf{E}_{ys} \times \mathbf{H}_{xs}^*\right\} = \frac{\beta_{10}}{2\omega\mu} E_0^2 \sin^2(\kappa_{10} x)\,\mathbf{a}_z \qquad \text{W/m}^2$$

**13.23.** Integrate the result of Problem 13.22 over the guide cross-section $0 < x < a$, $0 < y < b$, to show that the power in Watts transmitted down the guide is given as

$$P = \frac{\beta_{10} ab}{4\omega\mu} E_0^2 = \frac{ab}{4\eta} E_0^2 \sin\theta_{10} \ \ \text{W}$$

where $\eta = \sqrt{\mu/\epsilon}$, and $\theta_{10}$ is the wave angle associated with the $TE_{10}$ mode. Interpret. First, the integration:

$$P = \int_0^b \int_0^a \frac{\beta_{10}}{2\omega\mu} E_0^2 \sin^2(\kappa_{10} x)\,\mathbf{a}_z \cdot \mathbf{a}_z\, dx\, dy = \frac{\beta_{10} ab}{4\omega\mu} E_0^2$$

Next, from (54), we have $\beta_{10} = \omega\sqrt{\mu\epsilon}\sin\theta_{10}$, which, on substitution, leads to

$$P = \frac{ab}{4\eta} E_0^2 \sin\theta_{10} \ \ \text{W} \ \ \text{with } \eta = \sqrt{\frac{\mu}{\epsilon}}$$

The $\sin\theta_{10}$ dependence demonstrates the principle of group velocity as energy velocity (or power). This was considered in the discussion leading to Eq. (57).

**13.24.** Show that the group dispersion parameter, $d^2\beta/d\omega^2$, for given mode in a parallel-plate or rectangular waveguide is given by

$$\frac{d^2\beta}{d\omega^2} = -\frac{n}{\omega c}\left(\frac{\omega_c}{\omega}\right)^2 \left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^{-3/2}$$

where $\omega_c$ is the radian cutoff frequency for the mode in question (note that the first derivative form was already found, resulting in Eq. (57)). First, taking the reciprocal of (57), we find

$$\frac{d\beta}{d\omega} = \frac{n}{c}\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^{-1/2}$$

Taking the derivative of this equation with respect to $\omega$ leads to

$$\frac{d^2\beta}{d\omega^2} = \frac{n}{c}\left(-\frac{1}{2}\right)\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^{-3/2}\left(\frac{2\omega_c^2}{\omega^3}\right) = -\frac{n}{\omega c}\left(\frac{\omega_c}{\omega}\right)^2\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]^{-3/2}$$

**13.25.** Consider a transform-limited pulse of center frequency $f = 10$ GHz and of full-width $2T = 1.0$ ns. The pulse propagates in a lossless single mode rectangular guide which is air-filled and in which the 10 GHz operating frequency is 1.1 times the cutoff frequency of the $TE_{10}$ mode. Using the result of Problem 13.24, determine the length of the guide over which the pulse broadens to twice its initial width: The broadened pulse will have width given by $T' = \sqrt{T^2 + (\Delta\tau)^2}$, where $\Delta\tau = \beta_2 L/T$ for a transform limited pulse (assumed gaussian). $\beta_2$ is the Problem 13.24 result evaluated at the operating frequency, or

$$\beta_2 = \frac{d^2\beta}{d\omega^2}|_{\omega=10\,\mathrm{GHz}} = -\frac{1}{(2\pi \times 10^{10})(3 \times 10^8)}\left(\frac{1}{1.1}\right)^2\left[1 - \left(\frac{1}{1.1}\right)^2\right]^{-3/2}$$
$$= 6.1 \times 10^{-19}\ \mathrm{s^2/m} = 0.61\ \mathrm{ns^2/m}$$

Now $\Delta\tau = 0.61L/0.5 = 1.2L$ ns. For the pulse width to double, we have $T' = 1$ ns, and

$$\sqrt{(.05)^2 + (1.2L)^2} = 1 \quad \Rightarrow \quad L = 0.72\,\mathrm{m} = \underline{72\,\mathrm{cm}}$$

What simple step can be taken to reduce the amount of pulse broadening in this guide, while maintaining the same initial pulse width? It can be seen that $\beta_2$ can be reduced by increasing the operating frequency relative to the cutoff frequency; i.e., operate as far above cutoff as possible, without allowing the next higher-order modes to propagate.

**13.26.** A symmetric dielectric slab waveguide has a slab thickness $d = 10\ \mu$m, with $n_1 = 1.48$ and $n_2 = 1.45$. If the operating wavelength is $\lambda = 1.3\ \mu$m, what modes will propagate? We use the condition expressed through (141): $k_0 d\sqrt{n_1^2 - n_2^2} \geq (m-1)\pi$. Since $k_0 = 2\pi/\lambda$, the condition becomes

$$\frac{2d}{\lambda}\sqrt{n_1^2 - n_2^2} \geq (m-1) \quad \Rightarrow \quad \frac{2(10)}{1.3}\sqrt{(1.48)^2 - (1.45)^2} = 4.56 \geq m - 1$$

Therefore, $m_{max} = 5$, and we have TE and TM modes for which $m = 1, 2, 3, 4, 5$ propagating (ten total).

**13.27.** A symmetric slab waveguide is known to support only a single pair of TE and TM modes at wavelength $\lambda = 1.55\ \mu$m. If the slab thickness is 5 $\mu$m, what is the maximum value of $n_1$ if $n_2 = 3.30$? Using (142) we have

$$\frac{2\pi d}{\lambda}\sqrt{n_1^2 - n_2^2} < \pi \quad \Rightarrow \quad n_1 < \sqrt{\frac{\lambda}{2d} + n_2^2} = \sqrt{\frac{1.55}{2(5)} + (3.30)^2} = \underline{3.32}$$

**13.28.** In a symmetric slab waveguide, $n_1 = 1.50$, $n_2 = 1.45$, and $d = 10\ \mu$m.

a) What is the phase velocity of the $m = 1$ TE or TM mode at cutoff? At cutoff, the mode propagates in the slab at the critical angle, which means that the phase velocity will be equal to that of a plane wave in the upper or lower media of index $n_2$. Phase velocity will therefore be $v_p(\mathrm{cutoff}) = c/n_2 = 3 \times 10^8/1.45 = \underline{2.07 \times 10^8\ \mathrm{m/s}}$.

b) What is the phase velocity of the $m = 2$ TE or TM modes at cutoff? The reasoning of part $a$ applies to all modes, so the answer is the same, or $\underline{2.07 \times 10^8\ \mathrm{m/s}}$.

**13.29.** An *asymmetric* slab waveguide is shown in Fig. 13.26. In this case, the regions above and below the slab have unequal refractive indices, where $n_1 > n_3 > n_2$.

a) Write, in terms of the appropriate indices, an expression for the minimum possible wave angle, $\theta_1$, that a guided mode may have: The wave angle must be equal to or greater than the critical angle of total reflection at *both* interfaces. The minimum wave angle is thus determined by the *greater* of the two critical angles. Since $n_3 > n_2$, we find $\theta_{min} = \theta_{c,13} = \underline{\sin^{-1}(n_3/n_1)}$.

b) Write an expression for the maximum phase velocity a guided mode may have in this structure, using given or known parameters: We have $v_{p,max} = \omega/\beta_{min}$, where $\beta_{min} = n_1 k_0 \sin\theta_{1,min} = n_1 k_0 n_3/n_1 = n_3 k_0$. Thus $v_{p,max} = \omega/(n_3 k_0) = \underline{c/n_3}$.

**13.30.** A step index optical fiber is known to be single mode at wavelengths $\lambda > 1.2\ \mu$m. Another fiber is to be fabricated from the same materials, but is to be single mode at wavelengths $\lambda > 0.63\ \mu$m. By what percentage must the core radius of the new fiber differ from the old one, and should it be larger or smaller? We use the cutoff condition, given by (159):

$$\lambda > \lambda_c = \frac{2\pi a}{2.405}\sqrt{n_1^2 - n_2^2}$$

With $\lambda$ reduced, the core radius, $a$, must also be reduced by the same fraction. Therefore, the percentage *reduction* required in the core radius will be

$$\% = \frac{1.2 - .63}{1.2} \times 100 = \underline{47.5\%}$$

**13.31.** Is the mode field radius greater than or less than the fiber core radius in single-mode step-index fiber?

The answer to this can be found by inspecting Eq. (164). Clearly the mode field radius decreases with increasing $V$, so we can look at the extreme case of $V = 2.405$, which is the upper limit to single-mode operation. The equation evaluates as

$$\frac{\rho_0}{a} = 0.65 + \frac{1.619}{(2.405)^{3/2}} + \frac{2.879}{(2.405)^6} = 1.10$$

Therefore, $\rho_0$ is always greater than $a$ within the single-mode regime, $V < 2.405$.

