CHAPTER 1

1.1. Given the vectors $\mathbf{M} = -10\mathbf{a}_x + 4\mathbf{a}_y - 8\mathbf{a}_z$ and $\mathbf{N} = 8\mathbf{a}_x + 7\mathbf{a}_y - 2\mathbf{a}_z$, find:

a) a unit vector in the direction of $-\mathbf{M} + 2\mathbf{N}$.

$-\mathbf{M} + 2\mathbf{N} = 10\mathbf{a}_x - 4\mathbf{a}_y + 8\mathbf{a}_z + 16\mathbf{a}_x + 14\mathbf{a}_y - 4\mathbf{a}_z = (26, 10, 4)$

Thus

$$\mathbf{a} = \frac{(26, 10, 4)}{|(26, 10, 4)|} = \underline{(0.92, 0.36, 0.14)}$$

b) the magnitude of $5\mathbf{a}_x + \mathbf{N} - 3\mathbf{M}$:

$(5, 0, 0) + (8, 7, -2) - (-30, 12, -24) = (43, -5, 22)$, and $|(43, -5, 22)| = \underline{48.6}$.

c) $|\mathbf{M}||2\mathbf{N}|(\mathbf{M} + \mathbf{N})$:

$|(-10, 4, -8)||(16, 14, -4)|(-2, 11, -10) = (13.4)(21.6)(-2, 11, -10)$

$= \underline{(-580.5, 3193, -2902)}$

1.2. Vector $\mathbf{A}$ extends from the origin to (1,2,3) and vector $\mathbf{B}$ from the origin to (2,3,-2).

a) Find the unit vector in the direction of $(\mathbf{A} - \mathbf{B})$: First

$$\mathbf{A} - \mathbf{B} = (\mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z) - (2\mathbf{a}_x + 3\mathbf{a}_y - 2\mathbf{a}_z) = (-\mathbf{a}_x - \mathbf{a}_y + 5\mathbf{a}_z)$$

whose magnitude is $|\mathbf{A} - \mathbf{B}| = [(-\mathbf{a}_x - \mathbf{a}_y + 5\mathbf{a}_z) \cdot (-\mathbf{a}_x - \mathbf{a}_y + 5\mathbf{a}_z)]^{1/2} = \sqrt{1 + 1 + 25} = 3\sqrt{3} = 5.20$. The unit vector is therefore

$$\mathbf{a}_{AB} = \underline{(-\mathbf{a}_x - \mathbf{a}_y + 5\mathbf{a}_z)/5.20}$$

b) find the unit vector in the direction of the line extending from the origin to the midpoint of the line joining the ends of $\mathbf{A}$ and $\mathbf{B}$:

The midpoint is located at

$$P_{mp} = [1 + (2 - 1)/2,\ 2 + (3 - 2)/2,\ 3 + (-2 - 3)/2)] = (1.5, 2.5, 0.5)$$

The unit vector is then

$$\mathbf{a}_{mp} = \frac{(1.5\mathbf{a}_x + 2.5\mathbf{a}_y + 0.5\mathbf{a}_z)}{\sqrt{(1.5)^2 + (2.5)^2 + (0.5)^2}} = \underline{(1.5\mathbf{a}_x + 2.5\mathbf{a}_y + 0.5\mathbf{a}_z)/2.96}$$

1.3. The vector from the origin to the point A is given as $(6, -2, -4)$, and the unit vector directed from the origin toward point B is $(2, -2, 1)/3$. If points A and B are ten units apart, find the coordinates of point B.

With $\mathbf{A} = (6, -2, -4)$ and $\mathbf{B} = \frac{1}{3}B(2, -2, 1)$, we use the fact that $|\mathbf{B} - \mathbf{A}| = 10$, or

$|(6 - \frac{2}{3}B)\mathbf{a}_x - (2 - \frac{2}{3}B)\mathbf{a}_y - (4 + \frac{1}{3}B)\mathbf{a}_z| = 10$

Expanding, obtain

$36 - 8B + \frac{4}{9}B^2 + 4 - \frac{8}{3}B + \frac{4}{9}B^2 + 16 + \frac{8}{3}B + \frac{1}{9}B^2 = 100$

or $B^2 - 8B - 44 = 0$. Thus $B = \frac{8 \pm \sqrt{64 - 176}}{2} = 11.75$ (taking positive option) and so

$$\mathbf{B} = \frac{2}{3}(11.75)\mathbf{a}_x - \frac{2}{3}(11.75)\mathbf{a}_y + \frac{1}{3}(11.75)\mathbf{a}_z = \underline{7.83\mathbf{a}_x - 7.83\mathbf{a}_y + 3.92\mathbf{a}_z}$$

1.4. A circle, centered at the origin with a radius of 2 units, lies in the xy plane. Determine the unit vector in rectangular components that lies in the xy plane, is tangent to the circle at $(-\sqrt{3}, 1, 0)$, and is in the general direction of increasing values of y:

A unit vector tangent to this circle in the general increasing y direction is $\mathbf{t} = -\mathbf{a}_\phi$. Its x and y components are $\mathbf{t}_x = -\mathbf{a}_\phi \cdot \mathbf{a}_x = \sin\phi$, and $\mathbf{t}_y = -\mathbf{a}_\phi \cdot \mathbf{a}_y = -\cos\phi$. At the point $(-\sqrt{3}, 1)$, $\phi = 150°$, and so $\mathbf{t} = \sin 150° \mathbf{a}_x - \cos 150° \mathbf{a}_y = \underline{0.5(\mathbf{a}_x + \sqrt{3}\mathbf{a}_y)}$.

1.5. A vector field is specified as $\mathbf{G} = 24xy\mathbf{a}_x + 12(x^2+2)\mathbf{a}_y + 18z^2\mathbf{a}_z$. Given two points, $P(1, 2, -1)$ and $Q(-2, 1, 3)$, find:

a) $\mathbf{G}$ at P: $\mathbf{G}(1, 2, -1) = \underline{(48, 36, 18)}$

b) a unit vector in the direction of $\mathbf{G}$ at Q: $\mathbf{G}(-2, 1, 3) = (-48, 72, 162)$, so

$$\mathbf{a}_G = \frac{(-48, 72, 162)}{|(-48, 72, 162)|} = \underline{(-0.26, 0.39, 0.88)}$$

c) a unit vector directed from Q toward P:

$$\mathbf{a}_{QP} = \frac{\mathbf{P} - \mathbf{Q}}{|\mathbf{P} - \mathbf{Q}|} = \frac{(3, -1, 4)}{\sqrt{26}} = \underline{(0.59, 0.20, -0.78)}$$

d) the equation of the surface on which $|\mathbf{G}| = 60$: We write $60 = |(24xy, 12(x^2+2), 18z^2)|$, or $10 = |(4xy, 2x^2+4, 3z^2)|$, so the equation is

$$\underline{100 = 16x^2y^2 + 4x^4 + 16x^2 + 16 + 9z^4}$$

1.6. Find the acute angle between the two vectors $\mathbf{A} = 2\mathbf{a}_x + \mathbf{a}_y + 3\mathbf{a}_z$ and $\mathbf{B} = \mathbf{a}_x - 3\mathbf{a}_y + 2\mathbf{a}_z$ by using the definition of:

a) the dot product: First, $\mathbf{A} \cdot \mathbf{B} = 2 - 3 + 6 = 5 = AB\cos\theta$, where $A = \sqrt{2^2 + 1^2 + 3^2} = \sqrt{14}$, and where $B = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$. Therefore $\cos\theta = 5/14$, so that $\theta = \underline{69.1°}$.

b) the cross product: Begin with

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & 1 & 3 \\ 1 & -3 & 2 \end{vmatrix} = 11\mathbf{a}_x - \mathbf{a}_y - 7\mathbf{a}_z$$

and then $|\mathbf{A} \times \mathbf{B}| = \sqrt{11^2 + 1^2 + 7^2} = \sqrt{171}$. So now, with $|\mathbf{A} \times \mathbf{B}| = AB\sin\theta = \sqrt{171}$, find $\theta = \sin^{-1}\left(\sqrt{171}/14\right) = \underline{69.1°}$

1.7. Given the vector field $\mathbf{E} = 4zy^2\cos 2x\mathbf{a}_x + 2zy\sin 2x\mathbf{a}_y + y^2\sin 2x\mathbf{a}_z$ for the region $|x|$, $|y|$, and $|z|$ less than 2, find:

a) the surfaces on which $E_y = 0$. With $E_y = 2zy\sin 2x = 0$, the surfaces are 1) the plane $\underline{z = 0}$, with $|x| < 2$, $|y| < 2$; 2) the plane $\underline{y = 0}$, with $|x| < 2$, $|z| < 2$; 3) the plane $\underline{x = 0}$, with $|y| < 2$, $|z| < 2$; 4) the plane $\underline{x = \pi/2}$, with $|y| < 2$, $|z| < 2$.

b) the region in which $E_y = E_z$: This occurs when $2zy\sin 2x = y^2\sin 2x$, or on the plane $\underline{2z = y}$, with $|x| < 2$, $|y| < 2$, $|z| < 1$.

c) the region in which $\mathbf{E} = 0$: We would have $E_x = E_y = E_z = 0$, or $zy^2\cos 2x = zy\sin 2x = y^2\sin 2x = 0$. This condition is met on the plane $\underline{y = 0}$, with $|x| < 2$, $|z| < 2$.

1.8. Demonstrate the ambiguity that results when the cross product is used to find the angle between two vectors by finding the angle between $\mathbf{A} = 3\mathbf{a}_x - 2\mathbf{a}_y + 4\mathbf{a}_z$ and $\mathbf{B} = 2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z$. Does this ambiguity exist when the dot product is used?

We use the relation $\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}| \sin\theta \mathbf{n}$. With the given vectors we find

$$\mathbf{A} \times \mathbf{B} = 14\mathbf{a}_y + 7\mathbf{a}_z = 7\sqrt{5} \underbrace{\left[\frac{2\mathbf{a}_y + \mathbf{a}_z}{\sqrt{5}}\right]}_{\pm\mathbf{n}} = \sqrt{9+4+16}\sqrt{4+1+4}\,\sin\theta\,\mathbf{n}$$

where $\mathbf{n}$ is identified as shown; we see that $\mathbf{n}$ can be positive or negative, as $\sin\theta$ can be positive or negative. This apparent sign ambiguity is not the real problem, however, as we really want the magnitude of the angle anyway. Choosing the positive sign, we are left with $\sin\theta = 7\sqrt{5}/(\sqrt{29}\sqrt{9}) = 0.969$. *Two* values of θ ($75.7°$ and $104.3°$) satisfy this equation, and hence the real ambiguity.

In using the dot product, we find $\mathbf{A} \cdot \mathbf{B} = 6 - 2 - 8 = -4 = |\mathbf{A}||\mathbf{B}| \cos\theta = 3\sqrt{29}\cos\theta$, or $\cos\theta = -4/(3\sqrt{29}) = -0.248 \Rightarrow \theta = -75.7°$. Again, the minus sign is not important, as we care only about the angle magnitude. The main point is that *only one* θ value results when using the dot product, so no ambiguity.

1.9. A field is given as

$$\mathbf{G} = \frac{25}{(x^2+y^2)}(x\mathbf{a}_x + y\mathbf{a}_y)$$

Find:

a) a unit vector in the direction of $\mathbf{G}$ at $P(3, 4, -2)$: Have $\mathbf{G}_p = 25/(9+16) \times (3, 4, 0) = 3\mathbf{a}_x + 4\mathbf{a}_y$, and $|\mathbf{G}_p| = 5$. Thus $\mathbf{a}_G = \underline{(0.6, 0.8, 0)}$.

b) the angle between $\mathbf{G}$ and $\mathbf{a}_x$ at P: The angle is found through $\mathbf{a}_G \cdot \mathbf{a}_x = \cos\theta$. So $\cos\theta = (0.6, 0.8, 0) \cdot (1, 0, 0) = 0.6$. Thus $\underline{\theta = 53°}$.

c) the value of the following double integral on the plane $y = 7$:

$$\int_0^4 \int_0^2 \mathbf{G} \cdot \mathbf{a}_y dz dx$$

$$\int_0^4 \int_0^2 \frac{25}{x^2+y^2}(x\mathbf{a}_x + y\mathbf{a}_y) \cdot \mathbf{a}_y dz dx = \int_0^4 \int_0^2 \frac{25}{x^2+49} \times 7\, dz dx = \int_0^4 \frac{350}{x^2+49} dx$$

$$= 350 \times \frac{1}{7}\left[\tan^{-1}\left(\frac{4}{7}\right) - 0\right] = \underline{26}$$

1.10. By expressing diagonals as vectors and using the definition of the dot product, find the smaller angle between any two diagonals of a cube, where each diagonal connects diametrically opposite corners, and passes through the center of the cube:

Assuming a side length, b, two diagonal vectors would be $\mathbf{A} = b(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)$ and $\mathbf{B} = b(\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z)$. Now use $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta$, or $b^2(1 - 1 + 1) = (\sqrt{3}b)(\sqrt{3}b)\cos\theta \Rightarrow \cos\theta = 1/3 \Rightarrow \theta = \underline{70.53°}$. This result (in magnitude) is the same for *any* two diagonal vectors.

1.11. Given the points $M(0.1, -0.2, -0.1)$, $N(-0.2, 0.1, 0.3)$, and $P(0.4, 0, 0.1)$, find:

a) the vector $\mathbf{R}_{MN}$: $\mathbf{R}_{MN} = (-0.2, 0.1, 0.3) - (0.1, -0.2, -0.1) = \underline{(-0.3, 0.3, 0.4)}$.

b) the dot product $\mathbf{R}_{MN} \cdot \mathbf{R}_{MP}$: $\mathbf{R}_{MP} = (0.4, 0, 0.1) - (0.1, -0.2, -0.1) = (0.3, 0.2, 0.2)$. $\mathbf{R}_{MN} \cdot \mathbf{R}_{MP} = (-0.3, 0.3, 0.4) \cdot (0.3, 0.2, 0.2) = -0.09 + 0.06 + 0.08 = \underline{0.05}$.

c) the scalar projection of $\mathbf{R}_{MN}$ on $\mathbf{R}_{MP}$:

$$\mathbf{R}_{MN} \cdot \mathbf{a}_{RMP} = (-0.3, 0.3, 0.4) \cdot \frac{(0.3, 0.2, 0.2)}{\sqrt{0.09 + 0.04 + 0.04}} = \frac{0.05}{\sqrt{0.17}} = \underline{0.12}$$

d) the angle between $\mathbf{R}_{MN}$ and $\mathbf{R}_{MP}$:

$$\theta_M = \cos^{-1}\left(\frac{\mathbf{R}_{MN} \cdot \mathbf{R}_{MP}}{|\mathbf{R}_{MN}||\mathbf{R}_{MP}|}\right) = \cos^{-1}\left(\frac{0.05}{\sqrt{0.34}\sqrt{0.17}}\right) = \underline{78^\circ}$$

1.12. Write an expression in rectangular components for the vector that extends from (x_1, y_1, z_1) to (x_2, y_2, z_2) and determine the magnitude of this vector.

The two points can be written as vectors from the origin:

$$\mathbf{A}_1 = x_1\mathbf{a}_x + y_1\mathbf{a}_y + z_1\mathbf{a}_z \quad \text{and} \quad \mathbf{A}_2 = x_2\mathbf{a}_x + y_2\mathbf{a}_y + z_2\mathbf{a}_z$$

The desired vector will now be the difference:

$$\mathbf{A}_{12} = \mathbf{A}_2 - \mathbf{A}_1 = (x_2 - x_1)\mathbf{a}_x + (y_2 - y_1)\mathbf{a}_y + (z_2 - z_1)\mathbf{a}_z$$

whose magnitude is

$$|\mathbf{A}_{12}| = \sqrt{\mathbf{A}_{12} \cdot \mathbf{A}_{12}} = \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{1/2}$$

1.13. a) Find the vector component of $\mathbf{F} = (10, -6, 5)$ that is parallel to $\mathbf{G} = (0.1, 0.2, 0.3)$:

$$\mathbf{F}_{||G} = \frac{\mathbf{F} \cdot \mathbf{G}}{|\mathbf{G}|^2}\mathbf{G} = \frac{(10, -6, 5) \cdot (0.1, 0.2, 0.3)}{0.01 + 0.04 + 0.09}(0.1, 0.2, 0.3) = \underline{(0.93, 1.86, 2.79)}$$

b) Find the vector component of $\mathbf{F}$ that is perpendicular to $\mathbf{G}$:

$$\mathbf{F}_{pG} = \mathbf{F} - \mathbf{F}_{||G} = (10, -6, 5) - (0.93, 1.86, 2.79) = \underline{(9.07, -7.86, 2.21)}$$

c) Find the vector component of $\mathbf{G}$ that is perpendicular to $\mathbf{F}$:

$$\mathbf{G}_{pF} = \mathbf{G} - \mathbf{G}_{||F} = \mathbf{G} - \frac{\mathbf{G} \cdot \mathbf{F}}{|\mathbf{F}|^2}\mathbf{F} = (0.1, 0.2, 0.3) - \frac{1.3}{100 + 36 + 25}(10, -6, 5) = \underline{(0.02, 0.25, 0.26)}$$

1.14. Given that $\mathbf{A}+\mathbf{B}+\mathbf{C}=0$, where the three vectors represent line segments and extend from a common origin,

a) must the three vectors be coplanar?

In terms of the components, the vector sum will be

$$\mathbf{A}+\mathbf{B}+\mathbf{C}=(A_x+B_x+C_x)\mathbf{a}_x+(A_y+B_y+C_y)\mathbf{a}_y+(A_z+B_z+C_z)\mathbf{a}_z$$

which we require to be zero. Suppose the coordinate system is configured so that vectors $\mathbf{A}$ and $\mathbf{B}$ lie in the x-y plane; in this case $A_z=B_z=0$. Then C_z has to be zero in order for the three vectors to sum to zero. Therefore, the three vectors must be coplanar.

b) If $\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D}=0$, are the four vectors coplanar?

The vector sum is now

$$\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D}=(A_x+B_x+C_x+D_x)\mathbf{a}_x+(A_y+B_y+C_y+D_y)\mathbf{a}_y+(A_z+B_z+C_z+D_z)\mathbf{a}_z$$

Now, for example, if $\mathbf{A}$ and $\mathbf{B}$ lie in the x-y plane, $\mathbf{C}$ and $\mathbf{D}$ need not, as long as $C_z+D_z=0$. So the four vectors need not be coplanar to have a zero sum.

1.15. Three vectors extending from the origin are given as $\mathbf{r}_1=(7,3,-2)$, $\mathbf{r}_2=(-2,7,-3)$, and $\mathbf{r}_3=(0,2,3)$. Find:

a) a unit vector perpendicular to both $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\mathbf{a}_{p12}=\frac{\mathbf{r}_1\times\mathbf{r}_2}{|\mathbf{r}_1\times\mathbf{r}_2|}=\frac{(5,25,55)}{60.6}=\underline{(0.08,0.41,0.91)}$$

b) a unit vector perpendicular to the vectors $\mathbf{r}_1-\mathbf{r}_2$ and $\mathbf{r}_2-\mathbf{r}_3$: $\mathbf{r}_1-\mathbf{r}_2=(9,-4,1)$ and $\mathbf{r}_2-\mathbf{r}_3=(-2,5,-6)$. So $\mathbf{r}_1-\mathbf{r}_2\times\mathbf{r}_2-\mathbf{r}_3=(19,52,32)$. Then

$$\mathbf{a}_p=\frac{(19,52,32)}{|(19,52,32)|}=\frac{(19,52,32)}{63.95}=\underline{(0.30,0.81,0.50)}$$

c) the area of the triangle defined by $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\text{Area}=\frac{1}{2}|\mathbf{r}_1\times\mathbf{r}_2|=\underline{30.3}$$

d) the area of the triangle defined by the heads of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$:

$$\text{Area}=\frac{1}{2}|(\mathbf{r}_2-\mathbf{r}_1)\times(\mathbf{r}_2-\mathbf{r}_3)|=\frac{1}{2}|(-9,4,-1)\times(-2,5,-6)|=\underline{32.0}$$

1.16. If **A** represents a vector one unit in length directed due east, **B** represents a vector three units in length directed due north, and $\mathbf{A} + \mathbf{B} = 2\mathbf{C} - \mathbf{D}$ and $2\mathbf{A} - \mathbf{B} = \mathbf{C} + 2\mathbf{D}$, determine the length and direction of **C**. (difficulty 1)

Take north as the positive y direction, and then east as the positive x direction. Then we may write

$$\mathbf{A} + \mathbf{B} = \mathbf{a}_x + 3\mathbf{a}_y = 2\mathbf{C} - \mathbf{D}$$

and

$$2\mathbf{A} - \mathbf{B} = 2\mathbf{a}_x - 3\mathbf{a}_y = \mathbf{C} + 2\mathbf{D}$$

Multiplying the first equation by 2, and then adding the result to the second equation eliminates **D**, and we get

$$4\mathbf{a}_x + 3\mathbf{a}_y = 5\mathbf{C} \quad \Rightarrow \quad \mathbf{C} = \frac{4}{5}\mathbf{a}_x + \frac{3}{5}\mathbf{a}_y$$

The length of **C** is $|\mathbf{C}| = \left[(4/5)^2 + (3/5)^2\right]^{1/2} = \underline{1}$

C lies in the x-y plane at angle from due north (the y axis) given by $\alpha = \tan^{-1}(4/3) = 53.1^\circ$ (or 36.9° from the x axis). For those having nautical leanings, this is very close to the compass point NE$\frac{3}{4}$E (not required).

1.17. Point $A(-4, 2, 5)$ and the two vectors, $\mathbf{R}_{AM} = (20, 18, -10)$ and $\mathbf{R}_{AN} = (-10, 8, 15)$, define a triangle.

a) Find a unit vector perpendicular to the triangle: Use

$$\mathbf{a}_p = \frac{\mathbf{R}_{AM} \times \mathbf{R}_{AN}}{|\mathbf{R}_{AM} \times \mathbf{R}_{AN}|} = \frac{(350, -200, 340)}{527.35} = \underline{(0.664, -0.379, 0.645)}$$

The vector in the opposite direction to this one is also a valid answer.

b) Find a unit vector in the plane of the triangle and perpendicular to $\mathbf{R}_{AN}$:

$$\mathbf{a}_{AN} = \frac{(-10, 8, 15)}{\sqrt{389}} = (-0.507, 0.406, 0.761)$$

Then

$$\mathbf{a}_{pAN} = \mathbf{a}_p \times \mathbf{a}_{AN} = (0.664, -0.379, 0.645) \times (-0.507, 0.406, 0.761) = \underline{(-0.550, -0.832, 0.077)}$$

The vector in the opposite direction to this one is also a valid answer.

c) Find a unit vector in the plane of the triangle that bisects the interior angle at A: A non-unit vector in the required direction is $(1/2)(\mathbf{a}_{AM} + \mathbf{a}_{AN})$, where

$$\mathbf{a}_{AM} = \frac{(20, 18, -10)}{|(20, 18, -10)|} = (0.697, 0.627, -0.348)$$

Now

$$\frac{1}{2}(\mathbf{a}_{AM} + \mathbf{a}_{AN}) = \frac{1}{2}[(0.697, 0.627, -0.348) + (-0.507, 0.406, 0.761)] = (0.095, 0.516, 0.207)$$

Finally,

$$\mathbf{a}_{bis} = \frac{(0.095, 0.516, 0.207)}{|(0.095, 0.516, 0.207)|} = \underline{(0.168, 0.915, 0.367)}$$

1.18. A certain vector field is given as $\mathbf{G} = (y+1)\mathbf{a}_x + x\mathbf{a}_y$. a) Determine $\mathbf{G}$ at the point (3,-2,4): $\mathbf{G}(3,-2,4) = \underline{-\mathbf{a}_x + 3\mathbf{a}_y}$.

b) obtain a unit vector defining the direction of $\mathbf{G}$ at (3,-2,4).
$|\mathbf{G}(3,-2,4)| = [1+3^2]^{1/2} = \sqrt{10}$. So the unit vector is

$$\mathbf{a}_G(3,-2,4) = \frac{-\mathbf{a}_x + 3\mathbf{a}_y}{\sqrt{10}}$$

1.19. a) Express the field $\mathbf{D} = (x^2+y^2)^{-1}(x\mathbf{a}_x + y\mathbf{a}_y)$ in cylindrical components and cylindrical variables: Have $x = \rho\cos\phi$, $y = \rho\sin\phi$, and $x^2+y^2 = \rho^2$. Therefore

$$\mathbf{D} = \frac{1}{\rho}(\cos\phi\mathbf{a}_x + \sin\phi\mathbf{a}_y)$$

Then

$$D_\rho = \mathbf{D}\cdot\mathbf{a}_\rho = \frac{1}{\rho}\left[\cos\phi(\mathbf{a}_x\cdot\mathbf{a}_\rho) + \sin\phi(\mathbf{a}_y\cdot\mathbf{a}_\rho)\right] = \frac{1}{\rho}\left[\cos^2\phi + \sin^2\phi\right] = \frac{1}{\rho}$$

and

$$D_\phi = \mathbf{D}\cdot\mathbf{a}_\phi = \frac{1}{\rho}\left[\cos\phi(\mathbf{a}_x\cdot\mathbf{a}_\phi) + \sin\phi(\mathbf{a}_y\cdot\mathbf{a}_\phi)\right] = \frac{1}{\rho}\left[\cos\phi(-\sin\phi) + \sin\phi\cos\phi\right] = 0$$

Therefore

$$\underline{\mathbf{D} = \frac{1}{\rho}\mathbf{a}_\rho}$$

b) Evaluate $\mathbf{D}$ at the point where $\rho = 2$, $\phi = 0.2\pi$, and $z = 5$, expressing the result in cylindrical and cartesian coordinates: At the given point, and in cylindrical coordinates, $\underline{\mathbf{D} = 0.5\mathbf{a}_\rho}$. To express this in cartesian, we use

$$\mathbf{D} = 0.5(\mathbf{a}_\rho\cdot\mathbf{a}_x)\mathbf{a}_x + 0.5(\mathbf{a}_\rho\cdot\mathbf{a}_y)\mathbf{a}_y = 0.5\cos 36^\circ\mathbf{a}_x + 0.5\sin 36^\circ\mathbf{a}_y = \underline{0.41\mathbf{a}_x + 0.29\mathbf{a}_y}$$

1.20. If the three sides of a triangle are represented by the vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, all directed counter-clockwise, show that $|\mathbf{C}|^2 = (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B})$ and expand the product to obtain the law of cosines.

With the vectors drawn as described above, we find that $\mathbf{C} = -(\mathbf{A}+\mathbf{B})$ and so $|\mathbf{C}|^2 = C^2 = \mathbf{C}\cdot\mathbf{C} = (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B})$ So far so good. Now if we expand the product, obtain

$$(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}) = A^2 + B^2 + 2\mathbf{A}\cdot\mathbf{B}$$

where $\mathbf{A}\cdot\mathbf{B} = AB\cos(180^\circ - \alpha) = -AB\cos\alpha$ where α is the interior angle at the junction of $\mathbf{A}$ and $\mathbf{B}$. Using this, we have $C^2 = A^2 + B^2 - 2AB\cos\alpha$, which is the law of cosines.

1.21. Express in cylindrical components:

a) the vector from $C(3,2,-7)$ to $D(-1,-4,2)$:
$C(3,2,-7) \rightarrow C(\rho = 3.61, \phi = 33.7°, z = -7)$ and
$D(-1,-4,2) \rightarrow D(\rho = 4.12, \phi = -104.0°, z = 2)$.
Now $\mathbf{R}_{CD} = (-4,-6,9)$ and $R_\rho = \mathbf{R}_{CD} \cdot \mathbf{a}_\rho = -4\cos(33.7) - 6\sin(33.7) = -6.66$. Then $R_\phi = \mathbf{R}_{CD} \cdot \mathbf{a}_\phi = 4\sin(33.7) - 6\cos(33.7) = -2.77$. So $\mathbf{R}_{CD} = \underline{-6.66\mathbf{a}_\rho - 2.77\mathbf{a}_\phi + 9\mathbf{a}_z}$

b) a unit vector at D directed toward C:
$\mathbf{R}_{CD} = (4,6,-9)$ and $R_\rho = \mathbf{R}_{DC} \cdot \mathbf{a}_\rho = 4\cos(-104.0) + 6\sin(-104.0) = -6.79$. Then $R_\phi = \mathbf{R}_{DC} \cdot \mathbf{a}_\phi = 4[-\sin(-104.0)] + 6\cos(-104.0) = 2.43$. So $\mathbf{R}_{DC} = -6.79\mathbf{a}_\rho + 2.43\mathbf{a}_\phi - 9\mathbf{a}_z$
Thus $\mathbf{a}_{DC} = \underline{-0.59\mathbf{a}_\rho + 0.21\mathbf{a}_\phi - 0.78\mathbf{a}_z}$

c) a unit vector at D directed toward the origin: Start with $\mathbf{r}_D = (-1,-4,2)$, and so the vector toward the origin will be $-\mathbf{r}_D = (1,4,-2)$. Thus in cartesian the unit vector is $\mathbf{a} = (0.22, 0.87, -0.44)$. Convert to cylindrical:
$a_\rho = (0.22, 0.87, -0.44) \cdot \mathbf{a}_\rho = 0.22\cos(-104.0) + 0.87\sin(-104.0) = -0.90$, and
$a_\phi = (0.22, 0.87, -0.44) \cdot \mathbf{a}_\phi = 0.22[-\sin(-104.0)] + 0.87\cos(-104.0) = 0$, so that finally, $\mathbf{a} = \underline{-0.90\mathbf{a}_\rho - 0.44\mathbf{a}_z}$.

1.22. A sphere of radius a, centered at the origin, rotates about the z axis at angular velocity Ω rad/s. The rotation direction is clockwise when one is looking in the positive z direction.

a) Using spherical components, write an expression for the velocity field, $\mathbf{v}$, which gives the tangential velocity at any point within the sphere:

As in problem 1.20, we find the tangential velocity as the product of the angular velocity and the perperdicular distance from the rotation axis. With clockwise rotation, we obtain

$$\mathbf{v}(r,\theta) = \underline{\Omega r \sin\theta \, \mathbf{a}_\phi \ (r < a)}$$

b) Convert to rectangular components:

From here, the problem is the same as part c in Problem 1.20, except the rotation direction is reversed. The answer is $\mathbf{v}(x,y) = \underline{\Omega\,[-y\,\mathbf{a}_x + x\,\mathbf{a}_y]}$, where $(x^2+y^2+z^2)^{1/2} < a$.

1.23. The surfaces $\rho = 3$, $\rho = 5$, $\phi = 100°$, $\phi = 130°$, $z = 3$, and $z = 4.5$ define a closed surface.

a) Find the enclosed volume:

$$\text{Vol} = \int_3^{4.5} \int_{100°}^{130°} \int_3^5 \rho \, d\rho \, d\phi \, dz = \underline{6.28}$$

NOTE: The limits on the ϕ integration must be converted to radians (as was done here, but not shown).

b) Find the total area of the enclosing surface:

$$\text{Area} = 2\int_{100°}^{130°} \int_3^5 \rho \, d\rho \, d\phi \; + \int_3^{4.5} \int_{100°}^{130°} 3 \, d\phi \, dz$$
$$+ \int_3^{4.5} \int_{100°}^{130°} 5 \, d\phi \, dz \; + \; 2\int_3^{4.5} \int_3^5 d\rho \, dz = \underline{20.7}$$

1.23c) Find the total length of the twelve edges of the surfaces:

$$\text{Length} = 4 \times 1.5 \;+\; 4 \times 2 \;+\; 2 \times \left[\frac{30^\circ}{360^\circ} \times 2\pi \times 3 \;+\; \frac{30^\circ}{360^\circ} \times 2\pi \times 5 \right] = \underline{22.4}$$

d) Find the length of the longest straight line that lies entirely within the volume: This will be between the points A($\rho = 3$, $\phi = 100^\circ$, $z = 3$) and B($\rho = 5$, $\phi = 130^\circ$, $z = 4.5$). Performing point transformations to cartesian coordinates, these become A($x = -0.52$, $y = 2.95$, $z = 3$) and B($x = -3.21$, $y = 3.83$, $z = 4.5$). Taking A and B as vectors directed from the origin, the requested length is

$$\text{Length} = |\mathbf{B} - \mathbf{A}| = |(-2.69, 0.88, 1.5)| = \underline{3.21}$$

1.24. Two unit vectors, $\mathbf{a}_1$ and $\mathbf{a}_2$ lie in the xy plane and pass through the origin. They make angles ϕ_1 and ϕ_2 with the x axis respectively.

a) Express each vector in rectangular components; Have $\mathbf{a}_1 = A_{x1}\mathbf{a}_x + A_{y1}\mathbf{a}_y$, so that $A_{x1} = \mathbf{a}_1 \cdot \mathbf{a}_x = \cos\phi_1$. Then, $A_{y1} = \mathbf{a}_1 \cdot \mathbf{a}_y = \cos(90 - \phi_1) = \sin\phi_1$. Therefore,

$$\mathbf{a}_1 = \cos\phi_1\,\mathbf{a}_x + \sin\phi_1\,\mathbf{a}_y \quad \text{and similarly,} \quad \mathbf{a}_2 = \cos\phi_2\,\mathbf{a}_x + \sin\phi_2\,\mathbf{a}_y$$

b) take the dot product and verify the trigonometric identity, $\cos(\phi_1 - \phi_2) = \cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2$: From the definition of the dot product,

$$\begin{aligned}\mathbf{a}_1 \cdot \mathbf{a}_2 &= (1)(1)\cos(\phi_1 - \phi_2)\\ &= (\cos\phi_1\,\mathbf{a}_x + \sin\phi_1\,\mathbf{a}_y) \cdot (\cos\phi_2\,\mathbf{a}_x + \sin\phi_2\,\mathbf{a}_y) = \cos\phi_1\,\cos\phi_2 + \sin\phi_1\,\sin\phi_2\end{aligned}$$

c) take the cross product and verify the trigonometric identity $\sin(\phi_2 - \phi_1) = \sin\phi_2\cos\phi_1 - \cos\phi_2\sin\phi_1$: From the definition of the cross product, and since $\mathbf{a}_1$ and $\mathbf{a}_2$ both lie in the x-y plane,

$$\mathbf{a}_1 \times \mathbf{a}_2 = (1)(1)\sin(\phi_1 - \phi_2)\,\mathbf{a}_z = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \cos\phi_1 & \sin\phi_1 & 0 \\ \cos\phi_2 & \sin\phi_2 & 0 \end{vmatrix}$$
$$= [\sin\phi_2\cos\phi_1 - \cos\phi_2\sin\phi_1]\;\mathbf{a}_z$$

thus verified.

1.25. Given point $P(r = 0.8$, $\theta = 30^\circ$, $\phi = 45^\circ)$, and

$$\mathbf{E} = \frac{1}{r^2}\left(\cos\phi\,\mathbf{a}_r + \frac{\sin\phi}{\sin\theta}\,\mathbf{a}_\phi\right)$$

a) Find $\mathbf{E}$ at P: $\mathbf{E} = \underline{1.10\mathbf{a}_\rho + 2.21\mathbf{a}_\phi}$.
b) Find $|\mathbf{E}|$ at P: $|\mathbf{E}| = \sqrt{1.10^2 + 2.21^2} = \underline{2.47}$.
c) Find a unit vector in the direction of $\mathbf{E}$ at P:

$$\mathbf{a}_E = \frac{\mathbf{E}}{|\mathbf{E}|} = \underline{0.45\mathbf{a}_r + 0.89\mathbf{a}_\phi}$$

1.26. Express the uniform vector field, $\mathbf{F} = 5\,\mathbf{a}_x$ in

a) cylindrical components: $F_\rho = 5\,\mathbf{a}_x \cdot \mathbf{a}_\rho = 5\cos\phi$, and $F_\phi = 5\,\mathbf{a}_x \cdot \mathbf{a}_\phi = -5\sin\phi$. Combining, we obtain $\underline{\mathbf{F}(\rho,\phi) = 5(\cos\phi\,\mathbf{a}_\rho - \sin\phi\,\mathbf{a}_\phi)}$.

b) spherical components: $F_r = 5\,\mathbf{a}_x \cdot \mathbf{a}_r = 5\sin\theta\cos\phi$; $F_\theta = 5\,\mathbf{a}_x \cdot \mathbf{a}_\theta = 5\cos\theta\cos\phi$; $F_\phi = 5\,\mathbf{a}_x \cdot \mathbf{a}_\phi = -5\sin\phi$. Combining, we obtain $\underline{\mathbf{F}(r,\theta,\phi) = 5\left[\sin\theta\cos\phi\,\mathbf{a}_r + \cos\theta\cos\phi\,\mathbf{a}_\theta - \sin\phi\,\mathbf{a}_\phi\right]}$.

1.27. The surfaces $r = 2$ and 4, $\theta = 30°$ and $50°$, and $\phi = 20°$ and $60°$ identify a closed surface.

a) Find the enclosed volume: This will be

$$\text{Vol} = \int_{20°}^{60°}\int_{30°}^{50°}\int_{2}^{4} r^2 \sin\theta\, dr\, d\theta\, d\phi = \underline{2.91}$$

where degrees have been converted to radians.

b) Find the total area of the enclosing surface:

$$\text{Area} = \int_{20°}^{60°}\int_{30°}^{50°} (4^2 + 2^2)\sin\theta\, d\theta\, d\phi + \int_{2}^{4}\int_{20°}^{60°} r(\sin 30° + \sin 50°)\, dr\, d\phi$$
$$+ 2\int_{30°}^{50°}\int_{2}^{4} r\, dr\, d\theta = \underline{12.61}$$

c) Find the total length of the twelve edges of the surface:

$$\text{Length} = 4\int_2^4 dr \; + \; 2\int_{30°}^{50°} (4+2)\, d\theta + \int_{20°}^{60°} (4\sin 50° + 4\sin 30° + 2\sin 50° + 2\sin 30°)\, d\phi$$
$$= \underline{17.49}$$

d) Find the length of the longest straight line that lies entirely within the surface: This will be from $A(r = 2, \theta = 50°, \phi = 20°)$ to $B(r = 4, \theta = 30°, \phi = 60°)$ or

$$A(x = 2\sin 50°\cos 20°, y = 2\sin 50°\sin 20°, z = 2\cos 50°)$$

to

$$B(x = 4\sin 30°\cos 60°, y = 4\sin 30°\sin 60°, z = 4\cos 30°)$$

or finally $A(1.44, 0.52, 1.29)$ to $B(1.00, 1.73, 3.46)$. Thus $\mathbf{B} - \mathbf{A} = (-0.44, 1.21, 2.18)$ and

$$\text{Length} = |\mathbf{B} - \mathbf{A}| = \underline{2.53}$$

1.28. State whether or not $\mathbf{A} = \mathbf{B}$ and, if not, what conditions are imposed on $\mathbf{A}$ and $\mathbf{B}$ when

a) $\mathbf{A} \cdot \mathbf{a}_x = \mathbf{B} \cdot \mathbf{a}_x$: For this to be true, both $\mathbf{A}$ and $\mathbf{B}$ must be oriented at the same angle, θ, from the x axis. But this would allow either vector to lie anywhere along a conical surface of angle θ about the x axis. Therefore, $\mathbf{A}$ *can* be equal to $\mathbf{B}$, but not necessarily.

b) $\mathbf{A} \times \mathbf{a}_x = \mathbf{B} \times \mathbf{a}_x$: This is a more restrictive condition because the cross product gives a vector. For both cross products to lie in the same direction, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{a}_x$ must be coplanar. But if $\mathbf{A}$ lies at angle θ to the x axis, $\mathbf{B}$ could lie at θ *or* at $180° - \theta$ to give the same cross product. So again, $\mathbf{A}$ *can* be equal to $\mathbf{B}$, but not necessarily.

1.28c) $\mathbf{A}\cdot\mathbf{a}_x = \mathbf{B}\cdot\mathbf{a}_x$ *and* $\mathbf{A}\times\mathbf{a}_x = \mathbf{B}\times\mathbf{a}_x$: In this case, we need to satisfy both requirements in parts *a* and *b* – that is, **A**, **B**, and $\mathbf{a}_x$ must be coplanar, *and* **A** and **B** must lie at the same angle, θ, to $\mathbf{a}_x$. With coplanar vectors, this latter condition might imply that both $+\theta$ and $-\theta$ would therefore work. But the negative angle reverses the direction of the cross product direction. Therefore both vectors must lie in the same plane *and* lie at the same angle to x; i.e., **A** *must* be equal to **B**.

d) $\mathbf{A}\cdot\mathbf{C} = \mathbf{B}\cdot\mathbf{C}$ *and* $\mathbf{A}\times\mathbf{C} = \mathbf{B}\times\mathbf{C}$ where **C** is any vector except $\mathbf{C} = 0$: This is just the general case of part *c*. Since we can orient our coordinate system in any manner we choose, we can arrange it so that the x axis coincides with the direction of vector **C**. Thus all the arguments of part *c* apply, and again we conclude that **A** *must* be equal to **B**.

1.29. Express the unit vector $\mathbf{a}_x$ in spherical components at the point:

a) $r = 2$, $\theta = 1$ rad, $\phi = 0.8$ rad: Use

$$\mathbf{a}_x = (\mathbf{a}_x\cdot\mathbf{a}_r)\mathbf{a}_r + (\mathbf{a}_x\cdot\mathbf{a}_\theta)\mathbf{a}_\theta + (\mathbf{a}_x\cdot\mathbf{a}_\phi)\mathbf{a}_\phi =$$
$$\sin(1)\cos(0.8)\mathbf{a}_r + \cos(1)\cos(0.8)\mathbf{a}_\theta + (-\sin(0.8))\mathbf{a}_\phi = \underline{0.59\mathbf{a}_r + 0.38\mathbf{a}_\theta - 0.72\mathbf{a}_\phi}$$

b) $x = 3$, $y = 2$, $z = -1$: First, transform the point to spherical coordinates. Have $r = \sqrt{14}$, $\theta = \cos^{-1}(-1/\sqrt{14}) = 105.5^\circ$, and $\phi = \tan^{-1}(2/3) = 33.7^\circ$. Then

$$\mathbf{a}_x = \sin(105.5^\circ)\cos(33.7^\circ)\mathbf{a}_r + \cos(105.5^\circ)\cos(33.7^\circ)\mathbf{a}_\theta + (-\sin(33.7^\circ))\mathbf{a}_\phi$$
$$= \underline{0.80\mathbf{a}_r - 0.22\mathbf{a}_\theta - 0.55\mathbf{a}_\phi}$$

c) $\rho = 2.5$, $\phi = 0.7$ rad, $z = 1.5$: Again, convert the point to spherical coordinates. $r = \sqrt{\rho^2 + z^2} = \sqrt{8.5}$, $\theta = \cos^{-1}(z/r) = \cos^{-1}(1.5/\sqrt{8.5}) = 59.0^\circ$, and $\phi = 0.7\,\text{rad} = 40.1^\circ$. Now

$$\mathbf{a}_x = \sin(59^\circ)\cos(40.1^\circ)\mathbf{a}_r + \cos(59^\circ)\cos(40.1^\circ)\mathbf{a}_\theta + (-\sin(40.1^\circ))\mathbf{a}_\phi$$
$$= \underline{0.66\mathbf{a}_r + 0.39\mathbf{a}_\theta - 0.64\mathbf{a}_\phi}$$

1.30. Consider a problem analogous to the varying wind velocities encountered by transcontinental aircraft. We assume a constant altitude, a plane earth, a flight along the x axis from 0 to 10 units, no vertical velocity component, and no change in wind velocity with time. Assume $\mathbf{a}_x$ to be directed to the east and $\mathbf{a}_y$ to the north. The wind velocity at the operating altitude is assumed to be:

$$\mathbf{v}(x,y) = \frac{(0.01x^2 - 0.08x + 0.66)\mathbf{a}_x - (0.05x - 0.4)\mathbf{a}_y}{1 + 0.5y^2}$$

a) Determine the location and magnitude of the maximum tailwind encountered: Tailwind would be x-directed, and so we look at the x component only. Over the flight range, this function maximizes at a value of $0.86/(1+0.5y^2)$ at $x = 10$ (at the end of the trip). It reaches a local minimum of $0.50/(1+0.5y^2)$ at $x = 4$, and has another local maximum of $0.66/(1+0.5y^2)$ at the trip start, $x = 0$.

b) Repeat for headwind: The x component is always positive, and so therefore no headwind exists over the travel range.

c) Repeat for crosswind: Crosswind will be found from the y component, which is seen to maximize over the flight range at a value of $0.4/(1+0.5y^2)$ at the trip start ($x = 0$).

d) Would more favorable tailwinds be available at some other latitude? If so, where? Minimizing the denominator accomplishes this; in particular, the lattitude associated with $y = 0$ gives the strongest tailwind.

CHAPTER 2

2.1. Three point charges are positioned in the x-y plane as follows: 5nC at $y = 5$ cm, -10 nC at $y = -5$ cm, 15 nC at $x = -5$ cm. Find the required x-y coordinates of a 20-nC fourth charge that will produce a zero electric field at the origin.

With the charges thus configured, the electric field at the origin will be the superposition of the individual charge fields:

$$\mathbf{E}_0 = \frac{1}{4\pi\epsilon_0}\left[\frac{15}{(5)^2}\,\mathbf{a}_x - \frac{5}{(5)^2}\,\mathbf{a}_y - \frac{10}{(5)^2}\,\mathbf{a}_y\right] = \frac{1}{4\pi\epsilon_0}\left(\frac{3}{5}\right)[\mathbf{a}_x - \mathbf{a}_y] \quad \text{nC/m}$$

The field, $\mathbf{E}_{20}$, associated with the 20-nC charge (evaluated at the origin) must exactly cancel this field, so we write:

$$\mathbf{E}_{20} = \frac{-1}{4\pi\epsilon_0}\left(\frac{3}{5}\right)[\mathbf{a}_x - \mathbf{a}_y] = \frac{-20}{4\pi\epsilon_0\rho^2}\left(\frac{1}{\sqrt{2}}\right)[\mathbf{a}_x - \mathbf{a}_y]$$

From this, we identify the distance from the origin: $\rho = \sqrt{100/(3\sqrt{2})} = 4.85$. The x and y coordinates of the 20-nC charge will both be equal in magnitude to $4.85/\sqrt{2} = 3.43$. The coodinates of the 20-nC charge are then $\underline{(3.43, -3.43)}$.

2.2. Point charges of 1nC and -2nC are located at (0,0,0) and (1,1,1), respectively, in free space. Determine the vector force acting on each charge.

First, the electric field intensity associated with the 1nC charge, evalutated at the -2nC charge location is:

$$\mathbf{E}_{12} = \frac{1}{4\pi\epsilon_0(3)}\left(\frac{1}{\sqrt{3}}\right)(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z) \quad \text{nC/m}$$

in which the distance between charges is $\sqrt{3}$ m. The force on the -2nC charge is then

$$\mathbf{F}_{12} = q_2\mathbf{E}_{12} = \frac{-2}{12\sqrt{3}\,\pi\epsilon_0}(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z) = \underline{\frac{-1}{10.4\,\pi\epsilon_0}(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z) \quad \text{nN}}$$

The force on the 1nC charge at the origin is just the opposite of this result, or

$$\mathbf{F}_{21} = \underline{\frac{+1}{10.4\,\pi\epsilon_0}(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z) \quad \text{nN}}$$

2.3. Point charges of 50nC each are located at $A(1, 0, 0)$, $B(-1, 0, 0)$, $C(0, 1, 0)$, and $D(0, -1, 0)$ in free space. Find the total force on the charge at A.

The force will be:

$$\mathbf{F} = \frac{(50 \times 10^{-9})^2}{4\pi\epsilon_0}\left[\frac{\mathbf{R}_{CA}}{|\mathbf{R}_{CA}|^3} + \frac{\mathbf{R}_{DA}}{|\mathbf{R}_{DA}|^3} + \frac{\mathbf{R}_{BA}}{|\mathbf{R}_{BA}|^3}\right]$$

where $\mathbf{R}_{CA} = \mathbf{a}_x - \mathbf{a}_y$, $\mathbf{R}_{DA} = \mathbf{a}_x + \mathbf{a}_y$, and $\mathbf{R}_{BA} = 2\mathbf{a}_x$. The magnitudes are $|\mathbf{R}_{CA}| = |\mathbf{R}_{DA}| = \sqrt{2}$, and $|\mathbf{R}_{BA}| = 2$. Substituting these leads to

$$\mathbf{F} = \frac{(50 \times 10^{-9})^2}{4\pi\epsilon_0}\left[\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} + \frac{2}{8}\right]\mathbf{a}_x = \underline{21.5\mathbf{a}_x\ \mu\text{N}}$$

where distances are in meters.

2.5. Let a point charge $Q_1 = 25$ nC be located at $P_1(4,-2,7)$ and a charge $Q_2 = 60$ nC be at $P_2(-3,4,-2)$.

a) If $\epsilon = \epsilon_0$, find **E** at $P_3(1,2,3)$: This field will be

$$\mathbf{E} = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{25\mathbf{R}_{13}}{|\mathbf{R}_{13}|^3} + \frac{60\mathbf{R}_{23}}{|\mathbf{R}_{23}|^3}\right]$$

where $\mathbf{R}_{13} = -3\mathbf{a}_x + 4\mathbf{a}_y - 4\mathbf{a}_z$ and $\mathbf{R}_{23} = 4\mathbf{a}_x - 2\mathbf{a}_y + 5\mathbf{a}_z$. Also, $|\mathbf{R}_{13}| = \sqrt{41}$ and $|\mathbf{R}_{23}| = \sqrt{45}$. So

$$\mathbf{E} = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{25\times(-3\mathbf{a}_x + 4\mathbf{a}_y - 4\mathbf{a}_z)}{(41)^{1.5}} + \frac{60\times(4\mathbf{a}_x - 2\mathbf{a}_y + 5\mathbf{a}_z)}{(45)^{1.5}}\right]$$
$$= \underline{4.58\mathbf{a}_x - 0.15\mathbf{a}_y + 5.51\mathbf{a}_z}$$

b) At what point on the y axis is $E_x = 0$? P_3 is now at $(0, y, 0)$, so $\mathbf{R}_{13} = -4\mathbf{a}_x + (y+2)\mathbf{a}_y - 7\mathbf{a}_z$ and $\mathbf{R}_{23} = 3\mathbf{a}_x + (y-4)\mathbf{a}_y + 2\mathbf{a}_z$. Also, $|\mathbf{R}_{13}| = \sqrt{65 + (y+2)^2}$ and $|\mathbf{R}_{23}| = \sqrt{13 + (y-4)^2}$. Now the x component of **E** at the new P_3 will be:

$$E_x = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{25\times(-4)}{[65 + (y+2)^2]^{1.5}} + \frac{60\times 3}{[13 + (y-4)^2]^{1.5}}\right]$$

2.5b (continued) To obtain $E_x = 0$, we require the expression in the large brackets to be zero. This expression simplifies to the following quadratic:

$$0.48y^2 + 13.92y + 73.10 = 0$$

which yields the two values: $y = \underline{-6.89, -22.11}$

2.6. Two point charges of equal magnitude q are positioned at $z = \pm d/2$.

a) find the electric field everywhere on the z axis: For a point charge at any location, we have

$$\mathbf{E} = \frac{q(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|^3}$$

In the case of two charges, we would therefore have

$$\mathbf{E}_T = \frac{q_1(\mathbf{r} - \mathbf{r}_1')}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_1'|^3} + \frac{q_2(\mathbf{r} - \mathbf{r}_2')}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_2'|^3} \quad (1)$$

In the present case, we assign $q_1 = q_2 = q$, the observation point position vector as $\mathbf{r} = z\mathbf{a}_z$, and the charge position vectors as $\mathbf{r}_1' = (d/2)\mathbf{a}_z$, and $\mathbf{r}_2' = -(d/2)\mathbf{a}_z$ Therefore

$$\mathbf{r} - \mathbf{r}_1' = [z - (d/2)]\mathbf{a}_z, \quad \mathbf{r} - \mathbf{r}_2' = [z + (d/2)]\mathbf{a}_z,$$

then

$$|\mathbf{r} - \mathbf{r}_1|^3 = [z - (d/2)]^3 \quad \text{and} \quad |\mathbf{r} - \mathbf{r}_2|^3 = [z + (d/2)]^3$$

Substitute these results into (1) to obtain:

$$\mathbf{E}_T(z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{[z - (d/2)]^2} + \frac{1}{[z + (d/2)]^2}\right]\mathbf{a}_z \quad \text{V/m} \quad (2)$$

b) find the electric field everywhere on the x axis: We proceed as in part a, except that now $\mathbf{r} = x\mathbf{a}_x$. Eq. (1) becomes

$$\mathbf{E}_T(x) = \frac{q}{4\pi\epsilon_0}\left[\frac{x\mathbf{a}_x - (d/2)\mathbf{a}_z}{|x\mathbf{a}_x - (d/2)\mathbf{a}_z|^3} + \frac{x\mathbf{a}_x + (d/2)\mathbf{a}_z}{|x\mathbf{a}_x + (d/2)\mathbf{a}_z|^3}\right] \quad (3)$$

where

$$|x\mathbf{a}_x - (d/2)\mathbf{a}_z| = |x\mathbf{a}_x + (d/2)\mathbf{a}_z| = \left[x^2 + (d/2)^2\right]^{1/2}$$

Therefore (3) becomes

$$\mathbf{E}_T(x) = \frac{2qx\,\mathbf{a}_x}{4\pi\epsilon_0\left[x^2 + (d/2)^2\right]^{3/2}}$$

c) repeat parts a and b if the charge at $z = -d/2$ is $-q$ instead of $+q$: The field along the z axis is quickly found by changing the sign of the second term in (2):

$$\mathbf{E}_T(z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{[z - (d/2)]^2} - \frac{1}{[z + (d/2)]^2}\right]\mathbf{a}_z \quad \text{V/m}$$

In like manner, the field along the x axis is found from (3) by again changing the sign of the second term. The result is

$$-2qd\,\mathbf{a}_z$$

2.7. A 2 μC point charge is located at $A(4, 3, 5)$ in free space. Find E_ρ, E_ϕ, and E_z at $P(8, 12, 2)$. Have

$$\mathbf{E}_P = \frac{2 \times 10^{-6}}{4\pi\epsilon_0} \frac{\mathbf{R}_{AP}}{|\mathbf{R}_{AP}|^3} = \frac{2 \times 10^{-6}}{4\pi\epsilon_0} \left[\frac{4\mathbf{a}_x + 9\mathbf{a}_y - 3\mathbf{a}_z}{(106)^{1.5}} \right] = 65.9\mathbf{a}_x + 148.3\mathbf{a}_y - 49.4\mathbf{a}_z$$

Then, at point P, $\rho = \sqrt{8^2 + 12^2} = 14.4$, $\phi = \tan^{-1}(12/8) = 56.3°$, and $z = z$. Now,

$$E_\rho = \mathbf{E}_p \cdot \mathbf{a}_\rho = 65.9(\mathbf{a}_x \cdot \mathbf{a}_\rho) + 148.3(\mathbf{a}_y \cdot \mathbf{a}_\rho) = 65.9\cos(56.3°) + 148.3\sin(56.3°) = \underline{159.7}$$

and

$$E_\phi = \mathbf{E}_p \cdot \mathbf{a}_\phi = 65.9(\mathbf{a}_x \cdot \mathbf{a}_\phi) + 148.3(\mathbf{a}_y \cdot \mathbf{a}_\phi) = -65.9\sin(56.3°) + 148.3\cos(56.3°) = \underline{27.4}$$

Finally, $E_z = \underline{-49.4 \text{ V/m}}$

2.8. A crude device for measuring charge consists of two small insulating spheres of radius a, one of which is fixed in position. The other is movable along the x axis, and is subject to a restraining force kx, where k is a spring constant. The uncharged spheres are centered at $x = 0$ and $x = d$, the latter fixed. If the spheres are given equal and opposite charges of Q coulombs:

a) Obtain the expression by which Q may be found as a function of x: The spheres will attract, and so the movable sphere at $x = 0$ will move toward the other until the spring and Coulomb forces balance. This will occur at location x for the movable sphere. With equal and opposite forces, we have

$$\frac{Q^2}{4\pi\epsilon_0(d - x)^2} = kx$$

from which $\underline{Q = 2(d - x)\sqrt{\pi\epsilon_0 kx}}$.

b) Determine the maximum charge that can be measured in terms of ϵ_0, k, and d, and state the separation of the spheres then: With increasing charge, the spheres move toward each other until they just touch at $\underline{x_{max} = d - 2a}$. Using the part a result, we find the maximum measurable charge: $Q_{max} = \underline{4a\sqrt{\pi\epsilon_0 k(d - 2a)}}$. Presumably some form of stop mechanism is placed at $x = x_{max}^-$ to prevent the spheres from actually touching.

c) What happens if a larger charge is applied? No further motion is possible, so nothing happens.

2.9. A 100 nC point charge is located at $A(-1, 1, 3)$ in free space.

a) Find the locus of all points $P(x, y, z)$ at which $E_x = 500$ V/m: The total field at P will be:

$$\mathbf{E}_P = \frac{100 \times 10^{-9}}{4\pi\epsilon_0} \frac{\mathbf{R}_{AP}}{|\mathbf{R}_{AP}|^3}$$

where $\mathbf{R}_{AP} = (x+1)\mathbf{a}_x + (y-1)\mathbf{a}_y + (z-3)\mathbf{a}_z$, and where $|\mathbf{R}_{AP}| = [(x+1)^2 + (y-1)^2 + (z-3)^2]^{1/2}$. The x component of the field will be

$$E_x = \frac{100 \times 10^{-9}}{4\pi\epsilon_0} \left[\frac{(x + 1)}{[(x + 1)^2 + (y - 1)^2 + (z - 3)^2]^{1.5}} \right] = 500 \text{ V/m}$$

And so our condition becomes:

$$\underline{(x + 1) = 0.56\,[(x + 1)^2 + (y - 1)^2 + (z - 3)^2]^{1.5}}$$

2.9b) Find y_1 if $P(-2, y_1, 3)$ lies on that locus: At point P, the condition of part a becomes

$$3.19 = \left[1 + (y_1 - 1)^2\right]^3$$

from which $(y_1 - 1)^2 = 0.47$, or $y_1 = \underline{1.69 \text{ or } 0.31}$

2.11. A charge Q_0 located at the origin in free space produces a field for which $E_z = 1$ kV/m at point $P(-2, 1, -1)$.

a) Find Q_0: The field at P will be

$$\mathbf{E}_P = \frac{Q_0}{4\pi\epsilon_0}\left[\frac{-2\mathbf{a}_x + \mathbf{a}_y - \mathbf{a}_z}{6^{1.5}}\right]$$

Since the z component is of value 1 kV/m, we find $Q_0 = -4\pi\epsilon_0 6^{1.5} \times 10^3 = \underline{-1.63\ \mu\text{C}}$.

b) Find $\mathbf{E}$ at $M(1, 6, 5)$ in cartesian coordinates: This field will be:

$$\mathbf{E}_M = \frac{-1.63 \times 10^{-6}}{4\pi\epsilon_0}\left[\frac{\mathbf{a}_x + 6\mathbf{a}_y + 5\mathbf{a}_z}{[1 + 36 + 25]^{1.5}}\right]$$

or $\mathbf{E}_M = \underline{-30.11\mathbf{a}_x - 180.63\mathbf{a}_y - 150.53\mathbf{a}_z}$.

c) Find $\mathbf{E}$ at $M(1, 6, 5)$ in cylindrical coordinates: At M, $\rho = \sqrt{1 + 36} = 6.08$, $\phi = \tan^{-1}(6/1) = 80.54°$, and $z = 5$. Now

$$E_\rho = \mathbf{E}_M \cdot \mathbf{a}_\rho = -30.11\cos\phi - 180.63\sin\phi = -183.12$$

$$E_\phi = \mathbf{E}_M \cdot \mathbf{a}_\phi = -30.11(-\sin\phi) - 180.63\cos\phi = 0 \quad \text{(as expected)}$$

so that $\mathbf{E}_M = \underline{-183.12\mathbf{a}_\rho - 150.53\mathbf{a}_z}$.

d) Find $\mathbf{E}$ at $M(1, 6, 5)$ in spherical coordinates: At M, $r = \sqrt{1 + 36 + 25} = 7.87$, $\phi = 80.54°$ (as before), and $\theta = \cos^{-1}(5/7.87) = 50.58°$. Now, since the charge is at the origin, we expect to obtain only a radial component of $\mathbf{E}_M$. This will be:

$$E_r = \mathbf{E}_M \cdot \mathbf{a}_r = -30.11\sin\theta\cos\phi - 180.63\sin\theta\sin\phi - 150.53\cos\theta = \underline{-237.1}$$

2.12. Electrons are in random motion in a fixed region in space. During any 1μs interval, the probability of finding an electron in a subregion of volume 10^{-15} m^2 is 0.27. What volume charge density, appropriate for such time durations, should be assigned to that subregion?