**13.32.** The mode field radius of a step-index fiber is measured as 4.5 $\mu$m at free space wavelength $\lambda = 1.30$ $\mu$m. If the cutoff wavelength is specified as $\lambda_c = 1.20\ \mu$m, find the expected mode field radius at $\lambda = 1.55\ \mu$m.

In this problem it is helpful to use the relation $V = 2.405(\lambda_c/\lambda)$, and rewrite Eq. (164) to read:

$$\frac{\rho_0}{a} = 0.65 + 0.434\left(\frac{\lambda}{\lambda_c}\right)^{3/2} + 0.015\left(\frac{\lambda}{\lambda_c}\right)^6$$

At $\lambda = 1.30\,\mu$m, $\lambda/\lambda_c = 1.08$, and at 1.55 $\mu$m, $\lambda/\lambda_c = 1.29$. Using these values, along with our new equation, we write

$$\rho_0(1.55) = 4.5\left[\frac{0.65 + 0.434(1.29)^{3/2} + 0.015(1.29)^6}{0.65 + 0.434(1.08)^{3/2} + 0.015(1.08)^6}\right] = \underline{5.3\ \mu\text{m}}$$

## CHAPTER 14

**14.1.** A short dipole carrying current $I_0 \cos \omega t$ in the $\mathbf{a}_z$ direction is located at the origin in free space.

a) If $k = 1$ rad/m, $r = 2$ m, $\theta = 45°$, $\phi = 0$, and $t = 0$, give a unit vector in rectangular components that shows the instantaneous direction of $\mathbf{E}$: In spherical coordinates, the components of $\mathbf{E}$ are given by (13a) and (13b):

$$E_{rs} = \frac{I_0 d}{2\pi} \eta \cos\theta \, e^{-jkr} \left( \frac{1}{r^2} + \frac{1}{jkr^3} \right) \tag{13a}$$

$$E_{\theta s} = \frac{I_0 d}{4\pi} \eta \sin\theta \, e^{-jkr} \left( \frac{jk}{r} + \frac{1}{r^2} + \frac{1}{jkr^3} \right) \tag{13b}$$

Since we want a unit vector at $t = 0$, we need only the relative amplitudes of the two components, but we need the absolute phases. Since $\theta = 45°$, $\sin\theta = \cos\theta = 1/\sqrt{2}$. Also, with $k = 1 = 2\pi/\lambda$, it follows that $\lambda = 2\pi$ m. The above two equations can be simplified by these substitutions, while dropping all amplitude terms that are common to both. Obtain

$$A_r = \left( \frac{1}{r^2} + \frac{1}{jr^3} \right) e^{-jr}$$

$$A_\theta = \frac{1}{2} \left( j\frac{1}{r} + \frac{1}{r^2} + \frac{1}{jr^3} \right) e^{-jr}$$

Now with $r = 2$ m, we obtain

$$A_r = \left( \frac{1}{4} - j\frac{1}{8} \right) e^{-j2} = \frac{1}{4}(1.12)e^{-j26.6°} e^{-j2}$$

$$A_\theta = \left( j\frac{1}{4} + \frac{1}{8} - j\frac{1}{16} \right) e^{-j2} = \frac{1}{4}(0.90)e^{j56.3°} e^{-j2}$$

The total vector is now $\mathbf{A} = A_r \mathbf{a}_r + A_\theta \mathbf{a}_\theta$. We can normalize the vector by first finding the magnitude:

$$|\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}^*} = \frac{1}{4}\sqrt{(1.12)^2 + (0.90)^2} = 0.359$$

Dividing the field vector by this magnitude and converting 2 rad to 114.6°, we write the normalized vector as

$$\mathbf{A}_{Ns} = 0.780 e^{-j141.2°} \mathbf{a}_r + 0.627 e^{-58.3°} \mathbf{a}_\theta$$

In real instantaneous form, this becomes

$$\mathbf{A}_N(t) = \text{Re}\left( \mathbf{A}_{Ns} e^{j\omega t} \right) = 0.780 \cos(\omega t - 141.2°)\mathbf{a}_r + 0.627 \cos(\omega t - 58.3°)\mathbf{a}_\theta$$

We evaluate this at $t = 0$ to find

$$\mathbf{A}_N(0) = 0.780 \cos(141.2°)\mathbf{a}_r + 0.627 \cos(58.3°)\mathbf{a}_\theta = -0.608\mathbf{a}_r + 0.330\mathbf{a}_\theta$$

**14.1** a) (continued)

Dividing by the magnitude, $\sqrt{(0.608)^2 + (0.330)^2} = 0.692$, we obtain the unit vector at $t = 0$: $\mathbf{a}_N(0) = -0.879\mathbf{a}_r + 0.477\mathbf{a}_\theta$. We next convert this to rectangular components:

$$a_{Nx} = \mathbf{a}_N(0) \cdot \mathbf{a}_x = -0.879 \sin\theta \cos\phi + 0.477 \cos\theta \cos\phi = \frac{1}{\sqrt{2}}(-0.879 + 0.477) = -0.284$$

$$a_{Ny} = \mathbf{a}_N(0) \cdot \mathbf{a}_y = -0.879 \sin\theta \sin\phi + 0.477 \cos\theta \sin\phi = 0 \quad \text{since } \phi = 0$$

$$a_{Nz} = \mathbf{a}_N(0) \cdot \mathbf{a}_z = -0.879 \cos\theta - 0.477 \sin\theta = \frac{1}{\sqrt{2}}(-0.879 - 0.477) = -0.959$$

The final result is then

$$\mathbf{a}_N(0) = \underline{-0.284\mathbf{a}_x - 0.959\mathbf{a}_z}$$

b) What fraction of the total average power is radiated in the belt, $80° < \theta < 100°$? We use the far-zone phasor fields, (22) and (23), with $k = 2\pi/\lambda$, and first find the average power density:

$$P_{avg} = \frac{1}{2}\text{Re}[E_{\theta s} H^*_{\phi s}] = \frac{I_0^2 d^2 \eta}{8\lambda^2 r^2} \sin^2\theta \quad \text{W/m}^2$$

We integrate this over the given belt, an at radius $r$:

$$P_{belt} = \int_0^{2\pi} \int_{80°}^{100°} \frac{I_0^2 d^2 \eta}{8\lambda^2 r^2} \sin^2\theta \, r^2 \sin\theta \, d\theta \, d\phi = \frac{\pi I_0^2 d^2 \eta}{4\lambda^2} \int_{80°}^{100°} \sin^3\theta \, d\theta$$

Evaluating the integral, we find

$$P_{belt} = \frac{\pi I_0^2 d^2 \eta}{4\lambda^2} \left[ -\frac{1}{3} \cos\theta \left( \sin^2\theta + 2 \right) \right]_{80}^{100} = (0.344) \frac{\pi I_0^2 d^2 \eta}{4\lambda^2}$$

The total power is found by performing the same integral over $\theta$, where $0 < \theta < 180°$. Doing this, it is found that

$$P_{tot} = (1.333) \frac{\pi I_0^2 d^2 \eta}{4\lambda^2}$$

The fraction of the total power in the belt is then $f = 0.344/1.333 = \underline{0.258}$.

**14.2.** *For the Hertzian dipole*, prepare a curve, $r$ vs. $\theta$ in polar coordinates, showing the locus in the $\phi = 0$ plane where:

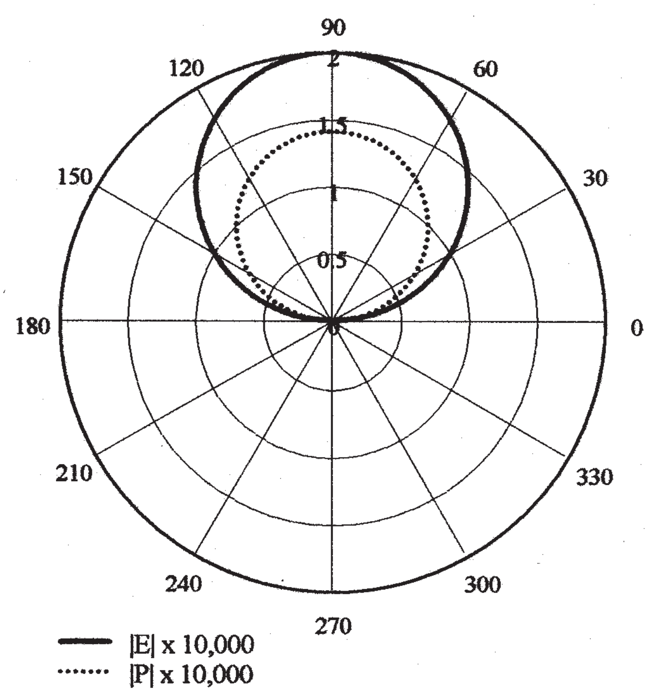
a) The radiation field $|E_{\theta s}|$ is one-half of its value at $r = 10^4$ m, $\theta = \pi/2$: Assuming the far field approximation, we use (22) with $k = 2\pi/\lambda$ to set up the equation:

$$|E_{\theta s}| = \frac{I_0 d\eta}{2\lambda r}\sin\theta = \frac{1}{2} \times \frac{I_0 d\eta}{2 \times 10^4 \lambda} \quad \Rightarrow \quad r = 2 \times 10^4 \sin\theta$$

b) The average radiated power density, $P_{r,av}$, is one-half of its value at $r = 10^4$ m, $\theta = \pi/2$. To find the average power, we use (22) and (23) in

$$P_{r,av} = \frac{1}{2}\text{Re}\{E_{\theta s}H^*_{\phi s}\} = \frac{1}{2}\frac{I_0^2 d^2 \eta}{4\lambda^2 r^2}\sin^2\theta = \frac{1}{2} \times \frac{1}{2}\frac{I_0^2 d^2\eta}{4\lambda^2(10^8)} \quad \Rightarrow \quad r = \sqrt{2} \times 10^4 \sin\theta$$

The polar plots for field ($r = 2 \times 10^4 \sin\theta$) and power ($r = \sqrt{2} \times 10^4 \sin\theta$) are shown below. Both are circles.