The finite probabilty effectively reduces the net charge quantity by the probability fraction. With $e = -1.602 \times 10^{-19}$ C, the density becomes

$$\rho_v = -\frac{0.27 \times 1.602 \times 10^{-19}}{10^{-15}} = \underline{-43.3\ \mu\text{C/m}^3}$$

2.13. A uniform volume charge density of 0.2 μC/m^3 is present throughout the spherical shell extending from $r = 3$ cm to $r = 5$ cm. If $\rho_v = 0$ elsewhere:

a) find the total charge present throughout the shell: This will be

$$Q = \int_0^{2\pi} \int_0^{\pi} \int_{.03}^{.05} 0.2\ r^2 \sin\theta\ dr\, d\theta\, d\phi = \left[4\pi(0.2)\frac{r^3}{3} \right]_{.03}^{.05} = 8.21 \times 10^{-5}\ \mu\text{C} = \underline{82.1\ \text{pC}}$$

b) find r_1 if half the total charge is located in the region $3\,\text{cm} < r < r_1$: If the integral over r in part a is taken to r_1, we would obtain

$$\left[4\pi(0.2)\frac{r^3}{3} \right]_{.03}^{r_1} = 4.105 \times 10^{-5}$$

Thus

$$r_1 = \left[\frac{3 \times 4.105 \times 10^{-5}}{0.2 \times 4\pi} + (.03)^3 \right]^{1/3} = \underline{4.24\ \text{cm}}$$

2.14. The electron beam in a certain cathode ray tube possesses cylindrical symmetry, and the charge density is represented by $\rho_v = -0.1/(\rho^2 + 10^{-8})$ pC/m^3 for $0 < \rho < 3 \times 10^{-4}$ m, and $\rho_v = 0$ for $\rho > 3 \times 10^{-4}$ m.

a) Find the total charge per meter along the length of the beam: We integrate the charge density over the cylindrical volume having radius 3×10^{-4} m, and length 1m.

$$q = \int_0^1 \int_0^{2\pi} \int_0^{3\times10^{-4}} \frac{-0.1}{(\rho^2 + 10^{-8})}\, \rho\, d\rho\, d\phi\, dz$$

From integral tables, this evaluates as

$$q = -0.2\pi \left(\frac{1}{2}\right) \ln\left(\rho^2 + 10^{-8}\right) \Big|_0^{3\times10^{-4}} = 0.1\pi \ln(10) = \underline{-0.23\pi\ \text{pC/m}}$$

b) if the electron velocity is 5×10^7 m/s, and with one ampere defined as 1C/s, find the beam current:

$$\text{Current} = \text{charge/m} \times v = -0.23\pi\,[\text{pC/m}] \times 5 \times 10^7\,[\text{m/s}] = -11.5\pi \times 10^6\,[\text{pC/s}] = \underline{-11.5\pi\,\mu\text{A}}$$

2.15. A spherical volume having a 2 μm radius contains a uniform volume charge density of 10^{15} C/m^3.

a) What total charge is enclosed in the spherical volume?
This will be $Q = (4/3)\pi(2 \times 10^{-6})^3 \times 10^{15} = \underline{3.35 \times 10^{-2}\ \text{C}}$.

b) Now assume that a large region contains one of these little spheres at every corner of a cubical grid 3mm on a side, and that there is no charge between spheres. What is the average volume charge density throughout this large region? Each cube will contain the equivalent of one little sphere. Neglecting the little sphere volume, the average density becomes

$$\rho_{v,avg} = \frac{3.35 \times 10^{-2}}{(0.003)^3} = \underline{1.24 \times 10^6\ \text{C/m}^3}$$

2.17. A uniform line charge of 16 nC/m is located along the line defined by $y = -2$, $z = 5$. If $\epsilon = \epsilon_0$:

a) Find **E** at $P(1, 2, 3)$: This will be

$$\mathbf{E}_P = \frac{\rho_l}{2\pi\epsilon_0} \frac{\mathbf{R}_P}{|\mathbf{R}_P|^2}$$

where $\mathbf{R}_P = (1, 2, 3) - (1, -2, 5) = (0, 4, -2)$, and $|\mathbf{R}_P|^2 = 20$. So

$$\mathbf{E}_P = \frac{16 \times 10^{-9}}{2\pi\epsilon_0} \left[\frac{4\mathbf{a}_y - 2\mathbf{a}_z}{20} \right] = \underline{57.5\mathbf{a}_y - 28.8\mathbf{a}_z\ \text{V/m}}$$

b) Find **E** at that point in the $z = 0$ plane where the direction of **E** is given by $(1/3)\mathbf{a}_y - (2/3)\mathbf{a}_z$: With $z = 0$, the general field will be

$$\mathbf{E}_{z=0} = \frac{\rho_l}{2\pi\epsilon_0} \left[\frac{(y+2)\mathbf{a}_y - 5\mathbf{a}_z}{(y+2)^2 + 25} \right]$$

We require $|E_z| = -|2E_y|$, so $2(y + 2) = 5$. Thus $y = 1/2$, and the field becomes:

$$\mathbf{E}_{z=0} = \frac{\rho_l}{2\pi\epsilon_0} \left[\frac{2.5\mathbf{a}_y - 5\mathbf{a}_z}{(2.5)^2 + 25} \right] = \underline{23\mathbf{a}_y - 46\mathbf{a}_z}$$

2.18. a) Find **E** in the plane $z = 0$ that is produced by a uniform line charge, ρ_L, extending along the z axis over the range $-L < z < L$ in a cylindrical coordinate system: We find **E** through

$$\mathbf{E} = \int_{-L}^{L} \frac{\rho_L dz(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^3}$$

where the observation point position vector is $\mathbf{r} = \rho\mathbf{a}_\rho$ (anywhere in the x-y plane), and where the position vector that locates any differential charge element on the z axis is $\mathbf{r}' = z\mathbf{a}_z$. So $\mathbf{r} - \mathbf{r}' = \rho\mathbf{a}_\rho - z\mathbf{a}_z$, and $|\mathbf{r} - \mathbf{r}'| = (\rho^2 + z^2)^{1/2}$. These relations are substituted into the integral to yield:

$$\mathbf{E} = \int_{-L}^{L} \frac{\rho_L dz(\rho\mathbf{a}_\rho - z\mathbf{a}_z)}{4\pi\epsilon_0(\rho^2 + z^2)^{3/2}} = \frac{\rho_L\,\rho\,\mathbf{a}_\rho}{4\pi\epsilon_0} \int_{-L}^{L} \frac{dz}{(\rho^2 + z^2)^{3/2}} = E_\rho\,\mathbf{a}_\rho$$

Note that the second term in the left-hand integral (involving $z\mathbf{a}_z$) has effectively vanished because it produces equal and opposite sign contributions when the integral is taken over symmetric limits (odd parity). Evaluating the integral results in

$$E_\rho = \frac{\rho_L\,\rho}{4\pi\epsilon_0}\, \frac{z}{\rho^2\sqrt{\rho^2 + z^2}}\Big|_{-L}^{L} = \frac{\rho_L}{2\pi\epsilon_0\rho}\, \frac{L}{\sqrt{\rho^2 + L^2}} = \frac{\rho_L}{2\pi\epsilon_0\rho}\, \frac{1}{\sqrt{1 + (\rho/L)^2}}$$

Note that as $L \to \infty$, the expression reduces to the expected field of the infinite line charge in free space, $\rho_L/(2\pi\epsilon_0\rho)$.

b) if the finite line charge is approximated by an infinite line charge ($L \to \infty$), by what percentage is E_ρ in error if $\rho = 0.5L$? The percent error in this situation will be

$$\%\,\text{error} = \left[1 - \frac{1}{\sqrt{1 + (\rho/L)^2}}\right] \times 100$$

For $\rho = 0.5L$, this becomes $\%\,\text{error} = \underline{10.6\,\%}$

c) repeat b with $\rho = 0.1L$. For this value, obtain $\%\,\text{error} = \underline{0.496\,\%}$.

2.19. A uniform line charge of 2 μC/m is located on the z axis. Find **E** in rectangular coordinates at $P(1, 2, 3)$ if the charge extends from

a) $-\infty < z < \infty$: With the infinite line, we know that the field will have only a radial component in cylindrical coordinates (or x and y components in cartesian). The field from an infinite line on the z axis is generally $\mathbf{E} = [\rho_l/(2\pi\epsilon_0\rho)]\mathbf{a}_\rho$. Therefore, at point P:

$$\mathbf{E}_P = \frac{\rho_l}{2\pi\epsilon_0}\frac{\mathbf{R}_{zP}}{|\mathbf{R}_{zP}|^2} = \frac{(2 \times 10^{-6})}{2\pi\epsilon_0}\ \frac{\mathbf{a}_x + 2\mathbf{a}_y}{5} = \underline{7.2\mathbf{a}_x + 14.4\mathbf{a}_y\ \text{kV/m}}$$

where $\mathbf{R}_{zP}$ is the vector that extends from the line charge to point P, and is perpendicular to the z axis; i.e., $\mathbf{R}_{zP} = (1, 2, 3) - (0, 0, 3) = (1, 2, 0)$.

b) $-4 \le z \le 4$: Here we use the general relation

$$\mathbf{E}_P = \int \frac{\rho_l dz}{4\pi\epsilon_0}\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}$$

2.19b (continued) where $\mathbf{r} = \mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$ and $\mathbf{r}' = z\mathbf{a}_z$. So the integral becomes

$$\mathbf{E}_P = \frac{(2 \times 10^{-6})}{4\pi\epsilon_0} \int_{-4}^{4} \frac{\mathbf{a}_x + 2\mathbf{a}_y + (3-z)\mathbf{a}_z}{[5 + (3-z)^2]^{1.5}}\, dz$$

Using integral tables, we obtain:

$$\mathbf{E}_P = 3597 \left[\frac{(\mathbf{a}_x + 2\mathbf{a}_y)(z-3) + 5\mathbf{a}_z}{(z^2 - 6z + 14)} \right]_{-4}^{4} \text{ V/m} = \underline{4.9\mathbf{a}_x + 9.8\mathbf{a}_y + 4.9\mathbf{a}_z \text{ kV/m}}$$

The student is invited to verify that when evaluating the above expression over the limits $-\infty < z < \infty$, the z component vanishes and the x and y components become those found in part a.

2.20. A line charge of uniform charge density ρ_0 C/m and of length ℓ, is oriented along the z axis at $-\ell/2 < z < \ell/2$.

a) Find the electric field strength, $\mathbf{E}$, in magnitude and direction at any position along the x axis: This follows the method in Problem 2.18. We find $\mathbf{E}$ through

$$\mathbf{E} = \int_{-\ell/2}^{\ell/2} \frac{\rho_0\, dz(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^3}$$

where the observation point position vector is $\mathbf{r} = x\mathbf{a}_x$ (anywhere on the x axis), and where the position vector that locates any differential charge element on the z axis is $\mathbf{r}' = z\mathbf{a}_z$. So $\mathbf{r} - \mathbf{r}' = x\mathbf{a}_x - z\mathbf{a}_z$, and $|\mathbf{r} - \mathbf{r}'| = (x^2 + z^2)^{1/2}$. These relations are substituted into the integral to yield:

$$\mathbf{E} = \int_{-\ell/2}^{\ell/2} \frac{\rho_0 dz(x\mathbf{a}_x - z\mathbf{a}_z)}{4\pi\epsilon_0 (x^2 + z^2)^{3/2}} = \frac{\rho_0\, x\, \mathbf{a}_x}{4\pi\epsilon_0} \int_{-\ell/2}^{\ell/2} \frac{dz}{(x^2 + z^2)^{3/2}} = E_x\, \mathbf{a}_x$$

Note that the second term in the left-hand integral (involving $z\mathbf{a}_z$) has effectively vanished because it produces equal and opposite sign contributions when the integral is taken over symmetric limits (odd parity). Evaluating the integral results in

$$E_x = \frac{\rho_0\, x}{4\pi\epsilon_0} \frac{z}{x^2\sqrt{x^2 + z^2}} \Big|_{-\ell/2}^{\ell/2} = \frac{\rho_0}{2\pi\epsilon_0 x} \frac{\ell/2}{\sqrt{x^2 + (\ell/2)^2}} = \frac{\rho_0}{2\pi\epsilon_0 x} \frac{1}{\sqrt{1 + (2x/\ell)^2}}$$

b) with the given line charge in position, find the force acting on an identical line charge that is oriented along the x axis at $\ell/2 < x < 3\ell/2$: The differential force on an element of the x-directed line charge will be $d\mathbf{F} = dq\mathbf{E} = (\rho_0\, dx)\mathbf{E}$, where $\mathbf{E}$ is the field as determined in part a. The net force is then the integral of the differential force over the length of the horizontal line charge, or

$$\mathbf{F} = \int_{\ell/2}^{3\ell/2} \frac{\rho_0^2}{2\pi\epsilon_0 x} \frac{1}{\sqrt{1 + (2x/\ell)^2}}\, dx\ \mathbf{a}_x$$

This can be re-written and then evaluated using integral tables as

$$\mathbf{F} = \frac{\rho_0^2\, \ell\, \mathbf{a}_x}{4\pi\epsilon_0} \int_{\ell/2}^{3\ell/2} \frac{dx}{x\sqrt{x^2 + (\ell/2)^2}} = \frac{-\rho_0^2\, \ell\, \mathbf{a}_x}{4\pi\epsilon_0} \left(\frac{1}{(\ell/2)} \ln \left[\frac{\ell/2 + \sqrt{x^2 + (\ell/2)^2}}{x} \right]_{\ell/2}^{3\ell/2} \right)$$

$$= \frac{-\rho_0^2\, \mathbf{a}_x}{2\pi\epsilon_0} \ln \left[\frac{(\ell/2)\left(1 + \sqrt{10}\right)}{3(\ell/2)\left(1 + \sqrt{2}\right)} \right] = \frac{\rho_0^2\, \mathbf{a}_x}{2\pi\epsilon_0} \ln \left[\frac{3(1 + \sqrt{2})}{1 + \sqrt{10}} \right] = \frac{0.55\rho_0^2}{2\pi\epsilon_0}\, \mathbf{a}_x \text{ N}$$

2.21. Two identical uniform line charges with $\rho_l = 75$ nC/m are located in free space at $x = 0$, $y = \pm 0.4$ m. What force per unit length does each line charge exert on the other? The charges are parallel to the z axis and are separated by 0.8 m. Thus the field from the charge at $y = -0.4$ evaluated at the location of the charge at $y = +0.4$ will be $\mathbf{E} = [\rho_l/(2\pi\epsilon_0(0.8))]\mathbf{a}_y$. The force on a differential length of the line at the positive y location is $d\mathbf{F} = dq\mathbf{E} = \rho_l dz\mathbf{E}$. Thus the force per unit length acting on the line at postive y arising from the charge at negative y is

$$\mathbf{F} = \int_0^1 \frac{\rho_l^2\,dz}{2\pi\epsilon_0(0.8)}\,\mathbf{a}_y = 1.26 \times 10^{-4}\,\mathbf{a}_y \text{ N/m} = \underline{126\,\mathbf{a}_y\ \mu\text{N/m}}$$

The force on the line at negative y is of course the same, but with $-\mathbf{a}_y$.

2.22. Two identical uniform sheet charges with $\rho_s = 100$ nC/m^2 are located in free space at $z = \pm 2.0$ cm. What force per unit area does each sheet exert on the other?

The field from the top sheet is $\mathbf{E} = -\rho_s/(2\epsilon_0)\,\mathbf{a}_z$ V/m. The differential force produced by this field on the bottom sheet is the charge density on the bottom sheet times the differential area there, multiplied by the electric field from the top sheet: $d\mathbf{F} = \rho_s da\mathbf{E}$. The force per unit area is then just $\mathbf{F} = \rho_s\mathbf{E} = (100 \times 10^{-9})(-100 \times 10^{-9})/(2\epsilon_0)\,\mathbf{a}_z = \underline{-5.6 \times 10^{-4}\,\mathbf{a}_z \text{ N/m}^2}$.

2.23. Given the surface charge density, $\rho_s = 2\,\mu$C/m^2, in the region $\rho < 0.2$ m, $z = 0$. Find $\mathbf{E}$ at:

a) $P_A(\rho = 0, z = 0.5)$: First, we recognize from symmetry that only a z component of $\mathbf{E}$ will be present. Considering a general point z on the z axis, we have $\mathbf{r} = z\mathbf{a}_z$. Then, with $\mathbf{r}' = \rho\mathbf{a}_\rho$, we obtain $\mathbf{r} - \mathbf{r}' = z\mathbf{a}_z - \rho\mathbf{a}_\rho$. The superposition integral for the z component of $\mathbf{E}$ will be:

$$E_{z,P_A} = \frac{\rho_s}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{0.2}\frac{z\,\rho\,d\rho\,d\phi}{(\rho^2+z^2)^{1.5}} = -\frac{2\pi\rho_s}{4\pi\epsilon_0}z\left[\frac{1}{\sqrt{z^2+\rho^2}}\right]_0^{0.2}$$
$$= \frac{\rho_s}{2\epsilon_0}z\left[\frac{1}{\sqrt{z^2}} - \frac{1}{\sqrt{z^2+0.04}}\right]$$

With $z = 0.5$ m, the above evaluates as $E_{z,P_A} = \underline{8.1 \text{ kV/m}}$.

b) $P_B(\rho = 0, z = -0.5)$. With z at -0.5 m, we evaluate the expression for E_z to obtain $E_{z,P_B} = \underline{-8.1 \text{ kV/m}}$.

c) Show that the field along the z axis reduces to that of an infinite sheet charge at small values of z: In general, the field can be expressed as

$$E_z = \frac{\rho_s}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+0.04}}\right]$$

At small z, this reduces to $E_z \doteq \rho_s/2\epsilon_0$, which is the infinite sheet charge field.

d) Show that the z axis field reduces to that of a point charge at large values of z: The development is as follows:

$$E_z = \frac{\rho_s}{2\epsilon_0}\left[1 - \frac{z}{\sqrt{z^2+0.04}}\right] = \frac{\rho_s}{2\epsilon_0}\left[1 - \frac{z}{z\sqrt{1+0.04/z^2}}\right] \doteq \frac{\rho_s}{2\epsilon_0}\left[1 - \frac{1}{1+(1/2)(0.04)/z^2}\right]$$

where the last approximation is valid if $z >> .04$. Continuing:

$$E_z \doteq \frac{\rho_s}{2\epsilon_0}\left[1 - [1 - (1/2)(0.04)/z^2]\right] = \frac{0.04\rho_s}{4\epsilon_0 z^2} = \frac{\pi(0.2)^2\rho_s}{4\pi\epsilon_0 z^2}$$

This the point charge field, where we identify $q = \pi(0.2)^2\rho_s$ as the total charge on the disk (which now looks like a point).

2.24. a) Find the electric field on the z axis produced by an annular ring of uniform surface charge density ρ_s in free space. The ring occupies the region $z = 0$, $a \le \rho \le b$, $0 \le \phi \le 2\pi$ in cylindrical coordinates: We find the field through

$$\mathbf{E} = \int\int \frac{\rho_s da(\mathbf{r}-\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^3}$$

where the integral is taken over the surface of the annular ring, and where $\mathbf{r} = z\mathbf{a}_z$ and $\mathbf{r}' = \rho\mathbf{a}_\rho$. The integral then becomes

$$\mathbf{E} = \int_0^{2\pi}\int_a^b \frac{\rho_s\,\rho\,d\rho\,d\phi\,(z\mathbf{a}_z - \rho\mathbf{a}_\rho)}{4\pi\epsilon_0(z^2+\rho^2)^{3/2}}$$

In evaluating this integral, we first note that the term involving $\rho\mathbf{a}_\rho$ integrates to zero over the ϕ integration range of 0 to 2π. This is because we need to introduce the ϕ dependence in $\mathbf{a}_\rho$ by writing it as $\mathbf{a}_\rho = \cos\phi\,\mathbf{a}_x + \sin\phi\,\mathbf{a}_y$, where $\mathbf{a}_x$ and $\mathbf{a}_y$ are invariant in their orientation as ϕ varies. So the integral now simplifies to

$$\mathbf{E} = \frac{2\pi\rho_s\,z\,\mathbf{a}_z}{4\pi\epsilon_0}\int_a^b \frac{\rho\,d\rho}{(z^2+\rho^2)^{3/2}} = \frac{\rho_s\,z\,\mathbf{a}_z}{2\epsilon_0}\left[\frac{-1}{\sqrt{z^2+\rho^2}}\right]_a^b$$

$$= \frac{\rho_s}{2\epsilon_0}\left[\frac{1}{\sqrt{1+(a/z)^2}} - \frac{1}{\sqrt{1+(b/z)^2}}\right]\mathbf{a}_z$$

b) from your part a result, obtain the field of an infinite uniform sheet charge by taking appropriate limits. The infinite sheet is obtained by letting $a \to 0$ and $b \to \infty$, in which case $\mathbf{E} \to \rho_s/(2\epsilon_0)\,\mathbf{a}_z$ as expected.

2.25. Find $\mathbf{E}$ at the origin if the following charge distributions are present in free space: point charge, 12 nC at $P(2,0,6)$; uniform line charge density, 3nC/m at $x = -2$, $y = 3$; uniform surface charge density, 0.2 nC/m^2 at $x = 2$. The sum of the fields at the origin from each charge in order is:

$$\mathbf{E} = \left[\frac{(12\times10^{-9})}{4\pi\epsilon_0}\,\frac{(-2\mathbf{a}_x - 6\mathbf{a}_z)}{(4+36)^{1.5}}\right] + \left[\frac{(3\times10^{-9})}{2\pi\epsilon_0}\,\frac{(2\mathbf{a}_x - 3\mathbf{a}_y)}{(4+9)}\right] - \left[\frac{(0.2\times10^{-9})\mathbf{a}_x}{2\epsilon_0}\right]$$
$$= \underline{-3.9\mathbf{a}_x - 12.4\mathbf{a}_y - 2.5\mathbf{a}_z \text{ V/m}}$$

2.27. Given the electric field $\mathbf{E} = (4x - 2y)\mathbf{a}_x - (2x + 4y)\mathbf{a}_y$, find:

a) the equation of the streamline that passes through the point $P(2, 3, -4)$: We write

$$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{-(2x+4y)}{(4x-2y)}$$

Thus

$$2(x\,dy + y\,dx) = y\,dy - x\,dx$$

or

$$2\,d(xy) = \frac{1}{2}\,d(y^2) - \frac{1}{2}\,d(x^2)$$

So

$$C_1 + 2xy = \frac{1}{2}y^2 - \frac{1}{2}x^2$$

or

$$y^2 - x^2 = 4xy + C_2$$

Evaluating at $P(2, 3, -4)$, obtain:

$$9 - 4 = 24 + C_2, \text{ or } C_2 = -19$$

Finally, at P, the requested equation is

$$\underline{y^2 - x^2 = 4xy - 19}$$

b) a unit vector specifying the direction of $\mathbf{E}$ at $Q(3, -2, 5)$: Have $\mathbf{E}_Q = [4(3) + 2(2)]\mathbf{a}_x - [2(3) - 4(2)]\mathbf{a}_y = 16\mathbf{a}_x + 2\mathbf{a}_y$. Then $|\mathbf{E}| = \sqrt{16^2 + 4} = 16.12$ So

$$\mathbf{a}_Q = \frac{16\mathbf{a}_x + 2\mathbf{a}_y}{16.12} = \underline{0.99\mathbf{a}_x + 0.12\mathbf{a}_y}$$

2.28 An electric dipole (discussed in detail in Sec. 4.7) consists of two point charges of equal and opposite magnitude $\pm Q$ spaced by distance d. With the charges along the z axis at positions $z = \pm d/2$ (with the positive charge at the positive z location), the electric field in spherical coordinates is given by $\mathbf{E}(r,\theta) = \left[Qd/(4\pi\epsilon_0 r^3)\right][2\cos\theta\mathbf{a}_r + \sin\theta\mathbf{a}_\theta]$, where $r >> d$. Using rectangular coordinates, determine expressions for the vector force on a point charge of magnitude q:

a) at $(0,0,z)$: Here, $\theta = 0$, $\mathbf{a}_r = \mathbf{a}_z$, and $r = z$. Therefore

$$\mathbf{F}(0,0,z) = \frac{qQd\,\mathbf{a}_z}{4\pi\epsilon_0 z^3} \text{ N}$$

b) at $(0,y,0)$: Here, $\theta = 90°$, $\mathbf{a}_\theta = -\mathbf{a}_z$, and $r = y$. The force is

$$\mathbf{F}(0,y,0) = \frac{-qQd\,\mathbf{a}_z}{4\pi\epsilon_0 y^3} \text{ N}$$

2.29. If $\mathbf{E} = 20e^{-5y}(\cos 5x\mathbf{a}_x - \sin 5x\mathbf{a}_y)$, find:

a) $|\mathbf{E}|$ at $P(\pi/6, 0.1, 2)$: Substituting this point, we obtain $\mathbf{E}_P = -10.6\mathbf{a}_x - 6.1\mathbf{a}_y$, and so $|\mathbf{E}_P| = \underline{12.2}$.

b) a unit vector in the direction of $\mathbf{E}_P$: The unit vector associated with $\mathbf{E}$ is $(\cos 5x\mathbf{a}_x - \sin 5x\mathbf{a}_y)$, which evaluated at P becomes $\mathbf{a}_E = \underline{-0.87\mathbf{a}_x - 0.50\mathbf{a}_y}$.

c) the equation of the direction line passing through P: Use

$$\frac{dy}{dx} = \frac{-\sin 5x}{\cos 5x} = -\tan 5x \quad \Rightarrow \quad dy = -\tan 5x\, dx$$

Thus $y = \frac{1}{5}\ln\cos 5x + C$. Evaluating at P, we find $C = 0.13$, and so

$$\underline{y = \frac{1}{5}\ln\cos 5x + 0.13}$$

2.30. For fields that do not vary with z in cylindrical coordinates, the equations of the streamlines are obtained by solving the differential equation $E_\rho/E_\phi = d\rho(\rho d\phi)$. Find the equation of the line passing through the point $(2, 30°, 0)$ for the field $\mathbf{E} = \rho\cos 2\phi\,\mathbf{a}_\rho - \rho\sin 2\phi\,\mathbf{a}_\phi$:

$$\frac{E_\rho}{E_\phi} = \frac{d\rho}{\rho d\phi} = \frac{-\rho\cos 2\phi}{\rho\sin 2\phi} = -\cot 2\phi \;\Rightarrow\; \frac{d\rho}{\rho} = -\cot 2\phi\, d\phi$$

Integrate to obtain

$$2\ln\rho = \ln\sin 2\phi + \ln C = \ln\left[\frac{C}{\sin 2\phi}\right] \;\Rightarrow\; \rho^2 = \frac{C}{\sin 2\phi}$$

At the given point, we have $4 = C/\sin(60°) \;\Rightarrow\; C = 4\sin 60° = 2\sqrt{3}$. Finally, the equation for the streamline is $\underline{\rho^2 = 2\sqrt{3}/\sin 2\phi}$.

CHAPTER 3

3.1. Suppose that the Faraday concentric sphere experiment is performed in free space using a central charge at the origin, Q_1, and with hemispheres of radius a. A second charge Q_2 (this time a point charge) is located at distance R from Q_1, where $R >> a$.

a) What is the force on the point charge before the hemispheres are assembled around Q_1? This will be simply the force beween two point charges, or

$$\mathbf{F} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}\,\mathbf{a}_r$$

b) What is the force on the point charge after the hemispheres are assembled but before they are discharged? The answer will be the <u>same as in part *a*</u> because induced charge Q_1 now resides as a surface charge layer on the sphere exterior. This produces the same electric field at the Q_2 location as before, and so the force acting on Q_2 is the same.

c) What is the force on the point charge after the hemispheres are assembled and after they are discharged? Discharging the hemispheres (connecting them to ground) neutralizes the positive outside surface charge layer, thus zeroing the net field outside the sphere. The force on Q_2 is now <u>zero</u>.

d) Qualitatively, describe what happens as Q_2 is moved toward the sphere assembly to the extent that the condition $R >> a$ is no longer valid. Q_2 itself begins to induce negative surface charge on the sphere. An attractive force thus begins to strengthen as the charge moves closer. The point charge field approximation used in parts *a* through *c* is no longer valid.

3.2. An electric field in free space is $\mathbf{E} = (5z^2/\epsilon_0)\,\hat{\mathbf{a}}_z$ V/m. Find the total charge contained within a cube, centered at the origin, of 4-m side length, in which all sides are parallel to coordinate axes (and therefore each side intersects an axis at ± 2.

The flux density is $\mathbf{D} = \epsilon_0 \mathbf{E} = 5z^2\,\mathbf{a}_z$. As $\mathbf{D}$ is z-directed only, it will intersect only the top and bottom surfaces (both parallel to the x-y plane). From Gauss' law, the charge in the cube is equal to the net outward flux of $\mathbf{D}$, which in this case is

$$Q_{encl} = \oint \mathbf{D}\cdot\mathbf{n}\,da = \int_{-2}^{2}\int_{-2}^{2} 5(2)^2\mathbf{a}_z\cdot\mathbf{a}_z\,dx\,dy + \int_{-2}^{2}\int_{-2}^{2} 5(-2)^2\mathbf{a}_z\cdot(-\mathbf{a}_z)\,dx\,dy = \underline{0}$$

where the first and second integrals on the far right are over the top and bottom surfaces respectively.

3.3. The cylindrical surface $\rho = 8$ cm contains the surface charge density, $\rho_s = 5e^{-20|z|}$ nC/m^2.

a) What is the total amount of charge present? We integrate over the surface to find:

$$Q = 2\int_0^\infty \int_0^{2\pi} 5e^{-20z}(.08)d\phi\, dz \text{ nC} = 20\pi(.08)\left(\frac{-1}{20}\right) e^{-20z}\Big|_0^\infty = \underline{0.25\,\text{nC}}$$

b) How much flux leaves the surface $\rho = 8\,\text{cm}$, $1\,\text{cm} < z < 5\text{cm}$, $30^\circ < \phi < 90^\circ$? We just integrate the charge density on that surface to find the flux that leaves it.

$$\Phi = Q' = \int_{.01}^{.05} \int_{30^\circ}^{90^\circ} 5e^{-20z}(.08)\, d\phi\, dz \text{ nC} = \left(\frac{90-30}{360}\right) 2\pi(5)(.08)\left(\frac{-1}{20}\right) e^{-20z}\Big|_{.01}^{.05}$$

$$= 9.45 \times 10^{-3}\,\text{nC} = \underline{9.45\,\text{pC}}$$

3.4. An electric field in free space is $\mathbf{E} = (5z^3/\epsilon_0)\,\hat{\mathbf{a}}_z$ V/m. Find the total charge contained within a sphere of 3-m radius, centered at the origin. Using Gauss' law, we set up the integral in free space over the sphere surface, whose outward unit normal is $\mathbf{a}_r$:

$$Q = \oint \epsilon_0 \mathbf{E}\cdot\mathbf{n}\, da = \int_0^{2\pi} \int_0^{\pi} 5z^3\, \mathbf{a}_z \cdot \mathbf{a}_r\, (3)^2 \sin\theta\, d\theta\, d\phi$$

where in this case $z = 3\cos\theta$ and (in all cases) $\mathbf{a}_z \cdot \mathbf{a}_r = \cos\theta$. These are substituted to yield

$$Q = 2\pi \int_0^{\pi} 5(3)^5 \cos^4\theta \sin\theta d\theta = -2\pi(5)(3)^5\left(\frac{1}{5}\right)\cos^5\theta\Big|_0^{2\pi} = \underline{972\pi}$$

3.5. Let $\mathbf{D} = 4xy\mathbf{a}_x + 2(x^2+z^2)\mathbf{a}_y + 4yz\mathbf{a}_z$ C/m^2 and evaluate surface integrals to find the total charge enclosed in the rectangular parallelepiped $0 < x < 2$, $0 < y < 3$, $0 < z < 5$ m: Of the 6 surfaces to consider, only 2 will contribute to the net outward flux. Why? First consider the planes at $y = 0$ and 3. The y component of $\mathbf{D}$ will penetrate those surfaces, but will be inward at $y = 0$ and outward at $y = 3$, while having the same magnitude in both cases. These fluxes will thus cancel. At the $x = 0$ plane, $D_x = 0$ and at the $z = 0$ plane, $D_z = 0$, so there will be no flux contributions from these surfaces. This leaves the 2 remaining surfaces at $x = 2$ and $z = 5$. The net outward flux becomes:

$$\Phi = \int_0^5 \int_0^3 \mathbf{D}\Big|_{x=2} \cdot \mathbf{a}_x\, dy\, dz + \int_0^3 \int_0^2 \mathbf{D}\Big|_{z=5} \cdot \mathbf{a}_z\, dx\, dy$$

$$= 5\int_0^3 4(2)y\, dy \;+\; 2\int_0^3 4(5)y\, dy = \underline{360 \text{ C}}$$

3.6. In free space, volume charge of constant density $\rho_v = \rho_0$ exists within the region $-\infty < x < \infty$, $-\infty < y < \infty$, and $-d/2 < z < d/2$. Find **D** and **E** everywhere.

From the symmetry of the configuration, we surmise that the field will be everywhere z-directed, and will be uniform with x and y at fixed z. For finding the field inside the charge, an appropriate Gaussian surface will be that which encloses a rectangular region defined by $-1 < x < 1$, $-1 < y < 1$, and $|z| < d/2$. The outward flux from this surface will be limited to that through the two parallel surfaces at $\pm z$:

$$\Phi_{in} = \oint \mathbf{D} \cdot d\mathbf{S} = 2\int_{-1}^{1}\int_{-1}^{1} D_z\, dxdy = Q_{encl} = \int_{-z}^{z}\int_{-1}^{1}\int_{-1}^{1} \rho_0\, dxdydz'$$

where the factor of 2 in the second integral account for the equal fluxes through the two surfaces. The above readily simplifies, as both D_z and ρ_0 are constants, leading to $\underline{\mathbf{D}_{in} = \rho_0 z\, \mathbf{a}_z \text{ C/m}^2 \quad (|z| < d/2)}$, and therefore $\underline{\mathbf{E}_{in} = (\rho_0 z/\epsilon_0)\, \mathbf{a}_z \text{ V/m} \quad (|z| < d/2)}$.

Outside the charge, the Gaussian surface is the same, except that the parallel boundaries at $\pm z$ occur at $|z| > d/2$. As a result, the calculation is nearly the same as before, with the only change being the limits on the total charge integral:

$$\Phi_{out} = \oint \mathbf{D} \cdot d\mathbf{S} = 2\int_{-1}^{1}\int_{-1}^{1} D_z\, dxdy = Q_{encl} = \int_{-d/2}^{d/2}\int_{-1}^{1}\int_{-1}^{1} \rho_0\, dxdydz'$$

Solve for D_z to find the constant values:

$$\mathbf{D}_{out} = \begin{cases} (\rho_0 d/2)\,\mathbf{a}_z & (z > d/2) \\ -(\rho_0 d/2)\,\mathbf{a}_z & (z < d/2) \end{cases} \text{ C/m}^2 \quad \text{and} \quad \mathbf{E}_{out} = \begin{cases} (\rho_0 d/2\epsilon_0)\,\mathbf{a}_z & (z > d/2) \\ -(\rho_0 d/2\epsilon_0)\,\mathbf{a}_z & (z < d/2) \end{cases} \text{ V/m}$$

3.7. Volume charge density is located in free space as $\rho_v = 2e^{-1000r}$ nC/m^3 for $0 < r < 1$ mm, and $\rho_v = 0$ elsewhere.

a) Find the total charge enclosed by the spherical surface $r = 1$ mm: To find the charge we integrate:

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_0^{.001} 2e^{-1000r} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Integration over the angles gives a factor of 4π. The radial integration we evaluate using tables; we obtain

$$Q = 8\pi\left[\frac{-r^2e^{-1000r}}{1000}\Big|_0^{.001} + \frac{2}{1000}\frac{e^{-1000r}}{(1000)^2}(-1000r - 1)\Big|_0^{.001}\right] = \underline{4.0 \times 10^{-9}\,\text{nC}}$$

b) By using Gauss's law, calculate the value of D_r on the surface $r = 1$ mm: The gaussian surface is a spherical shell of radius 1 mm. The enclosed charge is the result of part a. We thus write $4\pi r^2 D_r = Q$, or

$$D_r = \frac{Q}{4\pi r^2} = \frac{4.0 \times 10^{-9}}{4\pi(.001)^2} = \underline{3.2 \times 10^{-4}\text{ nC/m}^2}$$

3.8. Use Gauss's law in integral form to show that an inverse distance field in spherical coordinates, $\mathbf{D} = A\mathbf{a}_r/r$, where A is a constant, requires every spherical shell of 1 m thickness to contain $4\pi A$ coulombs of charge. Does this indicate a continuous charge distribution? If so, find the charge density variation with r.

The net outward flux of this field through a spherical surface of radius r is

$$\Phi = \oint \mathbf{D} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^{\pi} \frac{A}{r}\mathbf{a}_r \cdot \mathbf{a}_r \, r^2 \sin\theta \, d\theta \, d\phi = 4\pi A r = Q_{encl}$$

We see from this that with every increase in r by one m, the enclosed charge increases by $4\pi A$ (done). It is evident that the charge density is continuous, and we can find the density indirectly by constructing the integral for the enclosed charge, in which we already found the latter from Gauss's law:

$$Q_{encl} = 4\pi A r = \int_0^{2\pi} \int_0^{\pi} \int_0^{r} \rho(r')\,(r')^2 \sin\theta \, dr' \, d\theta \, d\phi = 4\pi \int_0^{r} \rho(r')\,(r')^2 \, dr'$$

To obtain the correct enclosed charge, the integrand must be $\rho(r) = \underline{A/r^2}$.

3.9. A uniform volume charge density of $80\,\mu\text{C/m}^3$ is present throughout the region $8\,\text{mm} < r < 10\,\text{mm}$. Let $\rho_v = 0$ for $0 < r < 8\,\text{mm}$.

a) Find the total charge inside the spherical surface $r = 10$ mm: This will be

$$Q = \int_0^{2\pi} \int_0^{\pi} \int_{.008}^{.010} (80 \times 10^{-6}) r^2 \sin\theta \, dr \, d\theta \, d\phi = 4\pi \times (80 \times 10^{-6}) \frac{r^3}{3}\Big|_{.008}^{.010}$$
$$= 1.64 \times 10^{-10}\,\text{C} = \underline{164\,\text{pC}}$$

b) Find D_r at $r = 10$ mm: Using a spherical gaussian surface at $r = 10$, Gauss' law is written as $4\pi r^2 D_r = Q = 164 \times 10^{-12}$, or

$$D_r(10\,\text{mm}) = \frac{164 \times 10^{-12}}{4\pi(.01)^2} = 1.30 \times 10^{-7}\,\text{C/m}^2 = \underline{130\,\text{nC/m}^2}$$

c) If there is no charge for $r > 10$ mm, find D_r at $r = 20$ mm: This will be the same computation as in part b, except the gaussian surface now lies at 20 mm. Thus

$$D_r(20\,\text{mm}) = \frac{164 \times 10^{-12}}{4\pi(.02)^2} = 3.25 \times 10^{-8}\,\text{C/m}^2 = \underline{32.5\,\text{nC/m}^2}$$

3.10. An infinitely long cylindrical dielectric of radius b contains charge within its volume of density $\rho_v = a\rho^2$, where a is a constant. Find the electric field strength, $\mathbf{E}$, both inside and outside the cylinder.

Inside, we note from symmetry that $\mathbf{D}$ will be radially-directed, in the manner of a line charge field. So we apply Gauss' law to a cylindrical surface of radius ρ, concentric with the charge distribution, having unit length in z, and where $\rho < b$. The outward normal to the surface is $\mathbf{a}_\rho$.

$$\oint \mathbf{D} \cdot \mathbf{n}\, da = \int_0^1 \int_0^{2\pi} D_\rho\, \mathbf{a}_\rho \cdot \mathbf{a}_\rho\, \rho\, d\phi\, dz = Q_{encl} = \int_0^1 \int_0^{2\pi} \int_0^{\rho} a(\rho')^2 \rho'\, d\rho'\, d\phi\, dz$$

in which the dummy variable ρ' must be used in the far-right integral because the upper radial limit is ρ. D_ρ is constant over the surface and can be factored outside the integral. Evaluating both integrals leads to

$$2\pi(1)\rho D_\rho = 2\pi a \left(\frac{1}{4}\right)\rho^4 \;\Rightarrow\; D_\rho = \frac{a\rho^3}{4} \quad \text{or} \quad \underline{\mathbf{E}_{in} = \frac{a\rho^3}{4\epsilon_0}\, \mathbf{a}_\rho \;\; (\rho < b)}$$

To find the field outside the cylinder, we apply Gauss' law to a cylinder of radius $\rho > b$. The setup now changes only by the upper radius limit for the charge integral, which is now the charge radius, b:

$$\oint \mathbf{D} \cdot \mathbf{n}\, da = \int_0^1 \int_0^{2\pi} D_\rho\, \mathbf{a}_\rho \cdot \mathbf{a}_\rho\, \rho\, d\phi\, dz = Q_{encl} = \int_0^1 \int_0^{2\pi} \int_0^{b} a\rho^2 \rho\, d\rho\, d\phi\, dz$$

where the dummy variable is no longer needed. Evaluating as before, the result is

$$D_\rho = \frac{ab^4}{4\rho} \quad \text{or} \quad \underline{\mathbf{E}_{out} = \frac{ab^4}{4\epsilon_0 \rho}\, \mathbf{a}_\rho \;\; (\rho > b)}$$

3.11. In cylindrical coordinates, let $\rho_v = 0$ for $\rho < 1$ mm, $\rho_v = 2\sin(2000\pi\rho)$ nC/m^3 for $1\,\text{mm} < \rho < 1.5\,\text{mm}$, and $\rho_v = 0$ for $\rho > 1.5\,\text{mm}$. Find $\mathbf{D}$ everywhere: Since the charge varies only with radius, and is in the form of a cylinder, symmetry tells us that the flux density will be radially-directed and will be constant over a cylindrical surface of a fixed radius. Gauss' law applied to such a surface of unit length in z gives:

a) for $\rho < 1$ mm, $\underline{D_\rho = 0}$, since no charge is enclosed by a cylindrical surface whose radius lies within this range.

b) for $1\,\text{mm} < \rho < 1.5\,\text{mm}$, we have

$$2\pi\rho D_\rho = 2\pi \int_{.001}^{\rho} 2 \times 10^{-9} \sin(2000\pi\rho')\rho'\, d\rho'$$
$$= 4\pi \times 10^{-9} \left[\frac{1}{(2000\pi)^2}\sin(2000\pi\rho) - \frac{\rho}{2000\pi}\cos(2000\pi\rho)\right]_{.001}^{\rho}$$

or finally,

$$D_\rho = \frac{10^{-15}}{2\pi^2\rho}\Big[\sin(2000\pi\rho) + 2\pi\left[1 - 10^3\rho\cos(2000\pi\rho)\right]\Big] \;\text{C/m}^2 \quad (1\,\text{mm} < \rho < 1.5\,\text{mm})$$

3.11c) for $\rho > 1.5\,\text{mm}$, the gaussian cylinder now lies at radius ρ *outside* the charge distribution, so the integral that evaluates the enclosed charge now includes the entire charge distribution. To accomplish this, we change the upper limit of the integral of part b from ρ to 1.5 mm, finally obtaining:

$$D_\rho = \frac{2.5 \times 10^{-15}}{\pi\rho} \text{ C/m}^2 \quad (\rho > 1.5\,\text{mm})$$

3.12. The sun radiates a total power of about 3.86×10^{26} watts (W). If we imagine the sun's surface to be marked off in latitude and longitude and assume uniform radiation,

a) What power is radiated by the region lying between latitude 50° N and 60° N and longitude 12° W and 27° W?

50° N lattitude and 60° N lattitude correspond respectively to $\theta = 40°$ and $\theta = 30°$. 12° and 27° correspond directly to the limits on ϕ. Since the sun for our purposes is spherically-symmetric, the flux density emitted by it is $\mathbf{I} = 3.86 \times 10^{26}/(4\pi r^2)\,\mathbf{a}_r$ W/m^2. The required power is now found through

$$P_1 = \int_{12°}^{27°}\int_{30°}^{40°} \frac{3.86 \times 10^{26}}{4\pi r^2}\,\mathbf{a}_r \cdot \mathbf{a}_r\, r^2 \sin\theta\, d\theta\, d\phi$$
$$= \frac{3.86 \times 10^{26}}{4\pi}\left[\cos(30°) - \cos(40°)\right](27° - 12°)\left(\frac{2\pi}{360}\right) = \underline{8.1 \times 10^{23} \text{ W}}$$

b) What is the power density on a spherical surface 93,000,000 miles from the sun in W/m^2?

First, 93,000,000 miles = 155,000,000 km = 1.55×10^{11} m. Use this distance in the flux density expression above to obtain

$$\mathbf{I} = \frac{3.86 \times 10^{26}}{4\pi(1.55 \times 10^{11})^2}\,\mathbf{a}_r = \underline{1200\,\mathbf{a}_r \text{ W/m}^2}$$

3.13. Spherical surfaces at $r = 2$, 4, and 6 m carry uniform surface charge densities of 20 nC/m^2, -4 nC/m^2, and ρ_{s0}, respectively.

a) Find $\mathbf{D}$ at $r = 1$, 3 and 5 m: Noting that the charges are spherically-symmetric, we ascertain that $\mathbf{D}$ will be radially-directed and will vary only with radius. Thus, we apply Gauss' law to spherical shells in the following regions: $r < 2$: Here, no charge is enclosed, and so $\underline{D_r = 0}$.

$$2 < r < 4: \quad 4\pi r^2 D_r = 4\pi(2)^2(20 \times 10^{-9}) \quad \Rightarrow \quad D_r = \frac{80 \times 10^{-9}}{r^2} \text{ C/m}^2$$

So $D_r(r = 3) = \underline{8.9 \times 10^{-9} \text{ C/m}^2}$.

$$4 < r < 6: \quad 4\pi r^2 D_r = 4\pi(2)^2(20 \times 10^{-9}) + 4\pi(4)^2(-4 \times 10^{-9}) \quad \Rightarrow \quad D_r = \frac{16 \times 10^{-9}}{r^2}$$

So $D_r(r = 5) = \underline{6.4 \times 10^{-10} \text{ C/m}^2}$.

b) Determine ρ_{s0} such that $\mathbf{D} = 0$ at $r = 7$ m. Since fields will decrease as $1/r^2$, the question could be re-phrased to ask for ρ_{s0} such that $\mathbf{D} = 0$ at *all* points where $r > 6$ m. In this region, the total field will be

$$D_r(r > 6) = \frac{16 \times 10^{-9}}{r^2} + \frac{\rho_{s0}(6)^2}{r^2}$$

Requiring this to be zero, we find $\rho_{s0} = \underline{-(4/9) \times 10^{-9} \text{ C/m}^2}$.

3.14. A certain light-emitting diode (LED) is centered at the origin with its surface in the xy plane. At far distances, the LED appears as a point, but the glowing surface geometry produces a far-field radiation pattern that follows a raised cosine law: That is, the optical power (flux) density in Watts/m^2 is given in spherical coordinates by

$$\mathbf{P}_d = P_0 \frac{\cos^2\theta}{2\pi r^2}\,\mathbf{a}_r \qquad \text{Watts/m}^2$$

where θ is the angle measured with respect to the normal to the LED surface (in this case, the z axis), and r is the radial distance from the origin at which the power is detected.

a) Find, in terms of P_0, the total power in Watts emitted in the upper half-space by the LED: We evaluate the surface integral of the power density over a hemispherical surface of radius r:

$$P_t = \int_0^{2\pi}\int_0^{\pi/2} P_0 \frac{\cos^2\theta}{2\pi r^2}\,\mathbf{a}_r \cdot \mathbf{a}_r\, r^2 \sin\theta\, d\theta\, d\phi = -\frac{P_0}{3}\cos^3\theta\Big|_0^{\pi/2} = \underline{\frac{P_0}{3}}$$

b) Find the cone angle, θ_1, within which half the total power is radiated; i.e., within the range $0 < \theta < \theta_1$: We perform the same integral as in part a except the upper limit for θ is now θ_1. The result must be one-half that of part a, so we write:

$$\frac{P_t}{2} = \frac{P_0}{6} = -\frac{P_0}{3}\cos^3\theta\Big|_0^{\theta_1} = \frac{P_0}{3}\left(1 - \cos^3\theta_1\right) \;\Rightarrow\; \theta_1 = \cos^{-1}\left(\frac{1}{2^{1/3}}\right) = \underline{37.5^\circ}$$

c) An optical detector, having a 1 mm^2 cross-sectional area, is positioned at $r = 1$ m and at $\theta = 45^\circ$, such that it faces the LED. If one nanowatt (stated in error as 1mW) is measured by the detector, what (to a very good estimate) is the value of P_0? Start with

$$\mathbf{P}_d(45^\circ) = P_0 \frac{\cos^2(45^\circ)}{2\pi r^2}\mathbf{a}_r = \frac{P_0}{4\pi r^2}\mathbf{a}_r$$

Then the detected power in a 1-mm^2 area at $r = 1$ m approximates as

$$P[W] \doteq \frac{P_0}{4\pi} \times 10^{-6} = 10^{-9} \;\Rightarrow\; P_0 \doteq \underline{4\pi \times 10^{-3}\ \text{W}}$$

If the originally stated 1mW value is used for the detected power, the answer would have been 4π kW (!).

3.15. Volume charge density is located as follows: $\rho_v = 0$ for $\rho < 1$ mm and for $\rho > 2$ mm, $\rho_v = 4\rho\ \mu$C/m^3 for $1 < \rho < 2$ mm.

a) Calculate the total charge in the region $0 < \rho < \rho_1$, $0 < z < L$, where $1 < \rho_1 < 2$ mm: We find,

$$Q = \int_0^L \int_0^{2\pi} \int_{.001}^{\rho_1} 4\rho\, \rho\, d\rho\, d\phi\, dz = \underline{\frac{8\pi L}{3}[\rho_1^3 - 10^{-9}]\ \mu\text{C}}$$

b) Use Gauss' law to determine D_ρ at $\rho = \rho_1$: Gauss' law states that $2\pi\rho_1 L D_\rho = Q$, where Q is the result of part a. So, with ρ_1 in meters,

$$D_\rho(\rho_1) = \underline{\frac{4(\rho_1^3 - 10^{-9})}{3\rho_1}\ \mu\text{C/m}^2}$$

3.15c) Evaluate D_ρ at $\rho = 0.8\,\text{mm},\ 1.6\,\text{mm}$, and $2.4\,\text{mm}$: At $\rho = 0.8\,\text{mm}$, no charge is enclosed by a cylindrical gaussian surface of that radius, so $D_\rho(0.8\text{mm}) = \underline{0}$. At $\rho = 1.6\,\text{mm}$, we evaluate the part b result at $\rho_1 = 1.6$ to obtain:

$$D_\rho(1.6\text{mm}) = \frac{4[(.0016)^3 - (.0010)^3]}{3(.0016)} = \underline{3.6 \times 10^{-6}\ \mu\text{C/m}^2}$$

At $\rho = 2.4$, we evaluate the charge integral of part a from .001 to .002, and Gauss' law is written as

$$2\pi\rho L D_\rho = \frac{8\pi L}{3}[(.002)^2 - (.001)^2]\ \mu\text{C}$$

from which $D_\rho(2.4\text{mm}) = \underline{3.9 \times 10^{-6}\ \mu\text{C/m}^2}$.

3.16. An electric flux density is given by $\mathbf{D} = D_0\,\mathbf{a}_\rho$, where D_0 is a given constant.

a) What charge density generates this field? Charge density is found by taking the divergence: With radial $\mathbf{D}$ only, we have

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{\rho}\frac{d}{d\rho}(\rho D_0) = \frac{D_0}{\rho}\ \text{C/m}^3$$

b) For the specified field, what total charge is contained within a cylinder of radius a and height b, where the cylinder axis is the z axis? We can either integrate the charge density over the specified volume, or integrate $\mathbf{D}$ over the surface that contains the specified volume:

$$Q = \int_0^b \int_0^{2\pi} \int_0^a \frac{D_0}{\rho}\,\rho\,d\rho\,d\phi\,dz = \int_0^b \int_0^{2\pi} D_0\,\mathbf{a}_\rho \cdot \mathbf{a}_\rho\,a\,d\phi\,dz = \underline{2\pi a b D_0}\ \text{C}$$

3.17. A cube is defined by $1 < x, y, z < 1.2$. If $\mathbf{D} = 2x^2y\mathbf{a}_x + 3x^2y^2\mathbf{a}_y\ \text{C/m}^2$:

a) apply Gauss' law to find the total flux leaving the closed surface of the cube. We call the surfaces at $x = 1.2$ and $x = 1$ the front and back surfaces respectively, those at $y = 1.2$ and $y = 1$ the right and left surfaces, and those at $z = 1.2$ and $z = 1$ the top and bottom surfaces. To evaluate the total charge, we integrate $\mathbf{D} \cdot \mathbf{n}$ over all six surfaces and sum the results. We note that there is no z component of $\mathbf{D}$, so there will be no outward flux contributions from the top and bottom surfaces. The fluxes through the remaining four are

$$\Phi = Q = \oint \mathbf{D} \cdot \mathbf{n}\,da = \underbrace{\int_1^{1.2}\int_1^{1.2} 2(1.2)^2 y\,dy\,dz}_{\text{front}} + \underbrace{\int_1^{1.2}\int_1^{1.2} -2(1)^2 y\,dy\,dz}_{\text{back}}$$
$$+ \underbrace{\int_1^{1.2}\int_1^{1.2} -3x^2(1)^2\,dx\,dz}_{\text{left}} + \underbrace{\int_1^{1.2}\int_1^{1.2} 3x^2(1.2)^2\,dx\,dz}_{\text{right}} = \underline{0.1028\,\text{C}}$$

b) evaluate $\nabla \cdot \mathbf{D}$ at the center of the cube: This is

$$\nabla \cdot \mathbf{D} = \left[4xy + 6x^2y\right]_{(1.1,1.1)} = 4(1.1)^2 + 6(1.1)^3 = \underline{12.83}$$

c) Estimate the total charge enclosed within the cube by using Eq. (8): This is

$$Q \doteq \nabla \cdot \mathbf{D}\Big|_{\text{center}} \times \Delta v = 12.83 \times (0.2)^3 = \underline{0.1026}\ \text{Close!}$$

3.18. State whether the divergence of the following vector fields is positive, negative, or zero:

a) the thermal energy flow in $\mathrm{J/(m^2-s)}$ at any point in a freezing ice cube: One way to visualize this is to consider that heat is escaping through the surface of the ice cube as it freezes. Therefore the net outward flux of thermal energy through the surface is positive. Calling the thermal flux density $\mathbf{F}$, the divergence theorem says

$$\oint_s \mathbf{F}\cdot d\mathbf{S} = \int_v \nabla\cdot\mathbf{F}\,dv$$

and so if we identify the left integral as positive, the right integral (and its integrand) must also be positive. Answer: positive.

b) the current density in $\mathrm{A/m^2}$ in a bus bar carrying direct current: In this case, we have no accumulation or dissipation of charge within any small volume, since current is dc; this also means that the net outward current flux through the surface that surrounds any small volume is zero. Therefore the divergence must be zero.

c) the mass flow rate in $\mathrm{kg/(m^2-s)}$ below the surface of water in a basin, in which the water is circulating clockwise as viewed from above: Here again, taking any small volume in the water, the net outward flow through the surface that surrounds the small volume is zero; i.e., there is no accumulation or dissipation of mass that would result in a change in density at any point. Divergence is therefore zero.

3.19. A spherical surface of radius 3 mm is centered at $P(4,1,5)$ in free space. Let $\mathbf{D} = x\mathbf{a}_x$ C/m^2. Use the results of Sec. 3.4 to estimate the net electric flux leaving the spherical surface: We use $\Phi \doteq \nabla\cdot\mathbf{D}\Delta v$, where in this case $\nabla\cdot\mathbf{D} = (\partial/\partial x)x = 1$ C/m^3. Thus

$$\Phi \doteq \frac{4}{3}\pi(.003)^3(1) = 1.13\times10^{-7}\,\mathrm{C} = \underline{113\,\mathrm{nC}}$$

3.20. A radial electric field distribution in free space is given in spherical coordinates as:

$$\begin{aligned}
\mathbf{E}_1 &= \frac{r\rho_0}{3\epsilon_0}\,\mathbf{a}_r \qquad (r\le a)\\
\mathbf{E}_2 &= \frac{(2a^3-r^3)\rho_0}{3\epsilon_0\,r^2}\,\mathbf{a}_r \qquad (a\le r\le b)\\
\mathbf{E}_3 &= \frac{(2a^3-b^3)\rho_0}{3\epsilon_0\,r^2}\,\mathbf{a}_r \qquad (r\ge b)
\end{aligned}$$

where ρ_0, a, and b are constants.

a) Determine the volume charge density in the entire region $(0\le r\le\infty)$ by appropriate use of $\nabla\cdot\mathbf{D} = \rho_v$. We find ρ_v by taking the divergence of $\mathbf{D}$ in all three regions, where $\mathbf{D} = \epsilon_0\mathbf{E}$. As $\mathbf{D}$ has only a radial component, the divergences become:

$$\rho_{v1} = \nabla\cdot\mathbf{D}_1 = \frac{1}{r^2}\frac{d}{dr}\left(r^2D_1\right) = \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^3\rho_0}{3}\right) = \underline{\rho_0} \qquad (r\le a)$$

$$\rho_{v2} = \frac{1}{r^2}\frac{d}{dr}\left(r^2D_2\right) = \frac{1}{r^2}\frac{d}{dr}\left(\frac{1}{3}(2a^3-r^3)\rho_0\right) = \underline{-\rho_0} \qquad (a\le r\le b)$$

$$\rho_{v3} = \frac{1}{r^2}\frac{d}{dr}\left(r^2D_3\right) = \frac{1}{r^2}\frac{d}{dr}\left(\frac{1}{3}(2a^3-b^3)\rho_0\right) = \underline{0} \qquad (r\ge b)$$

3.20b) Find, in terms of given parameters, the total charge, Q, within a sphere of radius r where $r > b$. We integrate the charge densities (piecewise) over the spherical volume of radius b:

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho_0\, r^2 \sin\theta\, dr\, d\theta\, d\phi - \int_0^{2\pi}\int_0^{\pi}\int_a^{b} \rho_0\, r^2 \sin\theta\, dr\, d\theta\, d\phi = \underline{\frac{4}{3}\pi\left(2a^3 - b^3\right)\rho_0}$$

3.21. Calculate the divergence of $\mathbf{D}$ at the point specified if

a) $\mathbf{D} = (1/z^2)\left[10xyz\,\mathbf{a}_x + 5x^2 z\,\mathbf{a}_y + (2z^3 - 5x^2 y)\,\mathbf{a}_z\right]$ at $P(-2,3,5)$: We find

$$\nabla\cdot\mathbf{D} = \left[\frac{10y}{z} + 0 + 2 + \frac{10x^2 y}{z^3}\right]_{(-2,3,5)} = \underline{8.96}$$

b) $\mathbf{D} = 5z^2\mathbf{a}_\rho + 10\rho z\,\mathbf{a}_z$ at $P(3,-45°,5)$: In cylindrical coordinates, we have

$$\nabla\cdot\mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial\phi} + \frac{\partial D_z}{\partial z} = \left[\frac{5z^2}{\rho} + 10\rho\right]_{(3,-45°,5)} = \underline{71.67}$$

c) $\mathbf{D} = 2r\sin\theta\sin\phi\,\mathbf{a}_r + r\cos\theta\sin\phi\,\mathbf{a}_\theta + r\cos\phi\,\mathbf{a}_\phi$ at $P(3,45°,-45°)$: In spherical coordinates, we have

$$\nabla\cdot\mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi}$$
$$= \left[6\sin\theta\sin\phi + \frac{\cos 2\theta\sin\phi}{\sin\theta} - \frac{\sin\phi}{\sin\theta}\right]_{(3,45°,-45°)} = \underline{-2}$$

3.22. (a) A flux density field is given as $\mathbf{F}_1 = 5\mathbf{a}_z$. Evaluate the outward flux of $\mathbf{F}_1$ through the hemispherical surface, $r = a$, $0 < \theta < \pi/2$, $0 < \phi < 2\pi$.