**14.3.** Two short antennas at the origin in free space carry identical currents of $5\cos\omega t$ A, one in the $\mathbf{a}_z$ direction, one in the $\mathbf{a}_y$ direction. Let $\lambda = 2\pi$ m and $d = 0.1$ m. Find $\mathbf{E}_s$ at the distant point:

a) $(x = 0, y = 1000, z = 0)$: This point lies along the axial direction of the $\mathbf{a}_y$ antenna, so its contribution to the field will be zero. This leaves the $\mathbf{a}_z$ antenna, and since $\theta = 90°$, only the $E_{\theta s}$ component will be present (as (136) and (137) show). Since we are in the far zone, (138) applies. We use $\theta = 90°$, $d = 0.1$, $\lambda = 2\pi$, $\eta = \eta_0 = 120\pi$, and $r = 1000$ to write:

$$\mathbf{E}_s = E_{\theta s}\mathbf{a}_\theta = j\frac{I_0 d\eta}{2\lambda r}\sin\theta e^{-j2\pi r/\lambda}\,\mathbf{a}_\theta = j\frac{5(0.1)(120\pi)}{4\pi(1000)}e^{-j1000}\,\mathbf{a}_\theta$$
$$= j(1.5\times 10^{-2})e^{-j1000}\,\mathbf{a}_\theta = \underline{-j(1.5\times 10^{-2})e^{-j1000}\,\mathbf{a}_z \text{ V/m}}$$

b) $(0, 0, 1000)$: Along the $z$ axis, only the $\mathbf{a}_y$ antenna will contribute to the field. Since the distance is the same, we can apply the part $a$ result, modified such the the field direction is in $-\mathbf{a}_y$: $\mathbf{E}_s = \underline{-j(1.5\times 10^{-2})e^{-j1000}\,\mathbf{a}_y \text{ V/m}}$

c) $(1000, 0, 0)$: Here, both antennas will contribute. Applying the results of parts $a$ and $b$, we find $\mathbf{E}_s = \underline{-j(1.5\times 10^{-2})(\mathbf{a}_y + \mathbf{a}_z)}$.

d) Find $\mathbf{E}$ at $(1000, 0, 0)$ at $t = 0$: This is found through

$$\mathbf{E}(t) = \text{Re}\left(\mathbf{E}_s e^{j\omega t}\right) = (1.5\times 10^{-2})\sin(\omega t - 1000)(\mathbf{a}_y + \mathbf{a}_z)$$

Evaluating at $t = 0$, we find
$\mathbf{E}(0) = (1.5\times 10^{-2})[-\sin(1000)](\mathbf{a}_y + \mathbf{a}_z) = \underline{-(1.24\times 10^{-2})(\mathbf{a}_y + \mathbf{a}_z) \text{ V/m}}$.

e) Find $|\mathbf{E}|$ at $(1000, 0, 0)$ at $t = 0$: Taking the magnitude of the part $d$ result, we find $|\mathbf{E}| = \underline{1.75\times 10^{-2} \text{ V/m}}$.

**14.4.** Write the Hertzian dipole electric field, whose components are given in Eqs. (15) and (16), in the near-zone in free space, where $kr << 1$. In this case, only a single term in each of the two equations survives, and the phases, $\delta_r$ and $\delta_\theta$, simplify to a single value. Construct the resulting electric field vector and compare your result to the static dipole result (Eq. (35) (not (36)) in Chapter 4). What relation must exist between the static dipole charge, $Q$, and the current amplitude, $I_0$, so that the two results are identical?

First, we evaluate the phase terms under the approximation, $kr << 1$: Using (17b) and (18) we have

$$\delta_r \doteq \tan^{-1}(0) - \frac{\pi}{2} \;=\; -\frac{\pi}{2} \quad \text{and} \quad \delta_\theta \doteq \tan^{-1}(-\infty) \;=\; -\frac{\pi}{2}$$

These are used in (15) and (16). In those equations, the $(kr)^{-2}$ term is retained in (15) and the $(kr)^{-4}$ term is kept in (16). The results are

$$E_{rs} \doteq \frac{I_0 d}{2\pi r^2 (kr)} \eta \cos\theta \, \exp(-j\pi/2) = -j\frac{I_0 \eta d}{2\pi k r^3} \cos\theta$$

$$E_{\theta s} \doteq \frac{I_0 k d}{4\pi r (kr)^2} \eta \sin\theta \exp(-j\pi/2) = -j\frac{I_0 \eta d}{4\pi k r^3} \sin\theta$$

where, in free space

$$\frac{\eta}{k} = \frac{\sqrt{\mu_0/\epsilon_0}}{\omega\sqrt{\mu_0\epsilon_0}} = \frac{1}{\omega\epsilon_0}$$

The field vector can now be constructed as

$$\mathbf{E}_s = \frac{(-jI_0/\omega)d}{4\pi\epsilon_0 r^3} \;\; [2\cos\theta \, \mathbf{a}_r + \sin\theta \, \mathbf{a}_\theta]$$

This expression is equivalent to the static dipole field (Eq. (35) in Chapter 4), provided that $\underline{I_0 = j\omega Q}$, which is just a statement that the current is the time derivative of the time-harmonic charge.

**14.5.** Consider the term in Eq. (14) (or in Eq. (10)) that gives the $1/r^2$ dependence in the Hertzian dipole magnetic field. Assuming this term dominates and that $kr << 1$, show that the resulting magnetic field is the same as that found by applying the Biot-Savart law (Eq. (2), Chapter 7) to a current element of differential length $d$, oriented along the $z$ axis, and centered at the origin.

We begin by writing Eq. (14) under the condition $kr << 1$, for which the phase term in (14) becomes, using Eq. (17a):

$$\exp[-j(kr - \delta_\phi)] = \exp[-j(kr - \tan^{-1}(kr))] \doteq \exp[-j(kr - kr)] = 0$$

With $kr << 1$, the $1/(kr)^2$ term in (14) dominates, and the field becomes:

$$H_\phi \doteq \frac{I_0 kd}{4\pi r}\frac{1}{(kr)}\sin\theta\exp(j0) = \frac{I_0 d}{4\pi r^2}\sin\theta$$

Now, using the Biot-Savart law for a short element of assumed differential length, $d$, we have

$$\mathbf{H} = \frac{I_0 d\mathbf{L}\times\mathbf{a}_R}{4\pi R^2}$$

where, with the element at the origin and oriented along $z$, we have $R = r$, $\mathbf{a}_R = \mathbf{a}_r$, and $d\mathbf{L} = d\mathbf{a}_z$. With these subsitutions:

$$\mathbf{H} = \frac{I_0 d\,(\mathbf{a}_z \times \mathbf{a}_r)}{4\pi r^2} = \frac{I_0 d}{4\pi r^2}\sin\theta\,\mathbf{a}_\phi \quad \text{(done)}$$

**14.6.** Evaluate the time-average Poynting vector, $< \mathbf{S} >= (1/2)\mathcal{R}e\left\{\mathbf{E}_s \times \mathbf{H}_s^*\right\}$ for the Hertzian dipole, assuming the general case that involves the field components as given by Eqs. (10), (13a), and (13b). Compare your result to the far-zone case, Eq. (26).

With radial and theta components of $\mathbf{E}$, and with only a phi component of $\mathbf{H}$, the Poynting vector becomes

$$< \mathbf{S} >= \frac{1}{2}\mathcal{R}e\left\{E_{\theta s}H_{\phi s}^*\,\mathbf{a}_r \;-\; E_{rs}H_{\phi s}^*\,\mathbf{a}_\theta\right\}$$

Substituting (10), (13a) and (13b), this becomes:

$$< \mathbf{S} > = \frac{1}{2}\mathcal{R}e\left\{\left(\frac{I_0 d}{4\pi}\right)^2 \eta\sin^2\theta\left[\frac{k^2}{r^2} - j\frac{k}{r^3} - \frac{1}{r^4} + j\frac{k}{r^3} + \frac{1}{r^4} - j\frac{k}{r^5}\right]\mathbf{a}_r \right.$$
$$\left. - \frac{(I_0 d)^2}{8\pi^2}\sin\theta\cos\theta\left[-j\frac{k}{r^3} - \frac{1}{r^4} + \frac{1}{r^4} - j\frac{k}{r^5}\right]\mathbf{a}_\theta\right\}$$

In taking the real part, the imaginary terms that do not cancel are removed, other real terms cancel, and only the first term in the radial component remains. The final result is

$$< \mathbf{S} >= \frac{1}{2}\left(\frac{I_0 d\,k}{4\pi r}\right)^2 \eta\sin^\theta\,\mathbf{a}_r \ \mathrm{W/m^2}$$

which we see to be identical to Eq. (26).

**14.7.** A short current element has $d = 0.03\lambda$. Calculate the radiation resistance for each of the following current distributions:

a) Uniform: In this case, (30) applies directly and we find

$$R_{rad} = 80\pi^2 \left(\frac{d}{\lambda}\right)^2 = 80\pi^2(.03)^2 = \underline{0.711\,\Omega}$$

b) Linear: $I(z) = I_0(0.5d - |z|)/0.5d$: Here, the average current is $0.5I_0$, and so the average power drops by a factor of 0.25. The radiation resistance therefore is down to one-fourth the value found in part $a$, or $R_{rad} = (0.25)(0.711) = \underline{0.178\,\Omega}$.

c) Step: $I_0$ for $0 < |z| < 0.25d$ and $0.5I_0$ for $0.25d < |z| < 0.5d$: In this case the average current on the wire is $0.75I_0$. The radiated power (and radiation resistance) are down to a factor of $(0.75)^2$ times their values for a uniform current, and so $R_{rad} = (0.75)^2(0.711) = \underline{0.400\,\Omega}$.

**14.8.** Evaluate the time-average Poynting vector, $< \mathbf{S} >= (1/2)\mathcal{R}e\,\{\mathbf{E}_s \times \mathbf{H}_s^*\}$ for the magnetic dipole antenna in the far-zone, in which all terms of order $1/r^2$ and $1/r^4$ are neglected in Eqs. (48), (49), and (50). Compare your result to the far-zone power density of the Hertzian dipole, Eq. (26). In this comparison, and assuming equal current amplitudes, what relation between loop radius, $a$, and dipole length, $d$, would result in equal radiated powers from the two devices?

First, neglecting the indicated terms in (48)-(50), only the terms in $1/r$ survive. We find:

$$E_{\phi s} \doteq -j\frac{\omega\mu_0\pi a^2 I_0 k}{4\pi r}\sin\theta\exp[-j(kr - \delta_\phi)]$$

$$H_{\theta s} \doteq j\frac{\omega\mu_0\pi a^2 I_0 k}{4\pi r}\frac{1}{\eta}\sin\theta\exp[-j(kr - \delta_\theta)]$$

where, in the approximation, $\delta_\phi \doteq \delta_\theta = \tan^{-1}(kr)$. So now

$$< \mathbf{S} >= (1/2)\mathcal{R}e\,\{\mathbf{E}_{\phi s} \times \mathbf{H}_{\theta s}^*\} = -\frac{1}{2}\mathcal{R}e\,\{E_{\phi s}H_{\theta s}^*\}\,\mathbf{a}_r$$

Using the above field expressions, we find

$$|< \mathbf{S} >| = S_{r(md)} = \frac{1}{2}\left(\frac{I_0 k}{4\pi r}\right)^2 [\omega\mu_0\pi a^2]^2\,\frac{1}{\eta}\sin^2\theta$$

which we now compare to the power density from the Hertzian dipole, Eq. (26):

$$S_{r(Hd)} = \frac{1}{2}\left(\frac{I_0 k}{4\pi r}\right)^2 d^2\eta\sin^2\theta$$

For equal power densities from the two structures, we thus require (assuming free space):

$$[\omega\mu_0\pi a^2]^2\,\frac{1}{\eta} = d^2\eta \;\Rightarrow\; d = \pi a^2\,\frac{\omega\mu_0}{\eta} = \pi a^2\,(\omega\sqrt{\mu_0\epsilon_0}) = \pi a^2\left(\frac{2\pi}{\lambda}\right)$$

or

$$d = \frac{(2\pi a)^2}{2\lambda} \qquad \text{(the square of the loop circumference over twice the wavelength)}$$

**14.9.** A dipole antenna in free space has a linear current distribution. If the length is $0.02\lambda$, what value of $I_0$ is required to:

a) provide a radiation-field amplitude of 100 mV/m at a distance of one mile, at $\theta = 90°$: With a linear current distribution, the peak current, $I_0$, occurs at the center of the dipole; current decreases linearly to zero at the two ends. The average current is thus $I_0/2$, and we use Eq. (138) to write:

$$|E_\theta| = \frac{I_0 d\eta_0}{4\lambda r}\sin(90°) = \frac{I_0(0.02)(120\pi)}{(4)(5280)(12)(0.0254)} = 0.1 \quad \Rightarrow \quad I_0 = \underline{85.4 \text{ A}}$$

b) radiate a total power of 1 watt? We use

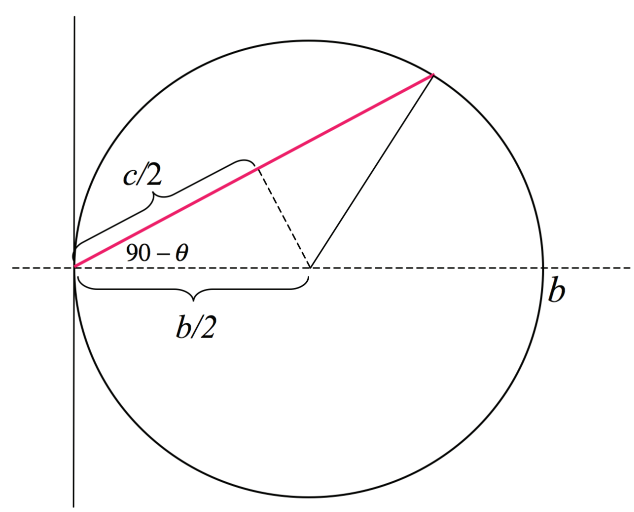
$$P_{avg} = \left(\frac{1}{4}\right)\left(\frac{1}{2} I_0^2 R_{rad}\right)$$

where the radiation resistance is given by Eq. (140), and where the factor of 1/4 arises from the average current of $I_0/2$: We obtain $P_{avg} = 10\pi^2 I_0^2 (0.02)^2 = 1 \quad \Rightarrow \quad I_0 = \underline{5.03 \text{ A}}$.

**14.10.** Show that the chord length in the E-plane plot of Fig. 14.4 is equal to $b\sin\theta$, where $b$ is the circle diameter.

Referring to the construction below, half the chord length, $c/2$, is given by

$$\frac{c}{2} = \frac{b}{2}\cos(90-\theta) = \frac{b}{2}\sin\theta \;\Rightarrow\; \underline{c = b\sin\theta} \;\; \text{(done)}$$



**14.11.** A monopole antenna in free space, extending vertically over a perfectly conducting plane, has a linear current distribution. If the length of the antenna is $0.01\lambda$, what value of $I_0$ is required to

a) provide a radiation field amplitude of 100 mV/m at a distance of 1 mi, at $\theta = 90°$: The image antenna below the plane provides a radiation pattern that is identical to a dipole antenna of length $0.02\lambda$. The radiation field is thus given by (22) in free space, where $k = 2\pi/\lambda$, $\theta = 90°$, and with an additional factor of 1/2 included to account for the linear current distribution:

$$|E_\theta| = \frac{1}{2}\frac{I_0 d\eta_0}{2\lambda r} \quad \Rightarrow \quad I_0 = \frac{4r|E_\theta|}{(d/\lambda)\eta_0} = \frac{4(5289)(12 \times .0254)(100 \times 10^{-3})}{(.02)(377)} = \underline{85.4\,\text{A}}$$

b) radiate a total power of 1W: For the monopole over the conducting plane, power is radiated only over the upper half-space. This reduces the radiation resistance of the equivalent dipole antenna by a factor of one-half. Additionally, the linear current distribution reduces the radiation resistance of a dipole having uniform current by a factor of one-fourth. Therefore, $R_{rad}$ is one-eighth the value obtained from (30), or $R_{rad} = 10\pi^2(d/\lambda)^2$. The current magnitude is now

$$I_0 = \left[\frac{2P_{av}}{R_{rad}}\right]^{1/2} = \left[\frac{2(1)}{10\pi^2(d/\lambda)^2}\right]^{1/2} = \frac{\sqrt{2}}{\sqrt{10}\,\pi(.02)} = \underline{7.1\,\text{A}}$$

**14.12.** Find the zeros in $\theta$ for the E-plane pattern of a dipole antenna for which (using Fig. 14.8 as a guide):

a) $\ell = \lambda$: We look for zeros in the pattern function, Eq. (59), for which $k\ell = (2\pi/\lambda)\lambda = 2\pi$.

$$F(\theta) = \left[\frac{\cos(k\ell\cos\theta) - \cos(k\ell)}{\sin\theta}\right] = \left[\frac{\cos(2\pi\cos\theta) - \cos(2\pi)}{\sin\theta}\right]$$

Zeros will occur whenever

$$\cos(2\pi\cos\theta) = 1 \quad \Rightarrow \quad \underline{\theta = (0,\, 90°,\, 180°,\, 270°)}$$

You are encouraged to show (using small argument approximations for cosine and sine) that $F(\theta)$ does approach zero when $\theta$ approaches zero or 180° (despite the $\sin\theta$ term in the denominator).

b) $2\ell = 1.3\lambda$: In this case $k\ell = (2\pi/\lambda)(1.3\lambda/2) = 1.3\pi$. The pattern function becomes

$$F(\theta) = \left[\frac{\cos(1.3\pi\cos\theta) - \cos(1.3\pi)}{\sin\theta}\right]$$

Zeros of this function will occur at $\underline{\theta = (0,\, 57.42°,\, 122.58°,\, 180°)}$.