The flux integral is

$$\Phi_1 = \int_{hem.} \mathbf{F}_1\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi/2} 5\,\underbrace{\mathbf{a}_z\cdot\mathbf{a}_r}_{\cos\theta}\, a^2\sin\theta\, d\theta\, d\phi = -2\pi(5)a^2\frac{\cos^2\theta}{2}\Big|_0^{\pi/2} = \underline{5\pi a^2}$$

b) What simple observation would have saved a lot of work in part a? The field is constant, and so the inward flux through the base of the hemisphere (of area πa^2) would be equal in magnitude to the outward flux through the upper surface (the flux through the base is a much easier calculation).

c) Now suppose the field is given by $\mathbf{F}_2 = 5z\mathbf{a}_z$. Using the appropriate surface integrals, evaluate the net outward flux of $\mathbf{F}_2$ through the closed surface consisting of the hemisphere of part a and its circular base in the xy plane:

Note that the integral over the base is zero, since $\mathbf{F}_2 = 0$ there. The remaining flux integral is that over the hemisphere:

$$\Phi_2 = \int_0^{2\pi}\int_0^{\pi/2} 5z\,\mathbf{a}_z\cdot\mathbf{a}_r\, a^2\sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi/2} 5(a\cos\theta)\cos\theta\, a^2\sin\theta\, d\theta\, d\phi$$
$$= 10\pi a^3\int_0^{\pi/2}\cos^2\theta\sin\theta\, d\theta\, d\phi = -\frac{10}{3}\pi a^3\cos^3\theta\Big|_0^{\pi/2} = \underline{\frac{10}{3}\pi a^3}$$

d) Repeat part c by using the divergence theorem and an appropriate volume integral:

The divergence of $\mathbf{F}_2$ is just $dF_2/dz = 5$. We then integrate this over the hemisphere volume, which in this case involves just multiplying 5 by $(2/3)\pi a^3$, giving the same answer as in part c.

3.23. a) A point charge Q lies at the origin. Show that div $\mathbf{D}$ is zero everywhere except at the origin. For a point charge at the origin we know that $\mathbf{D} = Q/(4\pi r^2)\,\mathbf{a}_r$. Using the formula for divergence in spherical coordinates (see problem 3.21 solution), we find in this case that

$$\nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{Q}{4\pi r^2}\right) = 0$$

The above is true provided $r > 0$. When $r = 0$, we have a singularity in $\mathbf{D}$, so its divergence is not defined.

b) Replace the point charge with a uniform volume charge density ρ_{v0} for $0 < r < a$. Relate ρ_{v0} to Q and a so that the total charge is the same. Find div $\mathbf{D}$ everywhere: To achieve the same net charge, we require that $(4/3)\pi a^3 \rho_{v0} = Q$, so $\underline{\rho_{v0} = 3Q/(4\pi a^3)\ \text{C/m}^3}$. Gauss' law tells us that inside the charged sphere

$$4\pi r^2 D_r = \frac{4}{3}\pi r^3 \rho_{v0} = \frac{Qr^3}{a^3}$$

Thus

$$D_r = \frac{Qr}{4\pi a^3}\ \text{C/m}^2 \text{ and } \nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}\left(\frac{Qr^3}{4\pi a^3}\right) = \frac{3Q}{4\pi a^3}$$

as expected. Outside the charged sphere, $\mathbf{D} = Q/(4\pi r^2)\,\mathbf{a}_r$ as before, and the divergence is zero.

3.24. In a region in free space, electric flux density is found to be:

$$\mathbf{D} = \begin{cases} \rho_0(z + 2d)\,\mathbf{a}_z \ \ \text{C/m}^2 & (-2d \le z \le 0) \\ -\rho_0(z - 2d)\,\mathbf{a}_z \ \ \text{C/m}^2 & (0 \le z \le 2d) \end{cases}$$

Everywhere else, $\mathbf{D} = 0$.

a) Using $\nabla \cdot \mathbf{D} = \rho_v$, find the volume charge density as a function of position everywhere: Use

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{dD_z}{dz} = \begin{cases} \rho_0 & (-2d \le z \le 0) \\ -\rho_0 & (0 \le z \le 2d) \end{cases}$$

b) determine the electric flux that passes through the surface defined by $z = 0$, $-a \le x \le a$, $-b \le y \le b$: In the x-y plane, $\mathbf{D}$ evaluates as the constant $\mathbf{D}(0) = 2d\rho_0\,\mathbf{a}_z$. Therefore the flux passing through the given area will be

$$\Phi = \int_{-a}^{a}\int_{-b}^{b} 2d\rho_0\, dx dy = \underline{8abd\,\rho_0}\ \text{C}$$

c) determine the total charge contained within the region $-a \le x \le a$, $-b \le y \le b$, $-d \le z \le d$: From part a, we have equal and opposite charge densities above and below the x-y plane. This means that within a region having equal volumes above and below the plane, the net charge is <u>zero</u>.

d) determine the total charge contained within the region $-a \le x \le a$, $-b \le y \le b$, $0 \le z \le 2d$. In this case,

$$Q = -\rho_0\,(2a)\,(2b)\,(2d) = \underline{-8abd\,\rho_0}\ \text{C}$$

This is equivalent to the net *inward* flux of $\mathbf{D}$ into the volume, as was found in part b.

3.25. Within the spherical shell, $3 < r < 4$ m, the electric flux density is given as

$$\mathbf{D} = 5(r-3)^3\,\mathbf{a}_r \text{ C/m}^2$$

a) What is the volume charge density at $r = 4$? In this case we have

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}(r^2 D_r) = \frac{5}{r}(r-3)^2(5r-6) \text{ C/m}^3$$

which we evaluate at $r = 4$ to find $\rho_v(r=4) = \underline{17.50 \text{ C/m}^3}$.

b) What is the electric flux density at $r = 4$? Substitute $r = 4$ into the given expression to find $\mathbf{D}(4) = \underline{5\,\mathbf{a}_r \text{ C/m}^2}$

c) How much electric flux leaves the sphere $r = 4$? Using the result of part b, this will be $\Phi = 4\pi(4)^2(5) = \underline{320\pi \text{ C}}$

d) How much charge is contained within the sphere, $r = 4$? From Gauss' law, this will be the same as the outward flux, or again, $Q = \underline{320\pi \text{ C}}$.

3.26. If we have a perfect gas of mass density ρ_m kg/m^3, and assign a velocity $\mathbf{U}$ m/s to each differential element, then the mass flow rate is $\rho_m \mathbf{U}$ kg/(m^2 – s). Physical reasoning then leads to the *continuity equation*, $\nabla \cdot (\rho_m \mathbf{U}) = -\partial \rho_m / \partial t$.

a) Explain in words the physical interpretation of this equation: The quantity $\rho_m \mathbf{U}$ is the flow (or flux) density of mass. Then the divergence of $\rho_m \mathbf{U}$ is the outward mass flux per unit volume at a point. This must be equivalent to the rate of depletion of mass per unit volume at the same point, as the continuity equation states.

b) Show that $\oint_s \rho_m \mathbf{U} \cdot d\mathbf{S} = -dM/dt$, where M is the total mass of the gas within the constant closed surface, S, and explain the physical significance of the equation.

Applying the divergence theorem, we have

$$\oint_s \rho_m \mathbf{U} \cdot d\mathbf{S} = \int_v \nabla \cdot (\rho_m \mathbf{U})\, dv = \int_v -\frac{\partial \rho_m}{\partial t}\, dv = -\frac{d}{dt}\int_v \rho_m\, dv = -\frac{dM}{dt}$$

This states in large-scale form what was already stated in part a. That is – the net outward mass flow (in kg/s) through a closed surface is equal to the negative time rate of change in total mass within the enclosed volume.

3.27. Let $\mathbf{D} = 5.00r^2\mathbf{a}_r$ mC/m^2 for $r \le 0.08$ m and $\mathbf{D} = 0.205\,\mathbf{a}_r/r^2$ μC/m^2 for $r \ge 0.08$ m (note error in problem statement).

a) Find ρ_v for $r = 0.06$ m: This radius lies within the first region, and so

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}(r^2 D_r) = \frac{1}{r^2}\frac{d}{dr}(5.00r^4) = 20r \text{ mC/m}^3$$

which when evaluated at $r = 0.06$ yields $\rho_v(r = .06) = \underline{1.20 \text{ mC/m}^3}$.

b) Find ρ_v for $r = 0.1$ m: This is in the region where the second field expression is valid. The $1/r^2$ dependence of this field yields a zero divergence (shown in Problem 3.23), and so the volume charge density is $\underline{\text{zero}}$ at 0.1 m.

3.27c) What surface charge density could be located at $r = 0.08$ m to cause $\mathbf{D} = 0$ for $r > 0.08$ m?

The total surface charge should be equal and opposite to the total volume charge. The latter is

$$Q = \int_0^{2\pi} \int_0^{\pi} \int_0^{.08} 20r(\text{mC/m}^3)\, r^2 \sin\theta\, dr\, d\theta\, d\phi = 2.57 \times 10^{-3} \text{ mC} = 2.57\,\mu\text{C}$$

So now

$$\rho_s = -\left[\frac{2.57}{4\pi(.08)^2}\right] = \underline{-32\,\mu\text{C/m}^2}$$

3.28. Repeat Problem 3.8, but use $\nabla \cdot \mathbf{D} = \rho_v$ and take an appropriate volume integral.

We begin by finding the charge density directly through

$$\rho_v = \nabla \cdot \mathbf{D} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{A}{r}\right) = \underline{\frac{A}{r^2}}$$

Then, within each spherical shell of unit thickness, the contained charge is

$$Q(1) = 4\pi \int_r^{r+1} \frac{A}{(r')^2}(r')^2\, dr' = 4\pi A(r + 1 - r) = \underline{4\pi A}$$

3.29. In the region of free space that includes the volume $2 < x, y, z < 3$,

$$\mathbf{D} = \frac{2}{z^2}(yz\,\mathbf{a}_x + xz\,\mathbf{a}_y - 2xy\,\mathbf{a}_z) \text{ C/m}^2$$

a) Evaluate the volume integral side of the divergence theorem for the volume defined above: In cartesian, we find $\nabla \cdot \mathbf{D} = 8xy/z^3$. The volume integral side is now

$$\int_{vol} \nabla \cdot \mathbf{D}\, dv = \int_2^3 \int_2^3 \int_2^3 \frac{8xy}{z^3}\, dxdydz = (9-4)(9-4)\left(\frac{1}{4} - \frac{1}{9}\right) = \underline{3.47 \text{ C}}$$

b. Evaluate the surface integral side for the corresponding closed surface: We call the surfaces at $x = 3$ and $x = 2$ the front and back surfaces respectively, those at $y = 3$ and $y = 2$ the right and left surfaces, and those at $z = 3$ and $z = 2$ the top and bottom surfaces. To evaluate the surface integral side, we integrate $\mathbf{D} \cdot \mathbf{n}$ over all six surfaces and sum the results. Note that since the x component of $\mathbf{D}$ does not vary with x, the outward fluxes from the front and back surfaces will cancel each other. The same is true for the left and right surfaces, since D_y does not vary with y. This leaves only the top and bottom surfaces, where the fluxes are:

$$\oint \mathbf{D} \cdot d\mathbf{S} = \underbrace{\int_2^3 \int_2^3 \frac{-4xy}{3^2}\, dxdy}_{\text{top}} - \underbrace{\int_2^3 \int_2^3 \frac{-4xy}{2^2}\, dxdy}_{\text{bottom}} = (9-4)(9-4)\left(\frac{1}{4} - \frac{1}{9}\right) = \underline{3.47 \text{ C}}$$

3.30 a) Use Maxwell's first equation, $\nabla \cdot \mathbf{D} = \rho_v$, to describe the variation of the electric field intensity with x in a region in which no charge density exists and in which a non-homogeneous dielectric has a permittivity that increases exponentially with x. The field has an x component only: The permittivity can be written as $\epsilon(x) = \epsilon_1 \exp(\alpha_1 x)$, where ϵ_1 and α_1 are constants. Then

$$\nabla \cdot \mathbf{D} = \nabla \cdot [\epsilon(x)\mathbf{E}(x)] = \frac{d}{dx}\left[\epsilon_1 e^{\alpha_1 x} E_x(x)\right] = \epsilon_1 \left[\alpha_1 e^{\alpha_1 x} E_x + e^{\alpha_1 x}\frac{dE_x}{dx}\right] = 0$$

This reduces to

$$\frac{dE_x}{dx} + \alpha_1 E_x = 0 \;\Rightarrow\; E_x(x) = \underline{E_0\, e^{-\alpha_1 x}}$$

where E_0 is a constant.

b) Repeat part a, but with a radially-directed electric field (spherical coordinates), in which again $\rho_v = 0$, but in which the permittivity *decreases* exponentially with r. In this case, the permittivity can be written as $\epsilon(r) = \epsilon_2 \exp(-\alpha_2 r)$, where ϵ_2 and α_2 are constants. Then

$$\nabla \cdot \mathbf{D} = \nabla \cdot [\epsilon(r)\mathbf{E}(r)] = \frac{1}{r^2}\frac{d}{dr}\left[r^2 \epsilon_2\, e^{-\alpha_2 r}\, E_r\right] = \frac{\epsilon_2}{r^2}\left[2rE_r - \alpha_2 r^2 E_r + r^2\frac{dE_r}{dr}\right] e^{-\alpha_2 r} = 0$$

This reduces to

$$\frac{dE_r}{dr} + \left(\frac{2}{r} - \alpha_2\right) E_r = 0$$

whose solution is

$$E_r(r) = E_0 \exp\left[-\int \left(\frac{2}{r} - \alpha_2\right) dr\right] = E_0 \exp\left[-2\ln r + \alpha_2 r\right] = \underline{\frac{E_0}{r^2} e^{\alpha_2 r}}$$

where E_0 is a constant.

3.31. Given the flux density

$$\mathbf{D} = \frac{16}{r}\cos(2\theta)\,\mathbf{a}_\theta \;\; \text{C/m}^2,$$

use two different methods to find the total charge within the region $1 < r < 2$ m, $1 < \theta < 2$ rad, $1 < \phi < 2$ rad: We use the divergence theorem and first evaluate the surface integral side. We are evaluating the net outward flux through a curvilinear "cube", whose boundaries are defined by the specified ranges. The flux contributions will be only through the surfaces of constant θ, however, since $\mathbf{D}$ has only a θ component. On a constant-theta surface, the differential area is $da = r \sin\theta dr d\phi$, where θ is fixed at the surface location. Our flux integral becomes

$$\oint \mathbf{D} \cdot d\mathbf{S} = -\underbrace{\int_1^2\int_1^2 \frac{16}{r}\cos(2)\, r\sin(1)\, dr d\phi}_{\theta=1} + \underbrace{\int_1^2\int_1^2 \frac{16}{r}\cos(4)\, r\sin(2)\, dr d\phi}_{\theta=2}$$

$$= -16\left[\cos(2)\sin(1) - \cos(4)\sin(2)\right] = \underline{-3.91\ \text{C}}$$

We next evaluate the volume integral side of the divergence theorem, where in this case,

$$\nabla \cdot \mathbf{D} = \frac{1}{r\sin\theta}\frac{d}{d\theta}(\sin\theta\, D_\theta) = \frac{1}{r\sin\theta}\frac{d}{d\theta}\left[\frac{16}{r}\cos 2\theta \sin\theta\right] = \frac{16}{r^2}\left[\frac{\cos 2\theta\cos\theta}{\sin\theta} - 2\sin 2\theta\right]$$

3.31 (continued) We now evaluate:

$$\int_{vol} \nabla \cdot \mathbf{D}\, dv = \int_1^2 \int_1^2 \int_1^2 \frac{16}{r^2} \left[\frac{\cos 2\theta \cos\theta}{\sin\theta} - 2\sin 2\theta \right] r^2 \sin\theta \, dr d\theta d\phi$$

The integral simplifies to

$$\int_1^2 \int_1^2 \int_1^2 16[\cos 2\theta \cos\theta - 2\sin 2\theta \sin\theta]\, dr d\theta d\phi = 8 \int_1^2 [3\cos 3\theta - \cos\theta]\, d\theta = \underline{-3.91 \text{ C}}$$

CHAPTER 4

4.1. The value of $\mathbf{E}$ at $P(\rho = 2, \phi = 40°, z = 3)$ is given as $\mathbf{E} = 100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z$ V/m. Determine the incremental work required to move a $20\,\mu$C charge a distance of 6 μm:

a) in the direction of $\mathbf{a}_\rho$: The incremental work is given by $dW = -q\,\mathbf{E}\cdot d\mathbf{L}$, where in this case, $d\mathbf{L} = d\rho\,\mathbf{a}_\rho = 6\times 10^{-6}\,\mathbf{a}_\rho$. Thus

$$dW = -(20\times 10^{-6}\,\mathrm{C})(100\,\mathrm{V/m})(6\times 10^{-6}\,\mathrm{m}) = -12\times 10^{-9}\ \mathrm{J} = \underline{-12\,\mathrm{nJ}}$$

b) in the direction of $\mathbf{a}_\phi$: In this case $d\mathbf{L} = 2\,d\phi\,\mathbf{a}_\phi = 6\times 10^{-6}\,\mathbf{a}_\phi$, and so

$$dW = -(20\times 10^{-6})(-200)(6\times 10^{-6}) = 2.4\times 10^{-8}\,\mathrm{J} = \underline{24\ \mathrm{nJ}}$$

c) in the direction of $\mathbf{a}_z$: Here, $d\mathbf{L} = dz\,\mathbf{a}_z = 6\times 10^{-6}\,\mathbf{a}_z$, and so

$$dW = -(20\times 10^{-6})(300)(6\times 10^{-6}) = -3.6\times 10^{-8}\,\mathrm{J} = \underline{-36\ \mathrm{nJ}}$$

d) in the direction of $\mathbf{E}$: Here, $d\mathbf{L} = 6\times 10^{-6}\,\mathbf{a}_E$, where

$$\mathbf{a}_E = \frac{100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z}{[100^2 + 200^2 + 300^2]^{1/2}} = 0.267\,\mathbf{a}_\rho - 0.535\,\mathbf{a}_\phi + 0.802\,\mathbf{a}_z$$

Thus

$$\begin{aligned} dW &= -(20\times 10^{-6})[100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z]\cdot[0.267\,\mathbf{a}_\rho - 0.535\,\mathbf{a}_\phi + 0.802\,\mathbf{a}_z](6\times 10^{-6}) \\ &= \underline{-44.9\,\mathrm{nJ}} \end{aligned}$$

e) In the direction of $\mathbf{G} = 2\,\mathbf{a}_x - 3\,\mathbf{a}_y + 4\,\mathbf{a}_z$: In this case, $d\mathbf{L} = 6\times 10^{-6}\,\mathbf{a}_G$, where

$$\mathbf{a}_G = \frac{2\mathbf{a}_x - 3\mathbf{a}_y + 4\mathbf{a}_z}{[2^2 + 3^2 + 4^2]^{1/2}} = 0.371\,\mathbf{a}_x - 0.557\,\mathbf{a}_y + 0.743\,\mathbf{a}_z$$

So now

$$\begin{aligned} dW &= -(20\times 10^{-6})[100\mathbf{a}_\rho - 200\mathbf{a}_\phi + 300\mathbf{a}_z]\cdot[0.371\,\mathbf{a}_x - 0.557\,\mathbf{a}_y + 0.743\,\mathbf{a}_z](6\times 10^{-6}) \\ &= -(20\times 10^{-6})\,[37.1(\mathbf{a}_\rho\cdot\mathbf{a}_x) - 55.7(\mathbf{a}_\rho\cdot\mathbf{a}_y) - 74.2(\mathbf{a}_\phi\cdot\mathbf{a}_x) + 111.4(\mathbf{a}_\phi\cdot\mathbf{a}_y) \\ &\quad + 222.9]\,(6\times 10^{-6}) \end{aligned}$$

where, at P, $(\mathbf{a}_\rho\cdot\mathbf{a}_x) = (\mathbf{a}_\phi\cdot\mathbf{a}_y) = \cos(40°) = 0.766$, $(\mathbf{a}_\rho\cdot\mathbf{a}_y) = \sin(40°) = 0.643$, and $(\mathbf{a}_\phi\cdot\mathbf{a}_x) = -\sin(40°) = -0.643$. Substituting these results in

$$dW = -(20\times 10^{-6})[28.4 - 35.8 + 47.7 + 85.3 + 222.9](6\times 10^{-6}) = \underline{-41.8\,\mathrm{nJ}}$$

4.2. A positive point charge of magnitude q_1 lies at the origin. Derive an expression for the incremental work done in moving a second point charge q_2 through a distance dx from the starting position (x, y, z), in the direction of $-\mathbf{a}_x$: The incremental work is given by

$$dW = -q_2 \mathbf{E}_{12} \cdot d\mathbf{L}$$

where $\mathbf{E}_{12}$ is the electric field arising from q_1 evaluated at the location of q_2, and where $d\mathbf{L} = -dx\, \mathbf{a}_x$. Taking the location of q_2 at spherical coordinates (r, θ, ϕ), we write:

$$dW = \frac{-q_2 q_1}{4\pi\epsilon_0 r^2} \mathbf{a}_r \cdot (-dx)\mathbf{a}_x$$

where $r^2 = x^2 + y^2 + z^2$, and where $\mathbf{a}_r \cdot \mathbf{a}_x = \sin\theta\cos\phi$. So

$$dW = \frac{q_2 q_1}{4\pi\epsilon_0 (x^2 + y^2 + z^2)} \underbrace{\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}}_{\sin\theta} \underbrace{\frac{x}{\sqrt{x^2+y^2}}}_{\cos\phi} dx = \underline{\frac{q_2 q_1\, x\, dx}{4\pi\epsilon_0 (x^2 + y^2 + z^2)^{3/2}}}$$

4.3. If $\mathbf{E} = 120\, \mathbf{a}_\rho$ V/m, find the incremental amount of work done in moving a $50\, \mu$m charge a distance of 2 mm from:

a) $P(1, 2, 3)$ toward $Q(2, 1, 4)$: The vector along this direction will be $Q - P = (1, -1, 1)$ from which $\mathbf{a}_{PQ} = [\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z]/\sqrt{3}$. We now write

$$dW = -q\mathbf{E} \cdot d\mathbf{L} = -(50 \times 10^{-6}) \left[120\mathbf{a}_\rho \cdot \frac{(\mathbf{a}_x - \mathbf{a}_y + \mathbf{a}_z}{\sqrt{3}} \right] (2 \times 10^{-3})$$
$$= -(50 \times 10^{-6})(120) \left[(\mathbf{a}_\rho \cdot \mathbf{a}_x) - (\mathbf{a}_\rho \cdot \mathbf{a}_y) \right] \frac{1}{\sqrt{3}} (2 \times 10^{-3})$$

At P, $\phi = \tan^{-1}(2/1) = 63.4°$. Thus $(\mathbf{a}_\rho \cdot \mathbf{a}_x) = \cos(63.4) = 0.447$ and $(\mathbf{a}_\rho \cdot \mathbf{a}_y) = \sin(63.4) = 0.894$. Substituting these, we obtain $dW = \underline{3.1\, \mu\text{J}}$.

b) $Q(2, 1, 4)$ toward $P(1, 2, 3)$: A little thought is in order here: Note that the field has only a radial component and does not depend on ϕ or z. Note also that P and Q are at the same radius ($\sqrt{5}$) from the z axis, but have different ϕ and z coordinates. We could just as well position the two points at the same z location and the problem would not change. If this were so, then moving along a straight line between P and Q would thus involve moving along a chord of a circle whose radius is $\sqrt{5}$. Halfway along this line is a point of symmetry in the field (make a sketch to see this). This means that when starting from either point, the initial force will be the same. Thus the answer is $dW = \underline{3.1\, \mu\text{J}}$ as in part a. This is also found by going through the same procedure as in part a, but with the direction (roles of P and Q) reversed.

4.4. An electric field in free space is given by $\mathbf{E} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$ V/m. Find the work done in moving a 1μC charge through this field

a) from (1,1,1) to (0,0,0): The work will be

$$W = -q\int \mathbf{E}\cdot d\mathbf{L} \;=\; -10^{-6}\left[\int_1^0 xdx \;+\; \int_1^0 ydy \;+\; \int_1^0 zdz\right] \text{J} \;=\; \underline{1.5\,\mu\text{J}}$$

b) from ($\rho = 2$, $\phi = 0$) to ($\rho = 2$, $\phi = 90°$): The path involves changing ϕ with ρ and z fixed, and therefore $d\mathbf{L} = \rho\, d\phi\, \mathbf{a}_\phi$. We set up the integral for the work as

$$W = -10^{-6}\int_0^{\pi/2} (x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z)\,\cdot\,\rho\, d\phi\, \mathbf{a}_\phi$$

where $\rho = 2$, $\mathbf{a}_x\cdot\mathbf{a}_\phi = -\sin\phi$, $\mathbf{a}_y\cdot\mathbf{a}_\phi = \cos\phi$, and $\mathbf{a}_z\cdot\mathbf{a}_\phi = 0$. Also, $x = 2\cos\phi$ and $y = 2\sin\phi$. Substitute all of these to get

$$W = -10^{-6}\int_0^{\pi/2}\left[-(2)^2\cos\phi\sin\phi + (2)^2\cos\phi\sin\phi\right]d\phi = \underline{0}$$

Given that the field is conservative (and so work is path-independent), can you see a much easier way to obtain this result?

c) from ($r = 10$, $\theta = \theta_0$) to ($r = 10$, $\theta = \theta_0 + 180°$): In this case, we are moving only in the $\mathbf{a}_\theta$ direction. The work is set up as

$$W = -10^{-6}\int_{\theta_0}^{\theta_0+\pi} (x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z)\,\cdot\, r\, d\theta\, \mathbf{a}_\theta$$

Now, substitute the following relations: $r = 10$, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, $\mathbf{a}_x\cdot\mathbf{a}_\theta = \cos\theta\cos\phi$, $\mathbf{a}_y\cdot\mathbf{a}_\theta = \cos\theta\sin\phi$, and $\mathbf{a}_z\cdot\mathbf{a}_\theta = -\sin\theta$. Obtain

$$W = -10^{-6}\int_{\theta_0}^{\theta_0+\pi}(10)^2\left[\sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi - \cos\theta\sin\theta\right]d\theta = \underline{0}$$

where we use $\cos^2\phi + \sin^2\phi = 1$.

4.5. Compute the value of $\int_A^P \mathbf{G}\cdot d\mathbf{L}$ for $\mathbf{G} = 2y\mathbf{a}_x$ with $A(1,-1,2)$ and $P(2,1,2)$ using the path:

a) straight-line segments $A(1,-1,2)$ to $B(1,1,2)$ to $P(2,1,2)$: In general we would have

$$\int_A^P \mathbf{G}\cdot d\mathbf{L} = \int_A^P 2y\, dx$$

The change in x occurs when moving between B and P, during which $y = 1$. Thus

$$\int_A^P \mathbf{G}\cdot d\mathbf{L} = \int_B^P 2y\, dx = \int_1^2 2(1)dx = \underline{2}$$

b) straight-line segments $A(1,-1,2)$ to $C(2,-1,2)$ to $P(2,1,2)$: In this case the change in x occurs when moving from A to C, during which $y = -1$. Thus

$$\int_A^P \mathbf{G}\cdot d\mathbf{L} = \int_A^C 2y\, dx = \int_1^2 2(-1)dx = \underline{-2}$$

4.6. An electric field in free space is given as $\mathbf{E} = x\,\hat{\mathbf{a}}_x + 4z\,\hat{\mathbf{a}}_y + 4y\,\hat{\mathbf{a}}_z$. Given $V(1,1,1) = 10$ V. Determine $V(3,3,3)$. The potential difference is expressed as

$$V(3,3,3) - V(1,1,1) = -\int_{1,1,1}^{3,3,3} (x\,\hat{\mathbf{a}}_x + 4z\,\hat{\mathbf{a}}_y + 4y\,\hat{\mathbf{a}}_z) \cdot (dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z)$$
$$= -\left[\int_1^3 x\,dx + \int_1^3 4z\,dy + \int_1^3 4y\,dz\right]$$

We choose the following path: 1) move along x from 1 to 3; 2) move along y from 1 to 3, holding x at 3 and z at 1; 3) move along z from 1 to 3, holding x at 3 and y at 3. The integrals become:

$$V(3,3,3) - V(1,1,1) = -\left[\int_1^3 x\,dx + \int_1^3 4(1)\,dy + \int_1^3 4(3)\,dz\right] = -36$$

So

$$V(3,3,3) = -36 + V(1,1,1) = -36 + 10 = \underline{-26} \text{ V}$$

4.7. Let $\mathbf{G} = 3xy^3\mathbf{a}_x + 2z\mathbf{a}_y$. Given an initial point $P(2,1,1)$ and a final point $Q(4,3,1)$, find $\int \mathbf{G} \cdot d\mathbf{L}$ using the path:

a) straight line: $y = x - 1$, $z = 1$: We obtain:

$$\int \mathbf{G} \cdot d\mathbf{L} = \int_2^4 3xy^2\,dx + \int_1^3 2z\,dy = \int_2^4 3x(x-1)^2\,dx + \int_1^3 2(1)\,dy = \underline{90}$$

b) parabola: $6y = x^2 + 2$, $z = 1$: We obtain:

$$\int \mathbf{G} \cdot d\mathbf{L} = \int_2^4 3xy^2\,dx + \int_1^3 2z\,dy = \int_2^4 \frac{1}{12}x(x^2+2)^2\,dx + \int_1^3 2(1)\,dy = \underline{82}$$

4.8. Given $\mathbf{E} = -x\mathbf{a}_x + y\mathbf{a}_y$, a) find the work involved in moving a unit positive charge on a circular arc, the circle centered at the origin, from $x = a$ to $x = y = a/\sqrt{2}$.

In moving along the arc, we start at $\phi = 0$ and move to $\phi = \pi/4$. The setup is

$$W = -q\int \mathbf{E} \cdot d\mathbf{L} = -\int_0^{\pi/4} \mathbf{E} \cdot a d\phi\,\mathbf{a}_\phi = -\int_0^{\pi/4} (-x\,\underbrace{\mathbf{a}_x \cdot \mathbf{a}_\phi}_{-\sin\phi} + y\,\underbrace{\mathbf{a}_y \cdot \mathbf{a}_\phi}_{\cos\phi})a\,d\phi$$
$$= -\int_0^{\pi/4} 2a^2 \sin\phi\cos\phi\,d\phi = -\int_0^{\pi/4} a^2\sin(2\phi)\,d\phi = \frac{a^2}{2}\cos(2\phi)\Big|_0^{\pi/4} = \underline{-\frac{a^2}{2}}$$

where $q = 1$, $x = a\cos\phi$, and $y = a\sin\phi$.

Note that the field is conservative, so we would get the same result by integrating along a two-segment path over x and y as shown:

$$W = -\int \mathbf{E} \cdot d\mathbf{L} = -\left[\int_a^{a/\sqrt{2}} (-x)\,dx + \int_0^{a/\sqrt{2}} y\,dy\right] = -a^2/2$$

4.8b) Verify that the work done in moving the charge around the full circle from $x = a$ is zero: In this case, the setup is the same, but the integration limits change:

$$W = -\int_0^{2\pi} a^2 \sin(2\phi)\, d\phi = \frac{a^2}{2}\cos(2\phi)\Big|_0^{2\pi} = \underline{0}$$

4.9. A uniform surface charge density of 20 nC/m^2 is present on the spherical surface $r = 0.6$ cm in free space.

a) Find the absolute potential at $P(r = 1\,\text{cm}, \theta = 25^\circ, \phi = 50^\circ)$: Since the charge density is uniform and is spherically-symmetric, the angular coordinates do not matter. The potential function for $r > 0.6$ cm will be that of a point charge of $Q = 4\pi a^2 \rho_s$, or

$$V(r) = \frac{4\pi(0.6 \times 10^{-2})^2(20 \times 10^{-9})}{4\pi\epsilon_0 r} = \frac{0.081}{r}\ \text{V} \quad \text{with } r \text{ in meters}$$

At $r = 1$ cm, this becomes $V(r = 1\,\text{cm}) = \underline{8.14\ \text{V}}$

b) Find V_{AB} given points $A(r = 2\,\text{cm}, \theta = 30^\circ, \phi = 60^\circ)$ and $B(r = 3\,\text{cm}, \theta = 45^\circ, \phi = 90^\circ)$: Again, the angles do not matter because of the spherical symmetry. We use the part a result to obtain

$$V_{AB} = V_A - V_B = 0.081\left[\frac{1}{0.02} - \frac{1}{0.03}\right] = \underline{1.36\ \text{V}}$$

4.10. A sphere of radius a carries a surface charge density of ρ_{s0} C/m^2.

a) Find the absolute potential at the sphere surface: The setup for this is

$$V_0 = -\int_\infty^a \mathbf{E} \cdot d\mathbf{L}$$

where, from Gauss' law:

$$\mathbf{E} = \frac{a^2 \rho_{s0}}{\epsilon_0 r^2}\,\mathbf{a}_r \quad \text{V/m}$$

So

$$V_0 = -\int_\infty^a \frac{a^2\rho_{s0}}{\epsilon_0 r^2}\,\mathbf{a}_r \cdot \mathbf{a}_r\, dr = \frac{a^2\rho_{s0}}{\epsilon_0 r}\Big|_\infty^a = \underline{\frac{a\rho_{s0}}{\epsilon_0}}\ \text{V}$$

b) A grounded conducting shell of radius b where $b > a$ is now positioned around the charged sphere. What is the potential at the inner sphere surface in this case? With the outer sphere grounded, the field exists only between the surfaces, and is zero for $r > b$. The potential is then

$$V_0 = -\int_b^a \frac{a^2\rho_{s0}}{\epsilon_0 r^2}\,\mathbf{a}_r \cdot \mathbf{a}_r\, dr = \frac{a^2\rho_{s0}}{\epsilon_0 r}\Big|_b^a = \underline{\frac{a^2\rho_{s0}}{\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]}\ \text{V}$$

4.11. Let a uniform surface charge density of $5\,\text{nC/m}^2$ be present at the $z = 0$ plane, a uniform line charge density of $8\,\text{nC/m}$ be located at $x = 0$, $z = 4$, and a point charge of $2\,\mu\text{C}$ be present at $P(2, 0, 0)$. If $V = 0$ at $M(0, 0, 5)$, find V at $N(1, 2, 3)$: We need to find a potential function for the combined charges which is zero at M. That for the point charge we know to be

$$V_p(r) = \frac{Q}{4\pi\epsilon_0 r}$$

Potential functions for the sheet and line charges can be found by taking indefinite integrals of the electric fields for those distributions. For the line charge, we have

$$V_l(\rho) = -\int \frac{\rho_l}{2\pi\epsilon_0\rho}\,d\rho + C_1 = -\frac{\rho_l}{2\pi\epsilon_0}\ln(\rho) + C_1$$

For the sheet charge, we have

$$V_s(z) = -\int \frac{\rho_s}{2\epsilon_0}\,dz + C_2 = -\frac{\rho_s}{2\epsilon_0}z + C_2$$

The total potential function will be the sum of the three. Combining the integration constants, we obtain:

$$V = \frac{Q}{4\pi\epsilon_0 r} - \frac{\rho_l}{2\pi\epsilon_0}\ln(\rho) - \frac{\rho_s}{2\epsilon_0}z + C$$

The terms in this expression are not referenced to a common origin, since the charges are at different positions. The parameters r, ρ, and z are *scalar distances* from the charges, and will be treated as such here. To evaluate the constant, C, we first look at point M, where $V_T = 0$. At M, $r = \sqrt{2^2 + 5^2} = \sqrt{29}$, $\rho = 1$, and $z = 5$. We thus have

$$0 = \frac{2 \times 10^{-6}}{4\pi\epsilon_0\sqrt{29}} - \frac{8 \times 10^{-9}}{2\pi\epsilon_0}\ln(1) - \frac{5 \times 10^{-9}}{2\epsilon_0}5 + C \quad \Rightarrow \quad C = -1.93 \times 10^3 \text{ V}$$

At point N, $r = \sqrt{1 + 4 + 9} = \sqrt{14}$, $\rho = \sqrt{2}$, and $z = 3$. The potential at N is thus

$$V_N = \frac{2 \times 10^{-6}}{4\pi\epsilon_0\sqrt{14}} - \frac{8 \times 10^{-9}}{2\pi\epsilon_0}\ln(\sqrt{2}) - \frac{5 \times 10^{-9}}{2\epsilon_0}(3) - 1.93 \times 10^3 = 1.98 \times 10^3 \text{ V} = \underline{1.98\,\text{kV}}$$

4.12. In spherical coordinates, $\mathbf{E} = 2r/(r^2 + a^2)^2 \, \mathbf{a}_r$ V/m. Find the potential at any point, using the reference

a) $V = 0$ at infinity: We write in general

$$V(r) = -\int \frac{2r\,dr}{(r^2+a^2)^2} + C = \frac{1}{r^2+a^2} + C$$

With a zero reference at $r \to \infty$, $C = 0$ and therefore $V(r) = \underline{1/(r^2 + a^2)}$.

b) $V = 0$ at $r = 0$: Using the general expression, we find

$$V(0) = \frac{1}{a^2} + C = 0 \quad \Rightarrow \quad C = -\frac{1}{a^2}$$

Therefore

$$V(r) = \frac{1}{r^2+a^2} - \frac{1}{a^2} = \frac{-r^2}{a^2(r^2+a^2)}$$

c) $V = 100\text{V}$ at $r = a$: Here, we find

$$V(a) = \frac{1}{2a^2} + C = 100 \quad \Rightarrow \quad C = 100 - \frac{1}{2a^2}$$

Therefore

$$V(r) = \frac{1}{r^2+a^2} - \frac{1}{2a^2} + 100 = \frac{a^2 - r^2}{2a^2(r^2+a^2)} + 100$$

4.13. Three identical point charges of 4 pC each are located at the corners of an equilateral triangle 0.5 mm on a side in free space. How much work must be done to move one charge to a point equidistant from the other two and on the line joining them? This will be the magnitude of the charge times the potential difference between the finishing and starting positions, or

$$W = \frac{(4 \times 10^{-12})^2}{2\pi\epsilon_0} \left[\frac{1}{2.5} - \frac{1}{5} \right] \times 10^4 = 5.76 \times 10^{-10}\,\text{J} = \underline{576\,\text{pJ}}$$

4.14. Given the electric field $\mathbf{E} = (y+1)\mathbf{a}_x + (x-1)\mathbf{a}_y + 2\mathbf{a}_z$, find the potential difference between the points

a) (2,-2,-1) and (0,0,0): We choose a path along which motion occurs in one coordinate direction at a time. Starting at the origin, first move along x from 0 to 2, where $y = 0$; then along y from 0 to -2, where x is 2; then along z from 0 to -1. The setup is

$$V_b - V_a = -\int_0^2 (y+1)\Big|_{y=0} dx \; - \int_0^{-2} (x-1)\Big|_{x=2} dy \; - \int_0^{-1} 2\,dz = \underline{2}$$

b) (3,2,-1) and (-2,-3,4): Following similar reasoning,

$$V_b - V_a = -\int_{-2}^3 (y+1)\Big|_{y=-3} dx \; - \int_{-3}^{2} (x-1)\Big|_{x=3} dy \; - \int_4^{-1} 2\,dz = \underline{10}$$

4.15. Two uniform line charges, 8 nC/m each, are located at $x = 1$, $z = 2$, and at $x = -1$, $y = 2$ in free space. If the potential at the origin is 100 V, find V at $P(4, 1, 3)$: The net potential function for the two charges would in general be:

$$V = -\frac{\rho_l}{2\pi\epsilon_0}\ln(R_1) - \frac{\rho_l}{2\pi\epsilon_0}\ln(R_2) + C$$

At the origin, $R_1 = R_2 = \sqrt{5}$, and $V = 100$ V. Thus, with $\rho_l = 8 \times 10^{-9}$,

$$100 = -2\frac{(8 \times 10^{-9})}{2\pi\epsilon_0}\ln(\sqrt{5}) + C \quad \Rightarrow \quad C = 331.6 \text{ V}$$

At $P(4, 1, 3)$, $R_1 = |(4, 1, 3) - (1, 1, 2)| = \sqrt{10}$ and $R_2 = |(4, 1, 3) - (-1, 2, 3)| = \sqrt{26}$. Therefore

$$V_P = -\frac{(8 \times 10^{-9})}{2\pi\epsilon_0}\left[\ln(\sqrt{10}) + \ln(\sqrt{26})\right] + 331.6 = \underline{-68.4 \text{ V}}$$

4.16. A spherically-symmetric charge distribution in free space (with $a < r < \infty$) – *note typo in problem statement*, which says $(0 < r < \infty)$ – is known to have a potential function $V(r) = V_0 a^2/r^2$, where V_0 and a are constants.

a) Find the electric field intensity: This is found through

$$\mathbf{E} = -\nabla V = -\frac{dV}{dr}\mathbf{a}_r = \underline{2V_0\frac{a^2}{r^3}\mathbf{a}_r \text{ V/m}}$$

b) Find the volume charge density: Use Maxwell's first equation:

$$\rho_v = \nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon_0 \mathbf{E}) = \frac{1}{r^2}\frac{d}{dr}\left[r^2\left(2\epsilon_0 V_0 \frac{a^2}{r^3}\right)\right] = \underline{-2\epsilon_0 V_0 \frac{a^2}{r^4} \text{ C/m}^3}$$

c) Find the charge contained inside radius a: Here, we do not know the charge density inside radius a, but we do know the flux density *at* that radius. We use Gauss' law to integrate $\mathbf{D}$ over the spherical surface at $r = a$ to find the charge enclosed:

$$Q_{encl} = \oint_{r=a} \mathbf{D} \cdot d\mathbf{S} = 4\pi a^2 D|_{r=a} = 4\pi a^2\left(2\epsilon_0 V_0 \frac{a^2}{a^3}\right) = \underline{8\pi\epsilon_0 a V_0} \text{ C}$$

d) Find the total energy stored in the charge (or equivalently, in its electric field) *in the region* $(a < r < \infty)$. We integrate the energy density in the field over the region:

$$W_e = \int_v \frac{1}{2}\mathbf{D} \cdot \mathbf{E}\, dv = \int_0^{2\pi}\int_0^{\pi}\int_a^{\infty} 2\epsilon_0 V_0^2 \frac{a^4}{r^6} r^2 \sin\theta\, dr\, d\theta\, d\phi$$
$$= -8\pi V_0^2 \epsilon_0 a^4 \frac{1}{3r^3}\Big|_a^{\infty} = \underline{8\pi\epsilon_0 a V_0^2/3} \text{ J}$$

4.17. Uniform surface charge densities of 6 and 2 nC/m^2 are present at $\rho = 2$ and 6 cm respectively, in free space. Assume $V = 0$ at $\rho = 4$ cm, and calculate V at:

a) $\rho = 5$ cm: Since $V = 0$ at 4 cm, the potential at 5 cm will be the potential difference between points 5 and 4:

$$V_5 = -\int_4^5 \mathbf{E} \cdot d\mathbf{L} = -\int_4^5 \frac{a\rho_{sa}}{\epsilon_0 \rho}\, d\rho = -\frac{(.02)(6 \times 10^{-9})}{\epsilon_0} \ln\left(\frac{5}{4}\right) = \underline{-3.026\,\mathrm{V}}$$

b) $\rho = 7$ cm: Here we integrate piecewise from $\rho = 4$ to $\rho = 7$:

$$V_7 = -\int_4^6 \frac{a\rho_{sa}}{\epsilon_0 \rho} d\rho - \int_6^7 \frac{(a\rho_{sa} + b\rho_{sb})}{\epsilon_0 \rho} d\rho$$

With the given values, this becomes

$$V_7 = -\left[\frac{(.02)(6 \times 10^{-9})}{\epsilon_0}\right] \ln\left(\frac{6}{4}\right) - \left[\frac{(.02)(6 \times 10^{-9}) + (.06)(2 \times 10^{-9})}{\epsilon_0}\right] \ln\left(\frac{7}{6}\right)$$
$$= \underline{-9.678\,\mathrm{V}}$$

4.18. Find the potential at the origin produced by a line charge $\rho_L = kx/(x^2 + a^2)$ extending along the x axis from $x = a$ to $+\infty$, where $a > 0$. Assume a zero reference at infinity.

Think of the line charge as an array of point charges, each of charge $dq = \rho_L dx$, and each having potential at the origin of $dV = \rho_L dx/(4\pi\epsilon_0 x)$. The total potential at the origin is then the sum of all these potentials, or

$$V = \int_a^\infty \frac{\rho_L\, dx}{4\pi\epsilon_0 x} = \int_a^\infty \frac{k\, dx}{4\pi\epsilon_0 (x^2 + a^2)} = \frac{k}{4\pi\epsilon_0 a} \tan^{-1}\left(\frac{x}{a}\right)_a^\infty = \frac{k}{4\pi\epsilon_0 a}\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = \frac{k}{16\epsilon_0 a}$$

4.19. The annular surface, $1\,\mathrm{cm} < \rho < 3\,\mathrm{cm}$, $z = 0$, carries the nonuniform surface charge density $\rho_s = 5\rho\,\mathrm{nC/m^2}$. Find V at $P(0, 0, 2\,\mathrm{cm})$ if $V = 0$ at infinity: We use the superposition integral form:

$$V_P = \int\int \frac{\rho_s\, da}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|}$$

where $\mathbf{r} = z\mathbf{a}_z$ and $\mathbf{r}' = \rho\mathbf{a}_\rho$. We integrate over the surface of the annular region, with $da = \rho\, d\rho\, d\phi$. Substituting the given values, we find

$$V_P = \int_0^{2\pi} \int_{.01}^{.03} \frac{(5 \times 10^{-9})\rho^2\, d\rho\, d\phi}{4\pi\epsilon_0 \sqrt{\rho^2 + z^2}}$$

Substituting $z = .02$, and using tables, the integral evaluates as

$$V_P = \left[\frac{(5 \times 10^{-9})}{2\epsilon_0}\right] \left[\frac{\rho}{2}\sqrt{\rho^2 + (.02)^2} - \frac{(.02)^2}{2} \ln(\rho + \sqrt{\rho^2 + (.02)^2})\right]_{.01}^{.03} = \underline{.081\,\mathrm{V}}$$

4.20. In a certain medium, the electric potential is given by

$$V(x) = \frac{\rho_0}{a\epsilon_0}\left(1 - e^{-ax}\right)$$

where ρ_0 and a are constants.

a) Find the electric field intensity, $\mathbf{E}$:

$$\mathbf{E} == -\nabla V = -\frac{d}{dx}\left[\frac{\rho_0}{a\epsilon_0}\left(1 - e^{-ax}\right)\right]\mathbf{a}_x = \underline{-\frac{\rho_0}{\epsilon_0}\, e^{-ax}\,\mathbf{a}_x} \text{ V/m}$$

b) find the potential difference between the points $x = d$ and $x = 0$:

$$V_{d0} = V(d) - V(0) = \underline{\frac{\rho_0}{a\epsilon_0}\left(1 - e^{-ad}\right)} \text{ V}$$

c) if the medium permittivity is given by $\epsilon(x) = \epsilon_0 e^{ax}$, find the electric flux density, $\mathbf{D}$, and the volume charge density, ρ_v, in the region:

$$\mathbf{D} = \epsilon\mathbf{E} = \epsilon_0 e^{ax}\left(-\frac{\rho_0}{\epsilon_0}\, e^{-ax}\,\mathbf{a}_x\right) = \underline{-\rho_0\,\mathbf{a}_x} \text{ C/m}^2$$

Then $\rho_v = \nabla \cdot \mathbf{D} = \underline{0}$.

d) Find the stored energy in the region $(0 < x < d)$, $(0 < y < 1)$, $(0 < z < 1)$:

$$W_e = \int_v \frac{1}{2}\mathbf{D}\cdot\mathbf{E}\,dv = \int_0^1\int_0^1\int_0^d \frac{\rho_0^2}{2\epsilon_0} e^{-ax}\,dx\,dy\,dz = \frac{-\rho_0^2}{2\epsilon_0 a}\, e^{-ax}\Big|_0^d = \underline{\frac{\rho_0^2}{2\epsilon_0 a}\left(1 - e^{-ad}\right)} \text{ J}$$

4.21. Let $V = 2xy^2z^3 + 3\ln(x^2 + 2y^2 + 3z^2)$ V in free space. Evaluate each of the following quantities at $P(3, 2, -1)$:

a) V: Substitute P directly to obtain: $V = \underline{-15.0\text{ V}}$

b) $|V|$. This will be just $\underline{15.0\text{ V}}$.

c) $\mathbf{E}$: We have

$$\mathbf{E}\Big|_P = -\nabla V\Big|_P = -\left[\left(2y^2z^3 + \frac{6x}{x^2+2y^2+3z^2}\right)\mathbf{a}_x + \left(4xyz^3 + \frac{12y}{x^2+2y^2+3z^2}\right)\mathbf{a}_y + \left(6xy^2z^2 + \frac{18z}{x^2+2y^2+3z^2}\right)\mathbf{a}_z\right]_P = \underline{7.1\mathbf{a}_x + 22.8\mathbf{a}_y - 71.1\mathbf{a}_z \text{ V/m}}$$

d) $|\mathbf{E}|_P$: taking the magnitude of the part c result, we find $|\mathbf{E}|_P = \underline{75.0\text{ V/m}}$.

e) $\mathbf{a}_N$: By definition, this will be

$$\mathbf{a}_N\Big|_P = -\frac{\mathbf{E}}{|\mathbf{E}|} = \underline{-0.095\,\mathbf{a}_x - 0.304\,\mathbf{a}_y + 0.948\,\mathbf{a}_z}$$

f) $\mathbf{D}$: This is $\mathbf{D}\Big|_P = \epsilon_0\mathbf{E}\Big|_P = \underline{62.8\,\mathbf{a}_x + 202\,\mathbf{a}_y - 629\,\mathbf{a}_z \text{ pC/m}^2}$.

4.22. A line charge of infinite length lies along the z axis, and carries a uniform linear charge density of ρ_ℓ C/m. A perfectly-conducting cylindrical shell, whose axis is the z axis, surrounds the line charge. The cylinder (of radius b), is at ground potential. Under these conditions, the potential function inside the cylinder ($\rho < b$) is given by

$$V(\rho) = k - \frac{\rho_\ell}{2\pi\epsilon_0} \ln(\rho)$$

where k is a constant.

a) Find k in terms of given or known parameters: At radius b,

$$V(b) = k - \frac{\rho_\ell}{2\pi\epsilon_0} \ln(b) = 0 \;\Rightarrow\; k = \underline{\frac{\rho_\ell}{2\pi\epsilon_0} \ln(b)}$$

b) find the electric field strength, **E**, for $\rho < b$:

$$\mathbf{E}_{in} = -\nabla V = -\frac{d}{d\rho}\left[\frac{\rho_\ell}{2\pi\epsilon_0} \ln(b) - \frac{\rho_\ell}{2\pi\epsilon_0} \ln(\rho)\right] \mathbf{a}_\rho = \underline{\frac{\rho_\ell}{2\pi\epsilon_0 \rho} \mathbf{a}_\rho \text{ V/m}}$$

c) find the electric field strength, **E**, for $\rho > b$: $\mathbf{E}_{out} = \underline{0}$ because the cylinder is at ground potential.

d) Find the stored energy in the electric field *per unit length* in the z direction within the volume defined by $\rho > a$, where $a < b$:

$$W_e = \int_v \frac{1}{2}\mathbf{D}\cdot\mathbf{E}\,dv = \int_0^1 \int_0^{2\pi} \int_a^b \frac{\rho_\ell^2}{8\pi^2\epsilon_0\rho^2}\,\rho\,d\rho\,d\phi\,dz = \underline{\frac{\rho_\ell^2}{4\pi\epsilon_0} \ln\left(\frac{b}{a}\right) \text{ J}}$$

4.23. It is known that the potential is given as $V = 80\rho^{.6}$ V. Assuming free space conditions, find:

a) **E**: We find this through

$$\mathbf{E} = -\nabla V = -\frac{dV}{d\rho}\mathbf{a}_\rho = \underline{-48\rho^{-.4} \text{ V/m}}$$

b) the volume charge density at $\rho = .5$ m: Using $\mathbf{D} = \epsilon_0 \mathbf{E}$, we find the charge density through

$$\rho_v \Big|_{.5} = [\nabla\cdot\mathbf{D}]_{.5} = \left(\frac{1}{\rho}\right)\frac{d}{d\rho}(\rho D_\rho)\Big|_{.5} = -28.8\epsilon_0\rho^{-1.4}\Big|_{.5} = \underline{-673 \text{ pC/m}^3}$$

c) the total charge lying within the closed surface $\rho = .6$, $0 < z < 1$: The easiest way to do this calculation is to evaluate D_ρ at $\rho = .6$ (noting that it is constant), and then multiply by the cylinder area: Using part a, we have $D_\rho\Big|_{.6} = -48\epsilon_0(.6)^{-.4} = -521 \text{ pC/m}^2$. Thus $Q = -2\pi(.6)(1)521 \times 10^{-12}\text{ C} = \underline{-1.96 \text{ nC}}$.

4.24. A certain spherically-symmetric charge configuration in free space produces an electric field given in spherical coordinates by:

$$\mathbf{E}(r) = \begin{cases} (\rho_0 r^2)/(100\epsilon_0)\,\mathbf{a}_r \text{ V/m} & (r \le 10) \\ (100\rho_0)/(\epsilon_0 r^2)\,\mathbf{a}_r \text{ V/m} & (r \ge 10) \end{cases}$$

where ρ_0 is a constant.

a) Find the charge density as a function of position:

$$\rho_v(r \le 10) = \nabla \cdot (\epsilon_0 \mathbf{E}_1) = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{\rho_0 r^2}{100}\right) = \underline{\frac{\rho_0 r}{25}} \text{ C/m}^3$$

$$\rho_v(r \ge 10) = \nabla \cdot (\epsilon_0 \mathbf{E}_2) = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{100\rho_0}{r^2}\right) = \underline{0}$$

b) find the absolute potential as a function of position in the two regions, $r \le 10$ and $r \ge 10$:

$$V(r \le 10) = -\int_\infty^{10} \frac{100\rho_0}{\epsilon_0 r^2}\mathbf{a}_r \cdot \mathbf{a}_r\,dr - \int_{10}^{r} \frac{\rho_0 (r')^2}{100\epsilon_0}\mathbf{a}_r \cdot \mathbf{a}_r\,dr'$$

$$= \frac{100\rho_0}{\epsilon_0 r}\Big|_\infty^{10} - \frac{\rho_0 (r')^3}{300\epsilon_0}\Big|_{10}^{r} = \underline{\frac{10\rho_0}{3\epsilon_0}\left[4 - (10^{-3}r^3)\right]} \text{ V}$$

$$V(r \ge 10) = -\int_\infty^{r} \frac{100\rho_0}{\epsilon_0 (r')^2}\mathbf{a}_r \cdot \mathbf{a}_r\,dr' = \frac{100\rho_0}{\epsilon_0 r'}\Big|_\infty^{r} = \underline{\frac{100\rho_0}{\epsilon_0 r}} \text{ V}$$

c) check your result of part *b* by using the gradient:

$$\mathbf{E}_1 = -\nabla V(r \le 10) = -\frac{d}{dr}\left[\frac{10\rho_0}{3\epsilon_0}\left[4 - (10^{-3}r^3)\right]\right]\mathbf{a}_r = \frac{10\rho_0}{3\epsilon_0}(3r^2)(10^{-3})\,\mathbf{a}_r = \frac{\rho_0 r^2}{100\epsilon_0}\mathbf{a}_r$$

$$\mathbf{E}_2 = -\nabla V(r \ge 10) = -\frac{d}{dr}\left[\frac{100\rho_0}{\epsilon_0 r}\right]\mathbf{a}_r = \frac{100\rho_0}{\epsilon_0 r^2}\mathbf{a}_r$$

d) find the stored energy in the charge by an integral of the form of Eq. (42) (not Eq. (43)):

$$W_e = \frac{1}{2}\int_v \rho_v V\,dv = \int_0^{2\pi}\int_0^{\pi}\int_0^{10} \frac{1}{2}\frac{\rho_0 r}{25}\left[\frac{10\rho_0}{3\epsilon_0}\left[4 - (10^{-3}r^3)\right]\right] r^2 \sin\theta\,dr\,d\theta\,d\phi$$

$$= \frac{4\pi\rho_0^2}{150\epsilon_0}\int_0^{10}\left[40r^3 - \frac{r^6}{100}\right]dr = \frac{4\pi\rho_0^2}{150\epsilon_0}\left[10r^4 - \frac{r^7}{700}\right]_0^{10} = \underline{7.18 \times 10^3\,\frac{\rho_0^2}{\epsilon_0}}$$

e) Find the stored energy in the field by an integral of the form of Eq. (44) (not Eq. (45)).

$$W_e = \int_{(r\le 10)} \frac{1}{2}\mathbf{D}_1 \cdot \mathbf{E}_1\,dv + \int_{(r \ge 10)} \frac{1}{2}\mathbf{D}_2 \cdot \mathbf{E}_2\,dv$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{10} \frac{\rho_0^2 r^4}{(2\times 10^4)\epsilon_0} r^2 \sin\theta\,dr\,d\theta\,d\phi + \int_0^{2\pi}\int_0^{\pi}\int_{10}^{\infty} \frac{10^4\rho_0^2}{2\epsilon_0 r^4} r^2 \sin\theta\,dr\,d\theta\,d\phi$$

$$= \frac{2\pi\rho_0^2}{\epsilon_0}\left[10^{-4}\int_0^{10} r^6\,dr + 10^4\int_{10}^{\infty}\frac{dr}{r^2}\right] = \frac{2\pi\rho_0^2}{\epsilon_0}\left[\frac{1}{7}(10^3) + 10^3\right] = \underline{7.18 \times 10^3\,\frac{\rho_0^2}{\epsilon_0}}$$

4.25. Within the cylinder $\rho = 2$, $0 < z < 1$, the potential is given by $V = 100 + 50\rho + 150\rho \sin\phi$ V.

a) Find V, $\mathbf{E}$, $\mathbf{D}$, and ρ_v at $P(1, 60°, 0.5)$ in free space: First, substituting the given point, we find $V_P = \underline{279.9\,\text{V}}$. Then,

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial \rho}\mathbf{a}_\rho - \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi = -\left[50 + 150\sin\phi\right]\mathbf{a}_\rho - \left[150\cos\phi\right]\mathbf{a}_\phi$$

Evaluate the above at P to find $\mathbf{E}_P = \underline{-179.9\mathbf{a}_\rho - 75.0\mathbf{a}_\phi \text{ V/m}}$

Now $\mathbf{D} = \epsilon_0 \mathbf{E}$, so $\mathbf{D}_P = \underline{-1.59\mathbf{a}_\rho - .664\mathbf{a}_\phi \text{ nC/m}^2}$. Then

$$\rho_v = \nabla \cdot \mathbf{D} = \left(\frac{1}{\rho}\right)\frac{d}{d\rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} = \left[-\frac{1}{\rho}(50 + 150\sin\phi) + \frac{1}{\rho}150\sin\phi\right]\epsilon_0 = -\frac{50}{\rho}\epsilon_0 \text{ C}$$

At P, this is $\rho_{vP} = \underline{-443 \text{ pC/m}^3}$.

b) How much charge lies within the cylinder? We will integrate ρ_v over the volume to obtain:

$$Q = \int_0^1 \int_0^{2\pi} \int_0^2 -\frac{50\epsilon_0}{\rho}\rho\, d\rho\, d\phi\, dz = -2\pi(50)\epsilon_0(2) = \underline{-5.56\,\text{nC}}$$

4.26. Let us assume that we have a very thin, square, imperfectly conducting plate 2m on a side, located in the plane $z = 0$ with one corner at the origin such that it lies entirely within the first quadrant. The potential at any point in the plate is given as $V = -e^{-x}\sin y$.

a) An electron enters the plate at $x = 0$, $y = \pi/3$ with zero initial velocity; in what direction is its initial movement? We first find the electric field associated with the given potential:

$$\mathbf{E} = -\nabla V = -e^{-x}[\sin y\, \mathbf{a}_x - \cos y\, \mathbf{a}_y]$$

Since we have an electron, its motion is opposite that of the field, so the direction on entry is that of $-\mathbf{E}$ at $(0, \pi/3)$, or $\underline{\sqrt{3}/2\,\mathbf{a}_x - 1/2\,\mathbf{a}_y}$.

b) Because of collisions with the particles in the plate, the electron achieves a relatively low velocity and little acceleration (the work that the field does on it is converted largely into heat). The electron therefore moves approximately along a streamline. Where does it leave the plate and in what direction is it moving at the time? Considering the result of part *a*, we would expect the exit to occur along the bottom edge of the plate. The equation of the streamline is found through

$$\frac{E_y}{E_x} = \frac{dy}{dx} = -\frac{\cos y}{\sin y} \quad \Rightarrow \quad x = -\int \tan y\, dy + C = \ln(\cos y) + C$$

At the entry point $(0, \pi/3)$, we have $0 = \ln[\cos(\pi/3)] + C$, from which $C = 0.69$. Now, along the bottom edge ($y = 0$), we find $x = 0.69$, and so the exit point is $\underline{(0.69, 0)}$. From the field expression evaluated at the exit point, we find the direction on exit to be $\underline{-\mathbf{a}_y}$.