**14.13.** The radiation field of a certain short vertical current element is $E_{\theta s} = (20/r)\sin\theta\, e^{-j10\pi r}$ V/m if it is located at the origin in free space.

a) Find $E_{\theta s}$ at $P(r = 100, \theta = 90°, \phi = 30°)$: Substituting these values into the given formula, find

$$E_{\theta s} = \frac{20}{100}\sin(90°)e^{-j10\pi(100)} = \underline{0.2e^{-j1000\pi} \text{ V/m}}$$

b) Find $E_{\theta s}$ at $P$ if the vertical element is located at $A(0.1, 90°, 90°)$: This places the element on the $y$ axis at $y = 0.1$. As a result of moving the antenna from the origin to $y = 0.1$, the change in distance to point $P$ is negligible when considering the change in field *amplitude*, but is not when considering the change in *phase*. Consider lines drawn from the origin to $P$ and from $y = 0.1$ to $P$. These lines can be considered essentially parallel, and so the difference in their lengths is $l \doteq 0.1\sin(30°)$, with the line from $y = 0.1$ being shorter by this amount. The construction and arguments are similar to those used in the discussion of the electric dipole in Sec. 4.7. The electric field is now the result of part $a$, modified by including a shorter distance, $r$, in the phase term only. We show this as an additional phase factor:

$$E_{\theta s} = 0.2e^{-j1000\pi}e^{j10\pi(0.1\sin 30} = \underline{0.2e^{-j1000\pi}e^{j0.5\pi} \text{ V/m}}$$

c) Find $E_{\theta s}$ at $P$ if identical elements are located at $A(0.1, 90°, 90°)$ and $B(0.1, 90°, 270°)$: The original element of part $b$ is still in place, but a new one has been added at $y = -0.1$. Again, constructing a line between $B$ and $P$, we find, using the same arguments as in part $b$, that the length of this line is approximately $0.1\sin(30°)$ *longer* than the distance from the origin to $P$. The part $b$ result is thus modified to include the contribution from the second element, whose field will add to that of the first:

$$E_{\theta s} = 0.2e^{-j1000\pi}\left(e^{j0.5\pi} + e^{-j0.5\pi}\right) = 0.2e^{-j1000\pi}2\cos(0.5\pi) = \underline{0}$$

The two fields are out of phase at $P$ under the approximations we have used.

**14.14.** For a dipole antenna of overall length $2\ell = \lambda$, evaluate the maximum directivity in dB, and the half-power beamwidth.

$D_{max}$ is found using Eq. (64) with $k\ell = \pi$, and involves a numerical integration. This I did using a Mathematica code to find $D_{max} = 2.41$, and in decibels this is $10\log_{10}(2.41) = \underline{3.82 \text{ dB}}$, with the maximum occurring in the direction $\theta = 90°$.

With $k\ell = \pi$, the pattern function, Eq. (59), reaches a maximum value of 2 at $\theta = 90°$. The half-power beamwidth is found numerically by setting the pattern function equal to $\sqrt{2}$ and thus solving

$$F(\theta) = \left[\frac{\cos(\pi\cos\theta) - \cos(\pi)}{\sin\theta}\right] = \sqrt{2}$$

The result is $\theta = 66.1°$, which means that the deviation angle from the direction along the maximum is $\theta_{1/2}/2 = 90 - 66.1 = 23.9°$. The half-power beamwidth is therefore $\theta_{1/2} = 2 \times 23.9 = \underline{47.8°}$.

**14.15.** For a dipole antenna of overall length $2\ell = 1.3\lambda$, determine the locations in $\theta$ and the peak intensity of the sidelobes, expressed as a fraction of the main lobe intensity. Express your result as the sidelobe level in decibels, given by $S_s[\text{dB}] = 10\log_{10}(S_{r,main}/S_{r,sidelobe})$. Again, use Fig. 14.8 as a guide.

Here, $k\ell = (2\pi/\lambda)(1.3\lambda/2) = 1.3\pi$. The angular intensity distribution is given by the square of the pattern function, Eq. (59), with $k\ell = 1.3\pi$ substituted:

$$|<\mathbf{S}>| \propto [F(\theta)]^2 = \left[\frac{\cos(1.3\pi\cos\theta) - \cos(1.3\pi)}{\sin\theta}\right]^2$$

To find the maxima, the easiest way is to create a polar plot of the function, as shown in Fig. 14.8, and then numerically determine the relative intensities of the peaks by searching for local maxima at locations indicated in the plot. The main lobe occurs at $\theta = 90°$, and has value $[F(\theta)]^2 = 2.52$. The secondary maxima occur at $\theta = 33.8°$ and $146.2°$, and both peaks have values $[F(\theta)]^2 = 0.47$. The sidelobe level in decibels is thus

$$S_s[dB] = 10\log_{10}\left(\frac{2.52}{0.47}\right) = \underline{7.3\text{ dB}}$$

**14.16.** For a dipole antenna of overall length, $2\ell = 1.5\lambda$:

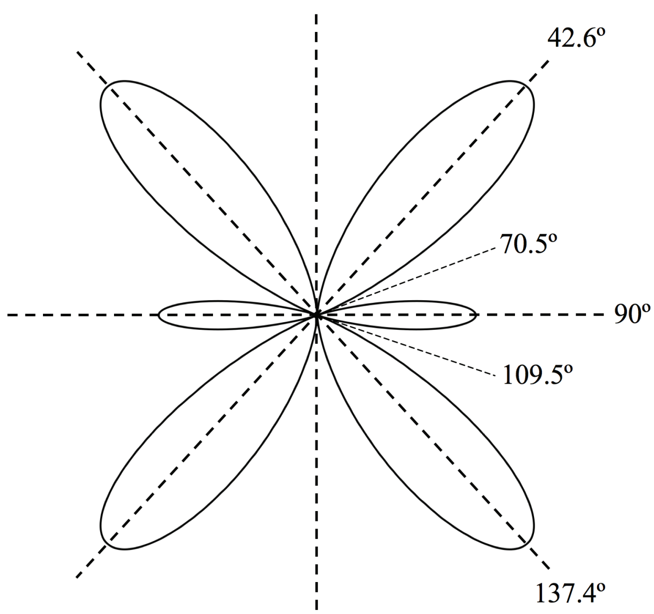
a) Evaluate the locations in $\theta$ at which the zeros and maxima in the E-plane pattern occur: In this case, $k\ell = (2\pi/\lambda)(1.5\lambda/2) = 1.5\pi$. The angular intensity distribution is given by the square of the pattern function, Eq. (59), with $k\ell = 1.3\pi$ substituted:

$$|<\mathbf{S}>| \propto [F(\theta)]^2 = \left[\frac{\cos(1.5\pi\cos\theta) - \cos(1.5\pi)}{\sin\theta}\right]^2 = \left[\frac{\cos(1.5\pi\cos\theta)}{\sin\theta}\right]^2$$

Zeros occur whenever $\cos(1.5\pi\cos\theta) = 0$, which in turn occurs whenever $1.5\pi\cos\theta$ is an odd multiple of $\pi/2$ (positive or negative). All zeros over the range $0 \le \theta \le \pi$ are found with just the first two odd multiples, or

$$\cos\theta_z = \pm\frac{1}{3},\ \pm 1 \quad \Rightarrow \quad \underline{\theta_z = (0, 70.5^\circ, 109.5^\circ, 180^\circ)}$$

The function does indeed zero at $\theta = 0$ and $180^\circ$, despite the presence of the $\sin\theta$ term in the denominator, as can be demonstrated using small argument approximations in the trig functions (show this!). The maxima are found numerically, as was done in Problem 14.15. The results are $\theta_{max,m} = (42.6^\circ, 137.4^\circ)$ (main lobes) and $\theta_{max,s} = 90^\circ$ (sidelobe). The pattern in plotted below.




b) Determine the sidelobe level, as per the definition in Problem 14.15: At the main lobe angle ($\theta = 42.6^\circ$), $[F(\theta)]^2$ evaluates as 1.96. The sidelobe at $\theta = 90^\circ$ gives a value of 1.00. The sidelobe level in decibles is thus

$$S_s[dB] = 10\log_{10}\left(\frac{1.96}{1.00}\right) = \underline{2.9\text{ dB}}$$

c) Determine the maximum directivity: This is found evaluating Eq. (64) numerically (which I did using a Mathematica code), and the result is $D_{max} = 2.23$, or in decibels this becomes $10\log_{10}(2.23) = \underline{3.5\text{ dB}}$.