4.27. Two point charges, 1 nC at $(0, 0, 0.1)$ and -1 nC at $(0, 0, -0.1)$, are in free space.

a) Calculate V at $P(0.3, 0, 0.4)$: Use

$$V_P = \frac{q}{4\pi\epsilon_0|\mathbf{R}^+|} - \frac{q}{4\pi\epsilon_0|\mathbf{R}^-|}$$

where $\mathbf{R}^+ = (.3, 0, .3)$ and $\mathbf{R}^- = (.3, 0, .5)$, so that $|\mathbf{R}^+| = 0.424$ and $|\mathbf{R}^-| = 0.583$. Thus

$$V_P = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{1}{.424} - \frac{1}{.583}\right] = \underline{5.78\,\text{V}}$$

b) Calculate $|\mathbf{E}|$ at P: Use

$$\mathbf{E}_P = \frac{q(.3\mathbf{a}_x + .3\mathbf{a}_z)}{4\pi\epsilon_0(.424)^3} - \frac{q(.3\mathbf{a}_x + .5\mathbf{a}_z)}{4\pi\epsilon_0(.583)^3} = \frac{10^{-9}}{4\pi\epsilon_0}[2.42\mathbf{a}_x + 1.41\mathbf{a}_z]\ \text{V/m}$$

Taking the magnitude of the above, we find $|\mathbf{E}_P| = \underline{25.2\,\text{V/m}}$.

c) Now treat the two charges as a dipole at the origin and find V at P: In spherical coordinates, P is located at $r = \sqrt{.3^2 + .4^2} = .5$ and $\theta = \sin^{-1}(.3/.5) = 36.9°$. Assuming a dipole in far-field, we have

$$V_P = \frac{qd\cos\theta}{4\pi\epsilon_0 r^2} = \frac{10^{-9}(.2)\cos(36.9°)}{4\pi\epsilon_0(.5)^2} = \underline{5.76\,\text{V}}$$

4.28. Use the electric field intensity of the dipole (Sec. 4.7, Eq. (36)) to find the difference in potential between points at θ_a and θ_b, each point having the same r and ϕ coordinates. Under what conditions does the answer agree with Eq. (34), for the potential at θ_a?

We perform a line integral of Eq. (36) along an arc of constant r and ϕ:

$$V_{ab} = -\int_{\theta_b}^{\theta_a} \frac{qd}{4\pi\epsilon_0 r^3}[2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta]\cdot\mathbf{a}_\theta\, r\,d\theta = -\int_{\theta_b}^{\theta_a}\frac{qd}{4\pi\epsilon_0 r^2}\sin\theta\,d\theta$$
$$= \frac{qd}{4\pi\epsilon_0 r^2}[\cos\theta_a - \cos\theta_b]$$

This result agrees with Eq. (34) if θ_a (the ending point in the path) is 90° (the xy plane). Under this condition, we note that if $\theta_b > 90°$, positive work is done when moving (against the field) to the xy plane; if $\theta_b < 90°$, negative work is done since we move with the field.

4.29. A dipole having a moment $\mathbf{p} = 3\mathbf{a}_x - 5\mathbf{a}_y + 10\mathbf{a}_z$ nC $\cdot$ m is located at $Q(1, 2, -4)$ in free space. Find V at $P(2, 3, 4)$: We use the general expression for the potential in the far field:

$$V = \frac{\mathbf{p}\cdot(\mathbf{r} - \mathbf{r}')}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|^3}$$

where $\mathbf{r} - \mathbf{r}' = P - Q = (1, 1, 8)$. So

$$V_P = \frac{(3\mathbf{a}_x - 5\mathbf{a}_y + 10\mathbf{a}_z)\cdot(\mathbf{a}_x + \mathbf{a}_y + 8\mathbf{a}_z)\times 10^{-9}}{4\pi\epsilon_0[1^2 + 1^2 + 8^2]^{1.5}} = \underline{1.31\ \text{V}}$$

4.30. A dipole for which $\mathbf{p} = 10\epsilon_0\,\mathbf{a}_z$ C · m is located at the origin. What is the equation of the surface on which $E_z = 0$ but $\mathbf{E} \neq 0$?

First we find the z component:

$$E_z = \mathbf{E}\cdot\mathbf{a}_z = \frac{10}{4\pi r^3}\left[2\cos\theta\,(\mathbf{a}_r\cdot\mathbf{a}_z) + \sin\theta\,(\mathbf{a}_\theta\cdot\mathbf{a}_z)\right] = \frac{5}{2\pi r^3}\left[2\cos^2\theta - \sin^2\theta\right]$$

This will be zero when $\left[2\cos^2\theta - \sin^2\theta\right] = 0$. Using identities, we write

$$2\cos^2\theta - \sin^2\theta = \frac{1}{2}[1 + 3\cos(2\theta)]$$

The above becomes zero on the cone surfaces, $\underline{\theta = 54.7^\circ}$ and $\underline{\theta = 125.3^\circ}$.

4.31. A potential field in free space is expressed as $V = 20/(xyz)$ V.

a) Find the total energy stored within the cube $1 < x, y, z < 2$. We integrate the energy density over the cube volume, where $w_E = (1/2)\epsilon_0\mathbf{E}\cdot\mathbf{E}$, and where

$$\mathbf{E} = -\nabla V = 20\left[\frac{1}{x^2yz}\mathbf{a}_x + \frac{1}{xy^2z}\mathbf{a}_y + \frac{1}{xyz^2}\mathbf{a}_z\right] \text{ V/m}$$

The energy is now

$$W_E = 200\epsilon_0\int_1^2\int_1^2\int_1^2\left[\frac{1}{x^4y^2z^2} + \frac{1}{x^2y^4z^2} + \frac{1}{x^2y^2z^4}\right]dx\,dy\,dz$$

The integral evaluates as follows:

$$\begin{aligned}
W_E &= 200\epsilon_0\int_1^2\int_1^2\left[-\left(\frac{1}{3}\right)\frac{1}{x^3y^2z^2} - \frac{1}{xy^4z^2} - \frac{1}{xy^2z^4}\right]_1^2 dy\,dz \\
&= 200\epsilon_0\int_1^2\int_1^2\left[\left(\frac{7}{24}\right)\frac{1}{y^2z^2} + \left(\frac{1}{2}\right)\frac{1}{y^4z^2} + \left(\frac{1}{2}\right)\frac{1}{y^2z^4}\right]dy\,dz \\
&= 200\epsilon_0\int_1^2\left[-\left(\frac{7}{24}\right)\frac{1}{yz^2} - \left(\frac{1}{6}\right)\frac{1}{y^3z^2} - \left(\frac{1}{2}\right)\frac{1}{yz^4}\right]_1^2 dz \\
&= 200\epsilon_0\int_1^2\left[\left(\frac{7}{48}\right)\frac{1}{z^2} + \left(\frac{7}{48}\right)\frac{1}{z^2} + \left(\frac{1}{4}\right)\frac{1}{z^4}\right]dz \\
&= 200\epsilon_0(3)\left[\frac{7}{96}\right] = \underline{387\,\text{pJ}}
\end{aligned}$$

b) What value would be obtained by assuming a uniform energy density equal to the value at the center of the cube? At $C(1.5, 1.5, 1.5)$ the energy density is

$$w_E = 200\epsilon_0(3)\left[\frac{1}{(1.5)^4(1.5)^2(1.5)^2}\right] = 2.07\times10^{-10}\ \text{J/m}^3$$

This, multiplied by a cube volume of 1, produces an energy value of $\underline{207\,\text{pJ}}$.

4.32. Using Eq. (36), a) find the energy stored in the dipole field in the region $r > a$:

We start with

$$\mathbf{E}(r,\theta) = \frac{qd}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta\right]$$

Then the energy will be

$$W_e = \int_{vol} \frac{1}{2}\epsilon_0 \mathbf{E}\cdot\mathbf{E}\,dv = \int_0^{2\pi}\int_0^{\pi}\int_a^{\infty} \frac{(qd)^2}{32\pi^2\epsilon_0 r^6}\underbrace{\left[4\cos^2\theta + \sin^2\theta\right]}_{3\cos^2\theta+1} r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \frac{-2\pi(qd)^2}{32\pi^2\epsilon_0}\left.\frac{1}{3r^3}\right|_a^{\infty}\int_0^{\pi}\left[3\cos^2\theta+1\right]\sin\theta\,d\theta = \frac{(qd)^2}{48\pi^2\epsilon_0 a^3}\underbrace{\left[-\cos^3\theta - \cos\theta\right]_0^{\pi}}_{4}$$

$$= \frac{(qd)^2}{12\pi\epsilon_0 a^3}\ \text{J}$$

b) Why can we not let a approach zero as a limit? From the above result, a singularity in the energy occurs as $a \to 0$. More importantly, a cannot be too small, or the original far-field assumption used to derive Eq. (36) ($a >> d$) will not hold, and so the field expression will not be valid.

4.33. A copper sphere of radius 4 cm carries a uniformly-distributed total charge of $5\,\mu$C in free space.

a) Use Gauss' law to find **D** external to the sphere: with a spherical Gaussian surface at radius r, D will be the total charge divided by the area of this sphere, and will be $\mathbf{a}_r$-directed. Thus

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r = \frac{5\times10^{-6}}{4\pi r^2}\mathbf{a}_r\ \text{C/m}^2$$

b) Calculate the total energy stored in the electrostatic field: Use

$$W_E = \int_{vol}\frac{1}{2}\mathbf{D}\cdot\mathbf{E}\,dv = \int_0^{2\pi}\int_0^{\pi}\int_{.04}^{\infty}\frac{1}{2}\,\frac{(5\times10^{-6})^2}{16\pi^2\epsilon_0 r^4}\,r^2\,\sin\theta\,dr\,d\theta\,d\phi$$

$$= (4\pi)\left(\frac{1}{2}\right)\frac{(5\times10^{-6})^2}{16\pi^2\epsilon_0}\int_{.04}^{\infty}\frac{dr}{r^2} = \frac{25\times10^{-12}}{8\pi\epsilon_0}\,\frac{1}{.04} = \underline{2.81\,\text{J}}$$

c) Use $W_E = Q^2/(2C)$ to calculate the capacitance of the isolated sphere: We have

$$C = \frac{Q^2}{2W_E} = \frac{(5\times10^{-6})^2}{2(2.81)} = 4.45\times10^{-12}\,\text{F} = \underline{4.45\,\text{pF}}$$

4.34. A sphere of radius a contains volume charge of uniform density ρ_0 C/m^3. Find the total stored energy by applying

a) Eq. (43): We first need the potential everywhere inside the sphere. The electric field inside and outside is readily found from Gauss's law:

$$\mathbf{E}_1 = \frac{\rho_0 r}{3\epsilon_0}\,\mathbf{a}_r \quad r \le a \quad \text{and} \quad \mathbf{E}_2 = \frac{\rho_0 a^3}{3\epsilon_0 r^2}\,\mathbf{a}_r \quad r \ge a$$

The potential at position r inside the sphere is now the work done in moving a unit positive point charge from infinity to position r:

$$V(r) = -\int_\infty^a \mathbf{E}_2 \cdot \mathbf{a}_r\, dr - \int_a^r \mathbf{E}_1 \cdot \mathbf{a}_r\, dr' = -\int_\infty^a \frac{\rho_0 a^3}{3\epsilon_0 r^2}\, dr - \int_a^r \frac{\rho_0 r'}{3\epsilon_0}\, dr' = \frac{\rho_0}{6\epsilon_0}\left(3a^2 - r^2\right)$$

Now, using this result in (43) leads to the energy associated with the charge in the sphere:

$$W_e = \frac{1}{2}\int_0^{2\pi}\int_0^\pi\int_0^a \frac{\rho_0^2}{6\epsilon_0}\left(3a^2 - r^2\right) r^2 \sin\theta\, dr\, d\theta\, d\phi = \frac{\pi\rho_0}{3\epsilon_0}\int_0^a \left(3a^2 r^2 - r^4\right)\, dr = \frac{4\pi a^5 \rho_0^2}{15\epsilon_0}$$

b) Eq. (45): Using the given fields we find the energy densities

$$w_{e1} = \frac{1}{2}\epsilon_0 \mathbf{E}_1 \cdot \mathbf{E}_1 = \frac{\rho_0^2 r^2}{18\epsilon_0} \quad r \le a \quad \text{and} \quad w_{e2} = \frac{1}{2}\epsilon_0 \mathbf{E}_2 \cdot \mathbf{E}_2 = \frac{\rho_0^2 a^6}{18\epsilon_0 r^4} \quad r \ge a$$

We now integrate these over their respective volumes to find the total energy:

$$W_e = \int_0^{2\pi}\int_0^\pi\int_0^a \frac{\rho_0^2 r^2}{18\epsilon_0} r^2 \sin\theta\, dr\, d\theta\, d\phi + \int_0^{2\pi}\int_0^\pi\int_a^\infty \frac{\rho_0^2 a^6}{18\epsilon_0 r^4} r^2 \sin\theta\, dr\, d\theta\, d\phi = \frac{4\pi a^5 \rho_0^2}{15\epsilon_0}$$

4.35. Four 0.8 nC point charges are located in free space at the corners of a square 4 cm on a side.

a) Find the total potential energy stored: This will be given by

$$W_E = \frac{1}{2}\sum_{n=1}^{4} q_n V_n$$

where V_n in this case is the potential at the location of any one of the point charges that arises from the other three. This will be (for charge 1)

$$V_1 = V_{21} + V_{31} + V_{41} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{.04} + \frac{1}{.04} + \frac{1}{.04\sqrt{2}}\right]$$

Taking the summation produces a factor of 4, since the situation is the same at all four points. Consequently,

$$W_E = \frac{1}{2}(4)q_1 V_1 = \frac{(.8\times 10^{-9})^2}{2\pi\epsilon_0(.04)}\left[2 + \frac{1}{\sqrt{2}}\right] = 7.79\times 10^{-7}\,\text{J} = \underline{0.779\,\mu\text{J}}$$

b) A fifth 0.8 nC charge is installed at the center of the square. Again find the total stored energy: This will be the energy found in part *a* plus the amount of work done in moving the fifth charge into position from infinity. The latter is just the potential at the square center arising from the original four charges, times the new charge value, or

$$\Delta W_E = \frac{4(.8\times 10^{-9})^2}{4\pi\epsilon_0(.04\sqrt{2}/2)} = .813\,\mu\text{J}$$

The total energy is now

$$W_{E\,net} = W_E(\text{part a}) + \Delta W_E = .779 + .813 = \underline{1.59\,\mu\text{J}}$$

4.36 Surface charge of uniform density ρ_s lies on a spherical shell of radius b, centered at the origin in free space.

a) Find the absolute potential everywhere, with zero reference at infinity: First, the electric field, found from Gauss' law, is

$$\mathbf{E} = \frac{b^2\rho_s}{\epsilon_0 r^2}\,\mathbf{a}_r \text{ V/m}$$

Then

$$V(r) = -\int_\infty^r \mathbf{E}\cdot d\mathbf{L} = -\int_\infty^r \frac{b^2\rho_s}{\epsilon_0 (r')^2}\,dr' = \underline{\frac{b^2\rho_s}{\epsilon_0 r}}\text{ V}$$

b) find the stored energy in the sphere by considering the charge density and the potential in a two-dimensional version of Eq. (42):

$$W_e = \frac{1}{2}\int_S \rho_s V(b)\,da = \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\rho_s\frac{b^2\rho_s}{\epsilon_0 b}\,b^2\sin\theta\,d\theta\,d\phi = \underline{\frac{2\pi\rho_s^2 b^3}{\epsilon_0}}$$

c) find the stored energy in the electric field and show that the results of parts *b* and *c* are identical.

$$W_e = \int_v \frac{1}{2}\mathbf{D}\cdot\mathbf{E}\,dv = \int_0^{2\pi}\int_0^{\pi}\int_b^{\infty}\frac{1}{2}\frac{b^4\rho_s^2}{\epsilon_0 r^4}\,r^2\sin\theta\,dr\,d\theta\,d\phi = \underline{\frac{2\pi\rho_s^2 b^3}{\epsilon_0}}$$

CHAPTER 5

5.1. Given the current density $\mathbf{J} = -10^4[\sin(2x)e^{-2y}\mathbf{a}_x + \cos(2x)e^{-2y}\mathbf{a}_y]\,\text{kA/m}^2$:

a) Find the total current crossing the plane $y = 1$ in the $\mathbf{a}_y$ direction in the region $0 < x < 1$, $0 < z < 2$: This is found through

$$I = \int\int_S \mathbf{J}\cdot\mathbf{n}\Big|_S da = \int_0^2\int_0^1 \mathbf{J}\cdot\mathbf{a}_y\Big|_{y=1} dx\,dz = \int_0^2\int_0^1 -10^4\cos(2x)e^{-2}\,dx\,dz$$
$$= -10^4(2)\frac{1}{2}\sin(2x)\Big|_0^1 e^{-2} = \underline{-1.23\,\text{MA}}$$

b) Find the total current leaving the region $0 < x, x < 1$, $2 < z < 3$ by integrating $\mathbf{J}\cdot\mathbf{dS}$ over the surface of the cube: Note first that current through the top and bottom surfaces will not exist, since $\mathbf{J}$ has no z component. Also note that there will be no current through the $x = 0$ plane, since $J_x = 0$ there. Current will pass through the three remaining surfaces, and will be found through

$$I = \int_2^3\int_0^1 \mathbf{J}\cdot(-\mathbf{a}_y)\Big|_{y=0} dx\,dz + \int_2^3\int_0^1 \mathbf{J}\cdot(\mathbf{a}_y)\Big|_{y=1} dx\,dz + \int_2^3\int_0^1 \mathbf{J}\cdot(\mathbf{a}_x)\Big|_{x=1} dy\,dz$$
$$= 10^4\int_2^3\int_0^1 \left[\cos(2x)e^{-0} - \cos(2x)e^{-2}\right] dx\,dz - 10^4\int_2^3\int_0^1 \sin(2)e^{-2y}\,dy\,dz$$
$$= 10^4\left(\frac{1}{2}\right)\sin(2x)\Big|_0^1(3-2)\left[1-e^{-2}\right] + 10^4\left(\frac{1}{2}\right)\sin(2)e^{-2y}\Big|_0^1(3-2) = \underline{0}$$

c) Repeat part b, but use the divergence theorem: We find the net outward current through the surface of the cube by integrating the divergence of $\mathbf{J}$ over the cube volume. We have

$$\nabla\cdot\mathbf{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} = -10^{-4}\left[2\cos(2x)e^{-2y} - 2\cos(2x)e^{-2y}\right] = \underline{0} \text{ as expected}$$

5.2. Given $\mathbf{J} = -10^{-4}(y\mathbf{a}_x + x\mathbf{a}_y)\,\text{A/m}^2$, find the current crossing the $y = 0$ plane in the $-\mathbf{a}_y$ direction between $z = 0$ and 1, and $x = 0$ and 2.

At $y = 0$, $\mathbf{J}(x,0) = -10^4 x\mathbf{a}_y$, so that the current through the plane becomes

$$I = \int \mathbf{J}\cdot d\mathbf{S} = \int_0^1\int_0^2 -10^4 x\,\mathbf{a}_y\cdot(-\mathbf{a}_y)\,dx\,dz = \underline{2\times 10^{-4}}\text{ A}$$

5.3. Let

$$\mathbf{J} = \frac{400 \sin\theta}{r^2+4}\, \mathbf{a}_r \ \text{A/m}^2$$

a) Find the total current flowing through that portion of the spherical surface $r = 0.8$, bounded by $0.1\pi < \theta < 0.3\pi$, $0 < \phi < 2\pi$: This will be

$$I = \int\int \mathbf{J}\cdot\mathbf{n}\Big|_S \, da = \int_0^{2\pi}\int_{.1\pi}^{.3\pi} \frac{400\sin\theta}{(.8)^2+4}(.8)^2 \sin\theta \, d\theta\, d\phi = \frac{400(.8)^2 2\pi}{4.64}\int_{.1\pi}^{.3\pi} \sin^2 \, d\theta$$

$$= 346.5 \int_{.1\pi}^{.3\pi} \frac{1}{2}[1-\cos(2\theta)]\, d\theta = \underline{77.4\,\text{A}}$$

b) Find the average value of $\mathbf{J}$ over the defined area. The area is

$$\text{Area} = \int_0^{2\pi}\int_{.1\pi}^{.3\pi} (.8)^2 \sin\theta\, d\theta\, d\phi = 1.46\,\text{m}^2$$

The average current density is thus $\mathbf{J}_{avg} = (77.4/1.46)\,\mathbf{a}_r = \underline{53.0\,\mathbf{a}_r\ \text{A/m}^2}$.

5.4. If volume charge density is given as $\rho_v = (\cos\omega t)/r^2\ \text{C/m}^3$ in spherical coordinates, find $\mathbf{J}$. It is reasonable to assume that $\mathbf{J}$ is not a function of θ or ϕ.

We use the continuity equation (5), along with the assumption of no angular variation to write

$$\nabla\cdot\mathbf{J} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 J_r\right) = -\frac{\partial\rho_v}{\partial t} = -\frac{\partial}{\partial t}\left(\frac{\cos\omega t}{r^2}\right) = \frac{\omega\sin\omega t}{r^2}$$

So we may now solve

$$\frac{\partial}{\partial r}\left(r^2 J_r\right) = \omega\sin\omega t$$

by direct integration to obtain:

$$\mathbf{J} = J_r\,\mathbf{a}_r = \underline{\frac{\omega\sin\omega t}{r}\,\mathbf{a}_r}\ \text{A/m}^2$$

where the integration constant is set to zero because a steady current will not be created by a time-varying charge density.

5.5. Let

$$\mathbf{J} = \frac{25}{\rho}\,\mathbf{a}_\rho - \frac{20}{\rho^2 + 0.01}\,\mathbf{a}_z \text{ A/m}^2$$

a) Find the total current crossing the plane $z = 0.2$ in the $\mathbf{a}_z$ direction for $\rho < 0.4$: Use

$$I = \int\int_S \mathbf{J}\cdot\mathbf{n}\Big|_{z=.2} da = \int_0^{2\pi}\int_0^{.4} \frac{-20}{\rho^2 + .01}\rho\,d\rho\,d\phi$$
$$= -\left(\frac{1}{2}\right) 20\ln(.01 + \rho^2)\Big|_0^{.4}(2\pi) = -20\pi\ln(17) = \underline{-178.0\,\text{A}}$$

b) Calculate $\partial\rho_v/\partial t$: This is found using the equation of continuity:

$$\frac{\partial\rho_v}{\partial t} = -\nabla\cdot\mathbf{J} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho J_\rho) + \frac{\partial J_z}{\partial z} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(25) + \frac{\partial}{\partial z}\left(\frac{-20}{\rho^2 + .01}\right) = \underline{0}$$

c) Find the outward current crossing the closed surface defined by $\rho = 0.01$, $\rho = 0.4$, $z = 0$, and $z = 0.2$: This will be

$$I = \int_0^{.2}\int_0^{2\pi} \frac{25}{.01}\mathbf{a}_\rho\cdot(-\mathbf{a}_\rho)(.01)\,d\phi\,dz + \int_0^{.2}\int_0^{2\pi}\frac{25}{.4}\mathbf{a}_\rho\cdot(\mathbf{a}_\rho)(.4)\,d\phi\,dz$$
$$+ \int_0^{2\pi}\int_0^{.4}\frac{-20}{\rho^2 + .01}\mathbf{a}_z\cdot(-\mathbf{a}_z)\,\rho\,d\rho\,d\phi + \int_0^{2\pi}\int_0^{.4}\frac{-20}{\rho^2 + .01}\mathbf{a}_z\cdot(\mathbf{a}_z)\,\rho\,d\rho\,d\phi = \underline{0}$$

since the integrals will cancel each other.

d) Show that the divergence theorem is satisfied for **J** and the surface specified in part *b*. In part *c*, the net outward flux was found to be zero, and in part *b*, the divergence of **J** was found to be zero (as will be its volume integral). Therefore, the divergence theorem is satisfied.

5.6. In spherical coordinates, a current density $\mathbf{J} = -k/(r\sin\theta)\,\mathbf{a}_\theta$ A/m^2 exists in a conducting medium, where k is a constant. Determine the total current in the $\mathbf{a}_z$ direction that crosses a circular disk of radius R, centered on the z axis and located at a) $z = 0$; b) $z = h$.

Integration over a disk means that we use cylindrical coordinates. The general flux integral assumes the form:

$$I = \int_s \mathbf{J}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^R \frac{-k}{r\sin\theta}\underbrace{\mathbf{a}_\theta\cdot\mathbf{a}_z}_{-\sin\theta}\,\rho\,d\rho\,d\phi$$

Then, using $r = \sqrt{\rho^2 + z^2}$, this becomes

$$I = \int_0^{2\pi}\int_0^R \frac{k\rho}{\sqrt{\rho^2 + z^2}} = 2\pi k\sqrt{\rho^2 + z^2}\Big|_0^R = 2\pi k\left[\sqrt{R^2 + z^2} - z\right]$$

At $z = 0$ (part *a*), we have $I(0) = \underline{2\pi kR}$, and at $z = h$ (part *b*): $I(h) = \underline{2\pi k\left[\sqrt{R^2 + h^2} - h\right]}$.

5.7. Assuming that there is no transformation of mass to energy or vice-versa, it is possible to write a continuity equation for mass.

a) If we use the continuity equation for charge as our model, what quantities correspond to **J** and ρ_v? These would be, respectively, mass flux density in $(\text{kg/m}^2 - \text{s})$ and mass density in (kg/m^3).

b) Given a cube 1 cm on a side, experimental data show that the rates at which mass is leaving each of the six faces are 10.25, -9.85, 1.75, -2.00, -4.05, and 4.45 mg/s. If we assume that the cube is an incremental volume element, determine an approximate value for the time rate of change of density at its center. We may write the continuity equation for mass as follows, also invoking the divergence theorem:

$$\int_v \frac{\partial \rho_m}{\partial t} dv = -\int_v \nabla \cdot \mathbf{J}_m \, dv = -\oint_s \mathbf{J}_m \cdot \mathbf{dS}$$

where

$$\oint_s \mathbf{J}_m \cdot \mathbf{dS} = 10.25 - 9.85 + 1.75 - 2.00 - 4.05 + 4.45 = 0.550 \text{ mg/s}$$

Treating our 1 cm^3 volume as differential, we find

$$\frac{\partial \rho_m}{\partial t} \doteq -\frac{0.550 \times 10^{-3} \,\text{g/s}}{10^{-6} \text{ m}^3} = \underline{-550 \,\text{g/m}^3 - \text{s}}$$

5.8. A truncated cone has a height of 16 cm. The circular faces on the top and bottom have radii of 2mm and 0.1mm, respectively. If the material from which this solid cone is constructed has a conductivity of 2×10^6 S/m, use some good approximations to determine the resistance between the two circular faces.

Consider the cone upside down and centered on the positive z axis. The 1-mm radius end is at distance $z = \ell$ from the x-y plane; the wide end (2-mm radius) lies at $z = \ell + 16\,\text{cm}$. ℓ is chosen such that if the cone were not truncated, its vertex would occur at the origin. The cone surface subtends angle θ_c from the z axis (in spherical coordinates). Therefore, we may write

$$\ell = \frac{0.1\text{mm}}{\tan\theta_c} \quad \text{and} \quad \tan\theta_c = \frac{2\,\text{mm}}{160 + \ell}$$

Solving these, we find $\ell = 8.4$ mm, $\tan\theta_c = 1.19 \times 10^{-2}$, and so $\theta_c = 0.68°$, which gives us a very thin cone! With this understanding, we can assume that the current density is uniform with θ and ϕ and will vary only with spherical radius, r. So the current density will be constant over a spherical cap (of constant r) anywhere within the cone. As the cone is thin, we can also assume constant current density over any *flat* surface within the cone at a specifed z. That is, any spherical cap looks flat if the cap radius, r, is large compared to its radius as measured from the z axis (ρ). This is our primary assumption.

Now, assuming constant current density at constant r, and net current, I, we may write

$$I = \int_0^{2\pi} \int_0^{\theta_c} J(r)\,\mathbf{a}_r \cdot \mathbf{a}_r\, r^2 \sin\theta \, d\theta \, d\phi = 2\pi r^2 J(r)(1 - \cos\theta_c)$$

or

$$\mathbf{J}(r) = \frac{I}{2\pi r^2 (1 - \cos\theta_c)}\,\mathbf{a}_r = \frac{(1.42 \times 10^4) I}{2\pi r^2}\,\mathbf{a}_r$$

5.8 (continued) The electric field is now

$$\mathbf{E}(r) = \frac{\mathbf{J}(r)}{\sigma} = \frac{(1.42 \times 10^4)I}{2\pi r^2 (2 \times 10^6)}\, \mathbf{a}_r \;=\; (7.1 \times 10^{-3})\frac{I}{2\pi r^2}\, \mathbf{a}_r \text{ V/m}$$

The voltage between the ends is now

$$V_0 = -\int_{r_{out}}^{r_{in}} \mathbf{E} \cdot \mathbf{a}_r \, dr$$

where $r_{in} = \ell / \cos\theta_c \doteq \ell$ and where $r_{out} = (160+\ell)/\cos\theta_c \doteq 160 + \ell$. The voltage is

$$V_0 = -\int_{(160+\ell)\times 10^{-3}}^{\ell \times 10^{-3}} (7.1 \times 10^{-3})\frac{I}{2\pi r^2}\, \mathbf{a}_r \cdot \mathbf{a}_r \, dr = (7.1 \times 10^{-3})\frac{I}{2\pi}\left[\frac{1}{.0084} - \frac{1}{.1684}\right]$$
$$= \frac{0.40}{\pi} I$$

from which we identify the resistance as $R = 0.40/\pi = \underline{0.128}$ ohms.