**14.17.** Consider a lossless half-wave dipole in free space, with radiation resistance, $R_{rad} = 73$ ohms, and maximum directivity $D_{max} = 1.64$. If the antenna carries a 1-A current amplitude,

a) How much total power (in watts) is radiated? We use the definition of radiation resistance, Eq. (29), and write the radiated power:

$$P_r = \frac{1}{2}I_0^2 R_{rad} = \frac{1}{2}(1)^2(73) = \underline{36.5 \text{ W}}$$

b) How much power is intercepted by a 1 $m^2$ aperture situated at distance $r = 1$ km away. The aperture is on the equatorial plane and squarely faces the antenna. Assume uniform power density over the aperture: The maximum directivity is given by $D_{max} = 4\pi K_{max}/P_r$, where the maximum radiation intensity, $K_{max}$, is related to the power density in W/$m^2$ at distance $r$ from the antenna as $S_r = K_{max}/r^2$. The power intercepted by the aperture is then the power density times the aperture area, or

$$P_{rec} = S_r \times Area = \frac{D_{max}P_r}{4\pi r^2} \times Area = \frac{1.64(36.5)}{4\pi(10^3)^2}(1)^2 = \underline{4.77 \; \mu\text{W}}$$

**14.18.** Repeat Problem 14.17, but with a full-wave antenna ($2\ell = \lambda$). Numerical integrals may be necessary. In this case, $k\ell = \pi$, and the radiation resistance is found by applying Eq. (65):

$$R_{rad} = 60 \int_0^{\pi} \left[\frac{\cos(\pi\cos\theta) - \cos(\pi)}{\sin\theta}\right]^2 \sin\theta \, d\theta = 199 \text{ ohms}$$

where the answer was found by numerical integration. With the same 1-A current amplitude as before, the total radiated power is now

$$P_r = \frac{1}{2}I_0^2 R_{rad} = \frac{1}{2}(1)^2(199) = \underline{99.5 \text{ W}}$$

The maximum directivity is the result of Problem 14.14 (involving a numerical integration in Eq. (64) with $k\ell = \pi$), and the result was found to be $D_{max} = 2.41$. The received power is now found as in Problem 14.17:

$$P_{rec} = \frac{D_{max}P_r}{4\pi r^2} \times Area = \frac{2.41(99.5)}{4\pi(10^3)^2}(1)^2 = \underline{19.1 \; \mu\text{W}}$$

**14.19.** Design a two-element dipole array that will radiate equal intensities in the $\phi = 0, \pi/2, \pi$, and $3\pi/2$ directions in the H-plane. Specify the smallest relative current phasing, $\xi$, and the smallest element spacing, $d$.

The array function is given by Eq. (81), with $n = 2$:

$$|A_2(\psi)| = \frac{1}{2}\left|\frac{\sin(\psi)}{\sin(\psi/2)}\right|$$

This has periodic maxima occurring at $\psi = 0, \pm 2m\pi$, and in the H-plane, $\psi = \xi + kd\cos\phi$. Now, for broadside operation, we have maxima at $\phi = \pm\pi/2$, at which $\psi = \xi$, so we set $\underline{\xi = 0}$ to get the array function principal maximum at $\psi = 0$. Having done this, we now have $\psi = kd\cos\phi$, which we need to have equal to $\pm 2\pi$ when $\phi = 0, \pi$, in order to get endfire operation. This will happen when $kd = 2\pi d/\lambda = 2\pi$, so we set $\underline{d = \lambda}$.

**14.20.** A two-element dipole array is configured to provide zero radiation in the broadside ($\phi = \pm 90°$) and endfire ($\phi = 0, 180°$) directions, but with maxima occurring at angles in between. Consider such a set-up with $\psi = \pi$ at $\phi = 0$ and $\psi = -3\pi$ at $\phi = \pi$, with both values determined in the H-plane.

a) Verify that these values give zero broadside and endfire radiation: For two elements, the array function is given by Eq. (81), with $n = 2$:

$$|A_2(\psi)| = \frac{1}{2}\left|\frac{\sin(\psi)}{\sin(\psi/2)}\right|$$

This is clearly zero at $\psi = \pi$ and at $\psi = -3\pi$.

b) Determine the required relative current phase, $\xi$: We know that $\psi = \xi + kd\cos\phi$ in the H-plane. Setting up the two given conditions, we have:

$$\pi = \xi + kd\cos(0) = \xi + kd$$

and

$$-3\pi = \xi + kd\cos(\pi) = \xi - kd$$

Adding these two equations gives $\underline{\xi = -\pi}$, which means that the contributions from the two elements will in fact completely cancel in the broadside direction, regardless of the element spacing.

c) Determine the required element spacing, $d$: We can now use the first condition, for example, substituting values that we know:

$$\psi(\phi = 0) = \pi = -\pi + kd\cos(0) = -\pi + kd$$

Therefore, $kd = 2\pi/\lambda = 2\pi$, or $\underline{d = \lambda}$.

d) Determine the values of $\phi$ at which maxima in the radiation pattern occur: With the values as found, the array function now becomes

$$|A_2(\phi)| = \frac{1}{2}\left|\frac{\sin[\pi(2\cos\phi - 1)]}{\sin[\frac{\pi}{2}(2\cos\phi - 1)]}\right|$$

This function maximizes at $\phi = 60°, 120°, 240°,$ and $300°$, as found numerically.

**14.21.** In the two-element endfire array of Example 14.4, consider the effect of varying the operating frequency, $f$, away from the original design frequency, $f_0$, while maintaining the original current phasing, $\xi = -\pi/2$. Determine the values of $\phi$ at which the maxima occur when the frequency is changed to

a) $f = 1.5f_0$: The original phase and length from the example are $\xi = -\pi/2$ and $d = \lambda_0/4$, where $\lambda_0 = c/f_0$. Thus $kd = (2\pi/\lambda_0)(\lambda_0/4) = \pi/2$. The H-plane array function in that case was found to be

$$A(\pi/2,\phi)\Big|_{\lambda_0} = \cos\left[\frac{\psi}{2}\right] = \cos\left[\frac{\xi}{2} + \frac{kd}{2}\cos\phi\right] = \cos\left[-\frac{\pi}{4} + \frac{\pi}{4}\cos\phi\right]$$

Now, if we change the frequency to $1.5f_0$, while leaving $\xi$ fixed (and the antenna physical length is unchanged), we obtain $\lambda = \lambda_0/1.5$, and $kd = (2\pi/\lambda_0)(1.5)(\lambda_0/4) = 3\pi/4$. The new array function is then:

$$A(\pi/2,\phi)\Big|_{\lambda_0/1.5} = \cos\left[\frac{\xi}{2} + \frac{kd}{2}\cos\phi\right] = \cos\left[-\frac{\pi}{4} + \frac{3\pi}{8}\cos\phi\right] = \cos\left[\frac{\pi}{8}(3\cos\phi - 2)\right]$$

This function amplitude maximizes when $3\cos\phi - 2 = 0$, or $\phi = \cos^{-1}(2/3) = \underline{\pm 48.2^\circ}$.

b) $f = 2f_0$: Now, $\lambda = \lambda_0/2$ and $kd = (2\pi/\lambda_0)(2)(\lambda_0/4) = \pi$. The array function becomes

$$A(\pi/2,\phi)\Big|_{\lambda_0/2} = \cos\left[\frac{\xi}{2} + \frac{kd}{2}\cos\phi\right] = \cos\left[-\frac{\pi}{4} + \frac{\pi}{2}\cos\phi\right] = \cos\left[\frac{\pi}{4}(2\cos\phi - 1)\right]$$

This function amplitude maximizes when $2\cos\phi - 1 = 0$, or $\phi = \cos^{-1}(1/2) = \underline{\pm 60.0^\circ}$.

**14.22.** Revisit Problem 14.21, but with the current phase allowed to vary with frequency (this will automatically occur if the phase difference is established by a simple time delay between the feed currents). Now, the current phase difference will be $\xi' = \xi\, f/f_0$, where $f_0$ is the original (design) frequency. Under this condition, radiation will maximize in the $\phi = 0$ direction regardless of frequency (show this). Backward radiation (along $\phi = \pi$) will develop however as the frequency is tuned away from $f_0$. Derive an expression for the *front-to-back ratio*, defined as the ratio of the radiation intensities at $\phi = 0$ and $\phi = \pi$, expressed in decibels. Express this result as a function of the frequency ratio $f/f_0$.

With the current phase and wavenumber both changing with frequency, we would have

$$\psi' = \left(\frac{f}{f_0}\right)\xi + \left(\frac{f}{f_0}\right) kd\cos\phi$$

With $\xi = -\pi/2$ and $d = \lambda_0/4$, The H-plane array function at the original frequency, $f_0$, is

$$A(\pi/2,\phi)\Big|_{f_0} = \cos\left[\frac{\psi}{2}\right] = \cos\left[\frac{\xi}{2} + \frac{kd}{2}\cos\phi\right] = \cos\left[\frac{\pi}{4}(\cos\phi - 1)\right]$$

The effect on this of changing the frequency is then

$$A(\pi/2,\phi)\Big|_{f} = \cos\left[\frac{\psi'}{2}\right] = \cos\left[\frac{f}{f_0}\left(\frac{\xi}{2} + \frac{kd}{2}\cos\phi\right)\right] = \cos\left[\left(\frac{f}{f_0}\right)\frac{\pi}{4}(\cos\phi - 1)\right]$$

The condition for the zero argument (which maximizes $A$) is $(\cos\phi - 1) = 0$ This term is unaffected by changing the frequency, and so the array function will always maximize in the $\phi = 0$ direction.

In the forward direction ($\phi = 0$), the value of $|A|^2$ (proportional to the radiation intensity) is unity. In the backward direction ($\phi = 180°$), we find

$$|A(\phi = \pi)|^2 = \cos^2\left(-\frac{f}{f_0}\frac{\pi}{2}\right) = \cos^2\left(\frac{\pi f}{2f_0}\right)$$

The front-to-back ratio is then

$$R_{fb} = 10\log_{10}\left[\frac{|A(\phi = 0)|^2}{|A(\phi = \pi)|^2}\right] = 10\log_{10}\left[\frac{1}{\cos^2(\pi f/2f_0)}\right]$$

Evaluate the front-to-back ratio for a) $f = 1.5f_0$, b) $f = 2f_0$, c) $f = 0.75f_0$. Substituting these values, we find:

$$f = 1.5f_0: \quad R_{fb} = 10\log_{10}\left[\frac{1}{\cos^2(1.5\pi/2)}\right] = \underline{3\,\text{dB}}$$

$$f = 2f_0: \quad R_{fb} = 10\log_{10}\left[\frac{1}{\cos^2(\pi)}\right] = \underline{0\,\text{dB}}$$

$$f = 0.75f_0: \quad R_{fb} = 10\log_{10}\left[\frac{1}{\cos^2(\pi/2.67)}\right] = \underline{8.3\,\text{dB}}$$

**14.23.** A *turnstile* antenna consists of two crossed dipole antennas, positioned in this case in the $xy$ plane. The dipoles are identical, lie along the $x$ and $y$ axes, and are both fed at the origin. Assume that equal currents are supplied to each antenna, and that a zero phase reference is applied to the $x$-directed antenna. determine the relative phase, $\xi$, of the $y$-directed antenna so that the net radiated electric field as measured on the positive $z$ axis is a) left circularly-polarized; b) linearly polarized along the 45° axis between $x$ and $y$.

When looking at the field along the $z$ axis, the expression for $\mathbf{E}$ can be constructed using Eq. (57), evaluated at $\theta = \pi/2$ (so that we consider the direction normal to the antenna), and in which $r$ is replaced by $z$. An additional "array" term includes the two polarization directions (unit vectors) which are out of phase by $\xi$:

$$\mathbf{E}(z, \theta = \pi/2) = J\frac{I_0\eta}{2\pi z}[1 - \cos(k\ell)]\, e^{-jkz}\left[\mathbf{a}_x + \mathbf{a}_y e^{j\xi}\right]$$

a) To achieve left circular polarization for propagation in the forward $z$ direction, the vector array function must be

$$\left[\mathbf{a}_x + \mathbf{a}_y e^{j\xi}\right] = [\mathbf{a}_x + j\mathbf{a}_y] \quad \Rightarrow \quad \underline{\xi = \pi/2}$$

b) To achieve 45° polarization, the vector array function must be

$$\left[\mathbf{a}_x + \mathbf{a}_y e^{j\xi}\right] = [\mathbf{a}_x \pm \mathbf{a}_y] \quad \Rightarrow \quad \underline{\xi = 0, \pi}$$

**14.24.** Consider a linear endfire array, designed for maximum radiation intensity at $\phi = 0$, using $\xi$ and $d$ values as suggested in Example 14.5. Determine an expression for the front-to-back ratio (defined in Problem 14.22) as a function of the number of elements, $n$, if $n$ is an odd number.

From Example 14.5, we use $\xi = -\pi/2$ and $d = \lambda/4$, so that $kd = \pi/2$. Then in the H-plane

$$\psi = \xi + kd\cos\phi = \frac{\pi}{2}(\cos\phi - 1)$$

The array function is then:

$$A(\psi) = \frac{1}{n}\left[\frac{\sin(n\psi/2)}{\sin(\psi/2)}\right] = \frac{1}{n}\left[\frac{\sin[(n\pi/4)(\cos\phi - 1)]}{\sin[(\pi/4)(\cos\phi - 1)]}\right]$$

In the forward direction ($\phi = 0$), $A = 1$. In the backward direction ($\phi = \pi$), $A = \pm 1/n$, where $n$ is an odd integer. The front-to-back ratio is therefore:

$$R_{fb} = 10\log_{10}\left(\frac{|A(\phi = 0)|^2}{|A(\phi = \pi)|^2}\right) = \underline{10\log_{10}(n^2)\ \text{dB}}$$

in which performance is clearly better as $n$ increases.

**14.25.** A six-element linear dipole array has element spacing $d = \lambda/2$.

a) Select the appropriate current phasing, $\xi$, to achieve maximum radiation along $\phi = \pm 60°$: With $d = \lambda/2$, we have $kd = \pi$ and in the H-plane,

$$\psi = \xi + kd\cos\phi = \xi + \pi\cos\phi$$

We set the principal maximum to occur at $\psi = 0$, and with $\phi = 60°$, the condition for this becomes

$$\psi = 0 = \xi + \pi\cos(60°) = \xi + \frac{\pi}{2} \quad \Rightarrow \quad \underline{\xi = -\frac{\pi}{2}}$$

b) With the phase set as in part *a*, evaluate the intensities (relative to the maximum) in the broadside and endfire directions. With the above result, the array function (Eq. (81)) in the H-plane becomes

$$A(-\pi/2, \phi) = \frac{1}{6}\left[\frac{\sin[3(\xi + kd\cos\phi)]}{\sin[(1/2)(\xi + kd\cos\phi)]}\right] = \frac{1}{6}\left[\frac{\sin[(3\pi/2)(2\cos\phi - 1)]}{\sin[(\pi/4)(2\cos\phi - 1)]}\right]$$

In the broadside direction ($\phi = 90°$) the array function becomes

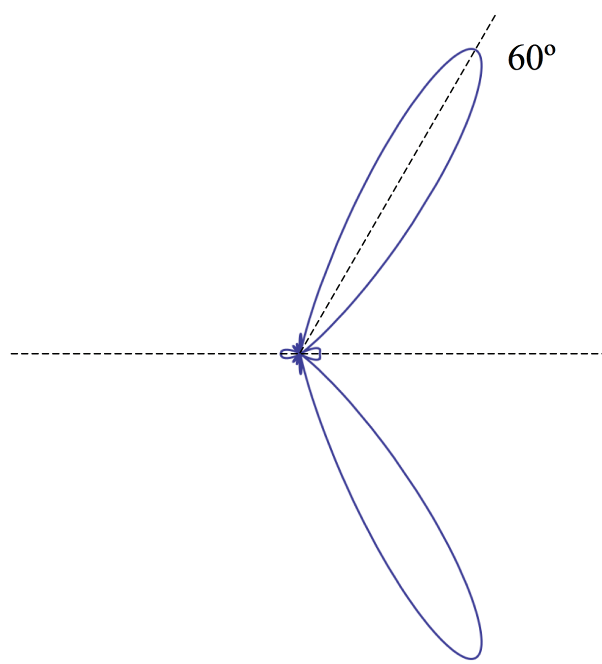
$$A(-\pi/2, \pi/2) = \frac{1}{6}\left[\frac{\sin[-3\pi/2]}{\sin[-\pi/4]}\right] = \frac{\sqrt{2}}{6}$$

The intensity in this direction (relative to that along $\phi = 60°$) is the square of this or $\underline{1/18}$

In the endfire direction ($\phi = 0°$) the array function becomes

$$A(-\pi/2, 0) = \frac{1}{6}\left[\frac{\sin[3\pi/2]}{\sin[\pi/4]}\right] = \frac{\sqrt{2}}{6}$$

The intensity in this direction (relative to that along $\phi = 60°$) is the square of this or $\underline{1/18}$, as before. The same result is found for $\phi = 180°$ and for $\phi = -90°$. The intensity pattern is shown below:



**14.26.** In a linear endfire array of $n$ elements, a choice of current phasing that improves the directivity is given by the Hansen-Woodyard condition:

$$\xi = \pm \left( \frac{2\pi d}{\lambda} + \frac{\pi}{n} \right)$$

where the plus or minus sign choices give maximum radiation along $\phi = 180°$ and $0°$ respectively. Applying this phasing may not necessarily lead to unidirectional endfire operation (zero backward radiation), but will do so with the proper choice of element spacing, $d$.

a) Determine this required spacing as a function of $n$ and $\lambda$: We first construct the expression for $\psi$ in the H-plane, using the given current phase expression, and in which the minus sign is chosen:

$$\psi = \xi + kd\cos\phi = -\frac{2\pi d}{\lambda} - \frac{\pi}{n} + \frac{2\pi d}{\lambda}\cos\phi$$

Note that $\psi(\phi = 0) = -\pi/n$. The array function, Eq. (81) is:

$$A(\xi, \phi) = \frac{1}{n}\left[\frac{\sin(n\psi/2)}{\sin(\psi/2)}\right] = \frac{1}{n}\left[\frac{\sin[(n\pi d/\lambda)(\cos\phi - 1) - \pi/2]}{\sin[(\pi d/\lambda)(\cos\phi - 1) - (\pi/2n)]}\right]$$

This function is equal to 1 when $\phi = 0$, regardless of the choice of $d$. When $\phi = \pi$, the array function becomes

$$A(\xi, \pi) = \frac{1}{n}\left[\frac{\sin[(-2n\pi d/\lambda) - (\pi/2)]}{\sin[(-2\pi d/\lambda) - (\pi/2n)]}\right]$$

which we require to be zero. This will happen when $\psi(\phi = \pi) = -\pi$. Using the above formula for $\psi$, we set up the condition:

$$\psi(\phi = \pi) = -\frac{4\pi d}{\lambda} - \frac{\pi}{n} = -\pi \quad \Rightarrow \quad \underline{d = \left(\frac{n-1}{n}\right)\frac{\lambda}{4}}$$

b) Show that the spacing as found in part $a$ approaches $\lambda/4$ for a large number of elements: It is readily seen from the part $a$ result that when $n \rightarrow \infty$, $d \rightarrow \lambda/4$.

c) Show that an even number of elements is required: This can be shown by substituting $d$ as found in part $a$ into the array function at $\phi = \pi$. The result is

$$A(\xi, \pi) = \frac{1}{n}\left[\frac{\sin(-n\pi/2)}{\sin(-\pi/2)}\right]$$

We require this result to be zero, which will only happen if $n$ is an even integer. This can be more quickly seen from the general expression:

$$A(\xi, \phi) = \frac{1}{n}\left[\frac{\sin(n\psi/2)}{\sin(\psi/2)}\right]$$

wherein if $\psi = -\pi$, we will get a zero result only if $n$ is even.

**14.27.** Consider an $n$-element broadside linear array. Increasing the number of elements has the effect of narrowing the main beam. Demonstrate this by evaluating the separation in $\phi$ between the zeros on either side of the principal maximum at $\phi = 90°$. Show that for large $n$ this separation is approximated by $\Delta\phi \doteq 2\lambda/L$, where $L \doteq nd$ is the overall length of the array.

For broadside operation, $\xi = 0$, and, with $kd = 2\pi d/\lambda$, we have

$$\psi = \xi + kd\cos\phi = \frac{2\pi d}{\lambda}\cos\phi$$

With this condition, the array function, Eq. (81) is:

$$A(\xi,\phi) = \frac{1}{n}\left[\frac{\sin(n\psi/2)}{\sin(\psi/2)}\right] = \frac{1}{n}\left[\frac{\sin[(n\pi d/\lambda)\cos\phi]}{\sin[(\pi d/\lambda)\cos\phi]}\right]$$

Zeros in this function will occur whenever

$$\frac{n\pi d}{\lambda}\cos\phi = \pm m\pi$$

where $m$ is an integer. Now, on either side of the principal maximum at $\psi = 0$, zeros will occur at $+\pi$ and $-\pi$ (giving a $2\pi$ separation), which we can associate with the two angles $\phi^+$ and $\phi^-$ respectively, which lie on either side of 90°. We can therefore write:

$$\frac{n\pi d}{\lambda}\left(\cos\phi^+ - \cos\phi^-\right) = 2\pi$$

Using a trig identity, this becomes:

$$\frac{2n\pi d}{\lambda}\underbrace{\sin\left[\frac{1}{2}(\phi^+ + \phi^-)\right]}_{1}\sin\left[\frac{1}{2}(\phi^- - \phi^+)\right] = 2\pi$$

Because we are considering the broadside direction, we have $(\phi^+ + \phi^-)/2 = \pi/2$ and so the first sine term is just unity. We find

$$\sin\left[\frac{1}{2}(\phi^- - \phi^+)\right] = \frac{\lambda}{nd}$$

Now as $n$ gets large, both sides of the equation become $<< 1$, so that we may approximate the sine term by just its argument:

$$\frac{1}{2}(\phi^- - \phi^+) \doteq \frac{\lambda}{nd}$$

Defining $\Delta\phi = \phi^- - \phi^+$, the above expression becomes

$$\Delta\phi \doteq \frac{2\lambda}{nd} \doteq \frac{2\lambda}{L} \quad \text{(done)}$$

**14.28.** A large ground-based transmitter radiates 10kW, and communicates with a mobile receiving station that dissipates 1mW on the matched load of its antenna. The receiver (not having moved) now transmits back to the ground station. If the mobile unit radiates 100W, what power is received (at a matched load) by the ground station?

We can use Eq. (93) and the fact that $Z_{21} = Z_{12}$ to write:

$$\frac{P_{L2}}{P_{r1}} = \frac{|Z_{21}|^2}{R_{11}R_{22}} = \frac{|Z_{12}|^2}{R_{11}R_{22}} = \frac{P_{L1}}{P_{r2}}$$

We have

$$\frac{P_{L2}}{P_{r1}} = \frac{10^{-3}}{10^4} = 10^{-7} = \frac{P_{L1}}{P_{r2}}$$

So

$$P_{L1} = P_{r2} \times 10^{-7} = 100 \times 10^{-7} = 10^{-5} = \underline{10\,\mu\text{W}}$$

**14.29.** Signals are transmitted at a 1m carrier wavelength between two identical half-wave dipole antennas spaced by 1km. The antennas are oriented such that they are exactly parallel to each other.

a) If the transmitting antenna radiates 100 watts, how much power is dissipated by a matched load at the receiving antenna? To find the dissipated power, use Eq. (106):

$$P_{L2} = \frac{P_{r1}\lambda^2}{(4\pi r)^2}\, D_1(\theta_1\phi_1)\, D_2(\theta_2\phi_2)$$

Since the antennas face each other squarely, the directivities are both the maximum values for the two, where for the half-wave dipole, we know that $D_{max} = 1.64$ So

$$P_{L2} = \frac{100(1)^2}{(4\pi \times 10^3)^2}\,(1.64)^2 = \underline{1.7\ \mu\text{W}}$$

b) Suppose the receiving antenna is tilted by 45° while the two antennas remain in the same plane. What is the received power in this case? With the receiving antenna tilted, its directivity is reduced accordingly. Using Eq. (63), with the help of the results of Example 14.2, we have

$$D_2(\theta_2 = 45^\circ) = D_{max} \times |F(45^\circ)|^2 = 1.64 \times \left|\frac{\cos[(\pi/2)\cos(45^\circ)]}{\sin(45^\circ)}\right|^2 = 0.643$$

So now,

$$P_{L2}(45^\circ) = \frac{100(1)^2}{(4\pi \times 10^3)^2}\,(1.64)(0.643) = \underline{672\ \text{nW}}$$

**14.30.** A half-wave dipole antenna is known to have a maximum effective area, given as $A_{max}$.

a) Write the maximum directivity of this antenna in terms of $A_{max}$ and wavelength $\lambda$: From Eq. (104), it follows that

$$D_{max} = \frac{4\pi}{\lambda^2} A_{max}$$

b) Express the current amplitude, $I_0$, needed to radiate total power, $P_r$, in terms of $P_r$, $A_{max}$, and $\lambda$: Combining Eqs. (64) and (65), we can write

$$\frac{2P_r}{I_0^2} = \frac{120[F(\theta)]^2_{max}}{D_{max}}$$

where, for a half-wave dipole, $[F(\theta)]^2_{max} = 1$. Then, using the part $a$ result, we find

$$I_0 = \sqrt{\frac{\pi P_r A_{max}}{15\lambda^2}}$$

c) At what values of $\theta$ and $\phi$ will the antenna effective area be equal to $A_{max}$? These will be the same angles over which the half-wave dipole directivity maximizes, or at which $F(\theta)$ for the antenna reaches its maximum. We know from Example 14.2 that this happens at $\theta = \pi/2$, and at all values of $\phi$.