A second method uses the idea that we can construct the cone from a stack of thin circular plates of linearly-increasing radius, ρ. Assuming each plate is of differential thickness, dz, the differential resistance of a plate will be

$$dR = \frac{dz}{\sigma \pi \rho^2}$$

where $\rho = z \tan\theta_c = z(1.19 \times 10^{-2})$. The cone resistance will be the resistance of the stack of plates (in series), found through

$$R = \int dR = \int_{\ell \times 10^{-3}}^{(160+\ell)\times 10^{-3}} \frac{dz}{\sigma \pi \rho^2} = \int_{\ell \times 10^{-3}}^{(160+\ell)\times 10^{-3}} \frac{dz}{(2 \times 10^6)\pi z^2 (1.19 \times 10^{-2})^2}$$
$$= \frac{3.53 \times 10^{-3}}{\pi}\left[\frac{1}{.0084} - \frac{1}{.1684}\right] = \underline{0.127} \text{ ohms}$$

5.9. a) Using data tabulated in Appendix C, calculate the required diameter for a 2-m long nichrome wire that will dissipate an average power of 450 W when 120 V rms at 60 Hz is applied to it:
The required resistance will be

$$R = \frac{V^2}{P} = \frac{l}{\sigma(\pi a^2)}$$

Thus the diameter will be

$$d = 2a = 2\sqrt{\frac{lP}{\sigma \pi V^2}} = 2\sqrt{\frac{2(450)}{(10^6)\pi(120)^2}} = 2.8 \times 10^{-4} \text{ m} = \underline{0.28\,\text{mm}}$$

b) Calculate the rms current density in the wire: The rms current will be $I = 450/120 = 3.75\,\text{A}$. Thus

$$J = \frac{3.75}{\pi\left(2.8 \times 10^{-4}/2\right)^2} = \underline{6.0 \times 10^7 \text{ A/m}^2}$$

5.10. A large brass washer has a 2-cm inside diameter, a 5-cm outside diameter, and is 0.5 cm thick. Its conductivity is $\sigma = 1.5 \times 10^7$ S/m. The washer is cut in half along a diameter, and a voltage is applied between the two rectangular faces of one part. The resultant electric field in the interior of the half-washer is $\mathbf{E} = (0.5/\rho)\,\mathbf{a}_\phi$ V/m in cylindrical coordinates, where the z axis is the axis of the washer.

a) What potential difference exists between the two rectangular faces? First, we orient the washer in the x-y plane with the cut faces aligned with the x axis. To find the voltage, we integrate $\mathbf{E}$ over a circular path of radius ρ inside the washer, between the two cut faces:

$$V_0 = -\int \mathbf{E} \cdot d\mathbf{L} = -\int_\pi^0 \frac{0.5\pi}{\rho}\,\mathbf{a}_\phi \cdot \mathbf{a}_\phi\,\rho\,d\phi = \underline{0.5\pi}\text{ V}$$

b) What total current is flowing? First, the current density is $\mathbf{J} = \sigma\mathbf{E}$, so

$$\mathbf{J} = \frac{1.5 \times 10^7(0.5)}{\rho}\,\mathbf{a}_\phi = \frac{7.5 \times 10^6}{\rho}\,\mathbf{a}_\phi\ \text{A/m}^2$$

Current is then found by integrating $\mathbf{J}$ over any transverse plane in the washer (the rectangular cross-section):

$$I = \int_s \mathbf{J} \cdot d\mathbf{S} = \int_0^{0.5\times10^{-2}} \int_{10^{-2}}^{2.5\times10^{-2}} \frac{7.5 \times 10^6}{\rho}\,\mathbf{a}_\phi \cdot \mathbf{a}_\phi\,d\rho\,dz$$
$$= 0.5 \times 10^{-2}(7.5 \times 10^6)\ln\left(\frac{2.5}{1}\right) = \underline{3.4 \times 10^4}\text{ A}$$

c) What is the resistance between the two faces?

$$R = \frac{V_0}{I} = \frac{0.5\pi}{3.4 \times 10^4} = \underline{4.6 \times 10^{-5}}\text{ ohms}$$

5.11. Two perfectly-conducting cylindrical surfaces of length l are located at $\rho = 3$ and $\rho = 5$ cm. The total current passing radially outward through the medium between the cylinders is 3 A dc.

a) Find the voltage and resistance between the cylinders, and $\mathbf{E}$ in the region between the cylinders, if a conducting material having $\sigma = 0.05$ S/m is present for $3 < \rho < 5$ cm: Given the current, and knowing that it is radially-directed, we find the current density by dividing it by the area of a cylinder of radius ρ and length l:

$$\mathbf{J} = \frac{3}{2\pi\rho l}\,\mathbf{a}_\rho\ \text{A/m}^2$$

5.11a) (continued)

Then the electric field is found by dividing this result by σ:

$$\mathbf{E} = \frac{3}{2\pi\sigma\rho l}\,\mathbf{a}_\rho = \underline{\frac{9.55}{\rho l}\mathbf{a}_\rho \text{ V/m}}$$

The voltage between cylinders is now:

$$V = -\int_5^3 \mathbf{E}\cdot\mathbf{dL} = \int_3^5 \frac{9.55}{\rho l}\mathbf{a}_\rho\cdot\mathbf{a}_\rho d\rho = \frac{9.55}{l}\ln\left(\frac{5}{3}\right) = \underline{\frac{4.88}{l}\text{ V}}$$

Now, the resistance will be

$$R = \frac{V}{I} = \frac{4.88}{3l} = \underline{\frac{1.63}{l}\ \Omega}$$

b) Show that integrating the power dissipated per unit volume over the volume gives the total dissipated power: We calculate

$$P = \int_v \mathbf{E}\cdot\mathbf{J}\,dv = \int_0^l\int_0^{2\pi}\int_{.03}^{.05}\frac{3^2}{(2\pi)^2\rho^2(.05)l^2}\,\rho\,d\rho\,d\phi\,dz = \frac{3^2}{2\pi(.05)l}\ln\left(\frac{5}{3}\right) = \frac{14.64}{l}\text{ W}$$

We also find the power by taking the product of voltage and current:

$$P = VI = \frac{4.88}{l}(3) = \frac{14.64}{l}\text{ W}$$

which is in agreement with the power density integration.

5.12. Two identical conducting plates, each having area A, are located at $z=0$ and $z=d$. The region between plates is filled with a material having z-dependent conductivity, $\sigma(z) = \sigma_0 e^{-z/d}$, where σ_0 is a constant. Voltage V_0 is applied to the plate at $z=d$; the plate at $z=0$ is at zero potential. Find, in terms of the given parameters:

a) the resistance of the material: We start with the differential resistance of a thin slab of the material of thickness dz, which is

$$dR = \frac{dz}{\sigma A} = \frac{e^{z/d}dz}{\sigma_0 A} \quad\text{so that}\quad R = \int dR = \int_0^d \frac{e^{z/d}dz}{\sigma_0 A} = \frac{d}{\sigma_0 A}(e-1) = \frac{1.72d}{\sigma_0 A}\ \Omega$$

b) the total current flowing between plates: We use

$$I = \frac{V_0}{R} = \frac{\sigma_0 A V_0}{1.72\,d}$$

c) the electric field intensity $\mathbf{E}$ within the material: First the current density is

$$\mathbf{J} = -\frac{I}{A}\,\mathbf{a}_z = \frac{-\sigma_0 V_0}{1.72\,d}\,\mathbf{a}_z \quad\text{so that}\quad \mathbf{E} = \frac{\mathbf{J}}{\sigma(z)} = \frac{-V_0 e^{z/d}}{1.72\,d}\,\mathbf{a}_z\text{ V/m}$$

5.13. A hollow cylindrical tube with a rectangular cross-section has external dimensions of 0.5 in by 1 in and a wall thickness of 0.05 in. Assume that the material is brass, for which $\sigma = 1.5 \times 10^7$ S/m. A current of 200 A dc is flowing down the tube.

a) What voltage drop is present across a 1m length of the tube? Converting all measurements to meters, the tube resistance over a 1 m length will be:

$$\begin{aligned} R_1 &= \frac{1}{(1.5 \times 10^7)\left[(2.54)(2.54/2) \times 10^{-4} - 2.54(1 - .1)(2.54/2)(1 - .2) \times 10^{-4}\right]} \\ &= 7.38 \times 10^{-4}\ \Omega \end{aligned}$$

The voltage drop is now $V = IR_1 = 200(7.38 \times 10^{-4} = \underline{0.147 \text{ V}}$.

b) Find the voltage drop if the interior of the tube is filled with a conducting material for which $\sigma = 1.5 \times 10^5$ S/m: The resistance of the filling will be:

$$R_2 = \frac{1}{(1.5 \times 10^5)(1/2)(2.54)^2 \times 10^{-4}(.9)(.8)} = 2.87 \times 10^{-2}\ \Omega$$

The total resistance is now the parallel combination of R_1 and R_2: $R_T = R_1 R_2/(R_1 + R_2) = 7.19 \times 10^{-4}\ \Omega$, and the voltage drop is now $V = 200 R_T = \underline{.144\,\text{V}}$.

5.14. A rectangular conducting plate lies in the xy plane, occupying the region $0 < x < a$, $0 < y < b$. An identical conducting plate is positioned directly above and parallel to the first, at $z = d$. The region between plates is filled with material having conductivity $\sigma(x) = \sigma_0 e^{-x/a}$, where σ_0 is a constant. Voltage V_0 is applied to the plate at $z = d$; the plate at $z = 0$ is at zero potential. Find, in terms of the given parameters:

a) the electric field intensity $\mathbf{E}$ within the material: We know that $\mathbf{E}$ will be z-directed, but the conductivity varies with x. We therefore expect no z variation in $\mathbf{E}$, and also note that the line integral of $\mathbf{E}$ between the bottom and top plates must always give V_0. Therefore $\mathbf{E} = \underline{-V_0/d\,\mathbf{a}_z \text{ V/m}}$.

b) the total current flowing between plates: We have

$$\mathbf{J} = \sigma(x)\mathbf{E} = \frac{-\sigma_0 e^{-x/a} V_0}{d}\,\mathbf{a}_z$$

Using this, we find

$$I = \int \mathbf{J} \cdot d\mathbf{S} = \int_0^b \int_0^a \frac{-\sigma_0 e^{-x/a} V_0}{d}\,\mathbf{a}_z \cdot (-\mathbf{a}_z)\,dx\,dy = \frac{\sigma_0 ab V_0}{d}(1 - e^{-1}) = \frac{0.63 ab \sigma_0 V_0}{d}\ \text{A}$$

c) the resistance of the material: We use

$$R = \frac{V_0}{I} = \frac{d}{0.63\,ab\,\sigma_0}\ \Omega$$

5.15. Let $V = 10(\rho+1)z^2\cos\phi\,\mathrm{V}$ in free space.

a) Let the equipotential surface $V = 20\,\mathrm{V}$ define a conductor surface. Find the equation of the conductor surface: Set the given potential function equal to 20, to find:

$$\underline{(\rho+1)z^2\cos\phi = 2}$$

b) Find ρ and $\mathbf{E}$ at that point on the conductor surface where $\phi = 0.2\pi$ and $z = 1.5$: At the given values of ϕ and z, we solve the equation of the surface found in part a for ρ, obtaining $\rho = \underline{.10}$. Then

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial\rho}\mathbf{a}_\rho - \frac{1}{\rho}\frac{\partial V}{\partial\phi}\mathbf{a}_\phi - \frac{\partial V}{\partial z}\mathbf{a}_z$$
$$= -10z^2\cos\phi\,\mathbf{a}_\rho + 10\,\frac{\rho+1}{\rho}z^2\sin\phi\,\mathbf{a}_\phi - 20(\rho+1)z\cos\phi\,\mathbf{a}_z$$

Then

$$\mathbf{E}(.10, .2\pi, 1.5) = \underline{-18.2\,\mathbf{a}_\rho + 145\,\mathbf{a}_\phi - 26.7\,\mathbf{a}_z\ \mathrm{V/m}}$$

c) Find $|\rho_s|$ at that point: Since $\mathbf{E}$ is at the perfectly-conducting surface, it will be normal to the surface, so we may write:

$$\rho_s = \epsilon_0\mathbf{E}\cdot\mathbf{n}\Big|_{\text{surface}} = \epsilon_0\frac{\mathbf{E}\cdot\mathbf{E}}{|\mathbf{E}|} = \epsilon_0\sqrt{\mathbf{E}\cdot\mathbf{E}} = \epsilon_0\sqrt{(18.2)^2+(145)^2+(26.7)^2} = \underline{1.32\ \mathrm{nC/m^2}}$$

5.16. A coaxial transmission line has inner and outer conductor radii a and b. Between conductors $(a < \rho < b)$ lies a conductive medium whose conductivity is $\sigma(\rho) = \sigma_0/\rho$, where σ_0 is a constant. The inner conductor is charged to potential V_0, and the outer conductor is grounded.

a) Assuming dc radial current I per unit length in z, determine the radial current density field $\mathbf{J}$ in $\mathrm{A/m^2}$: This will be the current divided by the cross-sectional area that is normal to the current direction:

$$\mathbf{J} = \frac{I}{2\pi\rho(1)}\,\mathbf{a}_\rho\ \mathrm{A/m^2}$$

b) Determine the electric field intensity $\mathbf{E}$ in terms of I and other parameters, given or known:

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} = \frac{I\rho}{2\pi\sigma_0\rho}\,\mathbf{a}_\rho = \frac{I}{2\pi\sigma_0}\,\mathbf{a}_\rho\ \mathrm{V/m}$$

c) by taking an appropriate line integral of $\mathbf{E}$ as found in part b, find an expression that relates V_0 to I:

$$V_0 = -\int_b^a \mathbf{E}\cdot d\mathbf{L} = -\int_b^a \frac{I}{2\pi\sigma_0}\mathbf{a}_\rho\cdot\mathbf{a}_\rho\,d\rho = \frac{I(b-a)}{2\pi\sigma_0}\ \mathrm{V}$$

d) find an expression for the conductance of the line per unit length, G:

$$G = \frac{I}{V_0} = \frac{2\pi\sigma_0}{(b-a)}\ \mathrm{S/m}$$

5.17. Given the potential field $V = 100xz/(x^2 + 4)$ V. in free space:

a) Find $\mathbf{D}$ at the surface $z = 0$: Use

$$\mathbf{E} = -\nabla V = -100z\frac{\partial}{\partial x}\left(\frac{x}{x^2+4}\right)\mathbf{a}_x - 0\,\mathbf{a}_y - \frac{100x}{x^2+4}\mathbf{a}_z \text{ V/m}$$

At $z = 0$, we use this to find $\mathbf{D}(z = 0) = \epsilon_0 \mathbf{E}(z = 0) = \underline{-100\epsilon_0 x/(x^2 + 4)\,\mathbf{a}_z \text{ C/m}^2}$.

b) Show that the $z = 0$ surface is an equipotential surface: There are two reasons for this: 1) $\mathbf{E}$ at $z = 0$ is everywhere z-directed, and so moving a charge around on the surface involves doing no work; 2) When evaluating the given potential function at $z = 0$, the result is 0 for all x and y.

c) Assume that the $z = 0$ surface is a conductor and find the total charge on that portion of the conductor defined by $0 < x < 2$, $-3 < y < 0$: We have

$$\rho_s = \mathbf{D} \cdot \mathbf{a}_z\Big|_{z=0} = -\frac{100\epsilon_0 x}{x^2+4} \text{ C/m}^2$$

So

$$Q = \int_{-3}^{0}\int_{0}^{2} -\frac{100\epsilon_0 x}{x^2+4}\,dx\,dy = -(3)(100)\epsilon_0\left(\frac{1}{2}\right)\ln(x^2+4)\Big|_0^2 = -150\epsilon_0 \ln 2 = \underline{-0.92\,\text{nC}}$$

5.18. Two parallel circular plates of radius a are located at $z = 0$ and $z = d$. The top plate ($z = d$) is raised to potential V_0; the bottom plate is grounded. Between the plates is a conducting material having radial-dependent conductivity, $\sigma(\rho) = \sigma_0 \rho$, where σ_0 is a constant.

a) Find the *ρ-independent* electric field strength, $\mathbf{E}$, between plates: The integral of $\mathbf{E}$ between plates must give V_0, independent of position on the plates. Therefore, it must be true that

$$\mathbf{E} = -\frac{V_0}{d}\mathbf{a}_z \text{ V/m} \quad (0 < \rho < a)$$

b) Find the current density, $\mathbf{J}$ between plates:

$$\mathbf{J} = \sigma\mathbf{E} = -\frac{\sigma_0 V_0 \rho}{d}\mathbf{a}_z \text{ A/m}^2$$

c) Find the total current, I, in the structure:

$$I = \int_0^{2\pi}\int_0^a -\frac{\sigma_0 V_0 \rho}{d}\mathbf{a}_z \cdot (-\mathbf{a}_z)\,\rho\,d\rho\,d\phi = \frac{2\pi a^3 \sigma_0 V_0}{3d} \text{ A}$$

d) Find the resistance between plates:

$$R = \frac{V_0}{I} = \frac{3d}{2\pi a^3 \sigma_0} \text{ ohms}$$

5.19. Let $V = 20x^2yz - 10z^2$ V in free space.

a) Determine the equations of the equipotential surfaces on which $V = 0$ and 60 V: Setting the given potential function equal to 0 and 60 and simplifying results in:

$$\text{At } 0\,\text{V}: \quad 2x^2y - z = 0$$

$$\text{At } 60\,\text{V}: \quad 2x^2y - z = \frac{6}{z}$$

b) Assume these are conducting surfaces and find the surface charge density at that point on the $V = 60$ V surface where $x = 2$ and $z = 1$. It is known that $0 \le V \le 60$ V is the field-containing region: First, on the 60 V surface, we have

$$2x^2y - z - \frac{6}{z} = 0 \;\Rightarrow\; 2(2)^2y(1) - 1 - 6 = 0 \;\Rightarrow\; y = \frac{7}{8}$$

Now

$$\mathbf{E} = -\nabla V = -40xyz\,\mathbf{a}_x - 20x^2z\,\mathbf{a}_y - [20xy - 20z]\,\mathbf{a}_z$$

Then, at the given point, we have

$$\mathbf{D}(2, 7/8, 1) = \epsilon_0\mathbf{E}(2, 7/8, 1) = -\epsilon_0[70\,\mathbf{a}_x + 80\,\mathbf{a}_y + 50\,\mathbf{a}_z]\,\text{C/m}^2$$

We know that since this is the higher potential surface, **D** must be directed away from it, and so the charge density would be positive. Thus

$$\rho_s = \sqrt{\mathbf{D}\cdot\mathbf{D}} = 10\epsilon_0\sqrt{7^2 + 8^2 + 5^2} = \underline{1.04\,\text{nC/m}^2}$$

c) Give the unit vector at this point that is normal to the conducting surface and directed toward the $V = 0$ surface: This will be in the direction of **E** and **D** as found in part *b*, or

$$\mathbf{a}_n = -\left[\frac{7\mathbf{a}_x + 8\mathbf{a}_y + 5\mathbf{a}_z}{\sqrt{7^2 + 8^2 + 5^2}}\right] = \underline{-[0.60\mathbf{a}_x + 0.68\mathbf{a}_y + 0.43\mathbf{a}_z]}$$

5.20. Two point charges of -100π μC are located at (2,-1,0) and (2,1,0). The surface $x = 0$ is a conducting plane.

a) Determine the surface charge density at the origin. I will solve the general case first, in which we find the charge density anywhere on the y axis. With the conducting plane in the yz plane, we will have two image charges, each of $+100\pi$ μC, located at (-2, -1, 0) and (-2, 1, 0). The electric flux density on the y axis from these four charges will be

$$\mathbf{D}(y) = \frac{-100\pi}{4\pi}\left[\underbrace{\frac{[(y-1)\,\mathbf{a}_y - 2\,\mathbf{a}_x]}{[(y-1)^2+4]^{3/2}} + \frac{[(y+1)\,\mathbf{a}_y - 2\,\mathbf{a}_x]}{[(y+1)^2+4]^{3/2}}}_{\text{given charges}}\right.$$

$$\left. - \underbrace{\frac{[(y-1)\,\mathbf{a}_y + 2\,\mathbf{a}_x]}{[(y-1)^2+4]^{3/2}} - \frac{[(y+1)\,\mathbf{a}_y + 2\,\mathbf{a}_x]}{[(y+1)^2+4]^{3/2}}}_{\text{image charges}}\right]\ \mu\text{C/m}^2$$

5.20 a) (continued)

In the expression, all y components cancel, and we are left with

$$\mathbf{D}(y) = 100\left[\frac{1}{[(y-1)^2+4]^{3/2}} + \frac{1}{[(y+1)^2+4]^{3/2}}\right]\mathbf{a}_x\ \mu\text{C/m}^2$$

We now find the charge density at the origin:

$$\rho_s(0,0,0) = \mathbf{D}\cdot\mathbf{a}_x\Big|_{y=0} = \underline{17.9\ \mu\text{C/m}^2}$$

b) Determine ρ_S at $P(0,h,0)$. This will be

$$\rho_s(0,h,0) = \mathbf{D}\cdot\mathbf{a}_x\Big|_{y=h} = 100\left[\frac{1}{[(h-1)^2+4]^{3/2}} + \frac{1}{[(h+1)^2+4]^{3/2}}\right]\ \mu\text{C/m}^2$$

5.21. Let the surface $y = 0$ be a perfect conductor in free space. Two uniform infinite line charges of 30 nC/m each are located at $x = 0$, $y = 1$, and $x = 0$, $y = 2$.

a) Let $V = 0$ at the plane $y = 0$, and find V at $P(1,2,0)$: The line charges will image across the plane, producing image line charges of -30 nC/m each at $x = 0$, $y = -1$, and $x = 0$, $y = -2$. We find the potential at P by evaluating the work done in moving a unit positive charge from the $y = 0$ plane (we choose the origin) to P: For each line charge, this will be:

$$V_P - V_{0,0,0} = -\frac{\rho_l}{2\pi\epsilon_0}\ln\left[\frac{\text{final distance from charge}}{\text{initial distance from charge}}\right]$$

where $V_{0,0,0} = 0$. Considering the four charges, we thus have

$$V_P = -\frac{\rho_l}{2\pi\epsilon_0}\left[\ln\left(\frac{1}{2}\right) + \ln\left(\frac{\sqrt{2}}{1}\right) - \ln\left(\frac{\sqrt{10}}{1}\right) - \ln\left(\frac{\sqrt{17}}{2}\right)\right]$$

$$= \frac{\rho_l}{2\pi\epsilon_0}\left[\ln(2) + \ln\left(\frac{1}{\sqrt{2}}\right) + \ln\left(\sqrt{10}\right) + \ln\left(\frac{\sqrt{17}}{2}\right)\right] = \frac{30\times10^{-9}}{2\pi\epsilon_0}\ln\left[\frac{\sqrt{10}\sqrt{17}}{\sqrt{2}}\right]$$

$$= \underline{1.20\,\text{kV}}$$

b) Find $\mathbf{E}$ at P: Use

$$\mathbf{E}_P = \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{(1,2,0)-(0,1,0)}{|(1,1,0)|^2} + \frac{(1,2,0)-(0,2,0)}{|(1,0,0)|^2} - \frac{(1,2,0)-(0,-1,0)}{|(1,3,0)|^2} - \frac{(1,2,0)-(0,-2,0)}{|(1,4,0)|^2}\right]$$

$$= \frac{\rho_l}{2\pi\epsilon_0}\left[\frac{(1,1,0)}{2} + \frac{(1,0,0)}{1} - \frac{(1,3,0)}{10} - \frac{(1,4,0)}{17}\right] = \underline{723\,\mathbf{a}_x - 18.9\,\mathbf{a}_y\ \text{V/m}}$$

5.22. The line segment $x = 0$, $-1 \le y \le 1$, $z = 1$, carries a linear charge density $\rho_L = \pi|y|\ \mu$C/m. Let $z = 0$ be a conducting plane and determine the surface charge density at: (a) (0,0,0); (b) (0,1,0).

We consider the line charge to be made up of a string of differential segments of length, dy', and of charge $dq = \rho_L\, dy'$. A given segment at location $(0, y', 1)$ will have a corresponding image charge segment at location $(0, y', -1)$. The differential flux density on the y axis that is associated with the segment-image pair will be

$$d\mathbf{D} = \frac{\rho_L\, dy'[(y-y')\,\mathbf{a}_y - \mathbf{a}_z]}{4\pi[(y-y')^2+1]^{3/2}} - \frac{\rho_L\, dy'[(y-y')\,\mathbf{a}_y + \mathbf{a}_z]}{4\pi[(y-y')^2+1]^{3/2}} = \frac{-\rho_L\, dy'\,\mathbf{a}_z}{2\pi[(y-y')^2+1]^{3/2}}$$

In other words, each charge segment and its image produce a net field in which the y components have cancelled. The total flux density from the line charge and its image is now

$$\mathbf{D}(y) = \int d\mathbf{D} = \int_{-1}^{1} \frac{-\pi|y'|\,\mathbf{a}_z\, dy'}{2\pi[(y-y')^2+1]^{3/2}}$$

$$= -\frac{\mathbf{a}_z}{2}\int_0^1 \left[\frac{y'}{[(y-y')^2+1]^{3/2}} + \frac{y'}{[(y+y')^2+1]^{3/2}}\right] dy'$$

$$= \frac{\mathbf{a}_z}{2}\left[\frac{y(y-y')+1}{[(y-y')^2+1]^{1/2}} + \frac{y(y+y')+1}{[(y+y')^2+1]^{1/2}}\right]_0^1$$

$$= \frac{\mathbf{a}_z}{2}\left[\frac{y(y-1)+1}{[(y-1)^2+1]^{1/2}} + \frac{y(y+1)+1}{[(y+1)^2+1]^{1/2}} - 2(y^2+1)^{1/2}\right]$$

Now, at the origin (part a), we find the charge density through

$$\rho_s(0,0,0) = \mathbf{D}\cdot\mathbf{a}_z\Big|_{y=0} = \frac{\mathbf{a}_z}{2}\left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 2\right] = \underline{-0.29\ \mu\text{C/m}^2}$$

Then, at (0,1,0) (part b), the charge density is

$$\rho_s(0,1,0) = \mathbf{D}\cdot\mathbf{a}_z\Big|_{y=1} = \frac{\mathbf{a}_z}{2}\left[1 + \frac{3}{\sqrt{5}} - 2\right] = \underline{-0.24\ \mu\text{C/m}^2}$$

5.23. A dipole with $\mathbf{p} = 0.1\mathbf{a}_z\ \mu\text{C}\cdot\text{m}$ is located at $A(1,0,0)$ in free space, and the $x = 0$ plane is perfectly-conducting.

a) Find V at $P(2,0,1)$. We use the far-field potential for a z-directed dipole:

$$V = \frac{p\cos\theta}{4\pi\epsilon_0 r^2} = \frac{p}{4\pi\epsilon_0}\frac{z}{[x^2+y^2+z^2]^{1.5}}$$

The dipole at $x = 1$ will image in the plane to produce a second dipole of the opposite orientation at $x = -1$. The potential at any point is now:

$$V = \frac{p}{4\pi\epsilon_0}\left[\frac{z}{[(x-1)^2+y^2+z^2]^{1.5}} - \frac{z}{[(x+1)^2+y^2+z^2]^{1.5}}\right]$$

Substituting $P(2,0,1)$, we find

$$V = \frac{.1\times 10^6}{4\pi\epsilon_0}\left[\frac{1}{2\sqrt{2}} - \frac{1}{10\sqrt{10}}\right] = \underline{289.5\ \text{V}}$$

5.23 b) Find the equation of the 200-V equipotential surface in cartesian coordinates: We just set the potential exression of part a equal to 200 V to obtain:

$$\left[\frac{z}{[(x-1)^2+y^2+z^2]^{1.5}} - \frac{z}{[(x+1)^2+y^2+z^2]^{1.5}}\right] = 0.222$$

5.24. At a certain temperature, the electron and hole mobilities in intrinsic germanium are given as 0.43 and 0.21 $m^2/V \cdot s$, respectively. If the electron and hole concentrations are both 2.3×10^{19} m^{-3}, find the conductivity at this temperature.

With the electron and hole charge magnitude of 1.6×10^{-19} C, the conductivity in this case can be written:

$$\sigma = |\rho_e|\mu_e + \rho_h\mu_h = (1.6 \times 10^{-19})(2.3 \times 10^{19})(0.43 + 0.21) = \underline{2.36 \text{ S/m}}$$

5.25. Electron and hole concentrations increase with temperature. For pure silicon, suitable expressions are $\rho_h = -\rho_e = 6200T^{1.5}e^{-7000/T}$ C/m^3. The functional dependence of the mobilities on temperature is given by $\mu_h = 2.3 \times 10^5 T^{-2.7}$ $m^2/V \cdot s$ and $\mu_e = 2.1 \times 10^5 T^{-2.5}$ $m^2/V \cdot s$, where the temperature, T, is in degrees Kelvin. The conductivity will thus be

$$\sigma = -\rho_e\mu_e + \rho_h\mu_h = 6200T^{1.5}e^{-7000/T}\left[2.1 \times 10^5 T^{-2.5} + 2.3 \times 10^5 T^{-2.7}\right]$$
$$= \frac{1.30 \times 10^9}{T}e^{-7000/T}\left[1 + 1.095T^{-.2}\right] \text{ S/m}$$

Find σ at:

a) 0° C: With $T = 273$°K, the expression evaluates as $\sigma(0) = \underline{4.7 \times 10^{-5} \text{ S/m}}$.

b) 40° C: With $T = 273 + 40 = 313$, we obtain $\sigma(40) = \underline{1.1 \times 10^{-3} \text{ S/m}}$.

c) 80° C: With $T = 273 + 80 = 353$, we obtain $\sigma(80) = \underline{1.2 \times 10^{-2} \text{ S/m}}$.

5.26. A semiconductor sample has a rectangular cross-section 1.5 by 2.0 mm, and a length of 11.0 mm. The material has electron and hole densities of 1.8×10^{18} and 3.0×10^{15} m^{-3}, respectively. If $\mu_e = 0.082$ $m^2/V \cdot s$ and $\mu_h = 0.0021$ $m^2/V \cdot s$, find the resistance offered between the end faces of the sample.

Using the given values along with the electron charge, the conductivity is

$$\sigma = (1.6 \times 10^{-19})\left[(1.8 \times 10^{18})(0.082) + (3.0 \times 10^{15})(0.0021)\right] = 0.0236 \text{ S/m}$$

The resistance is then

$$R = \frac{\ell}{\sigma A} = \frac{0.011}{(0.0236)(0.002)(0.0015)} = \underline{155 \text{ k}\Omega}$$

5.27. Atomic hydrogen contains 5.5×10^{25} atoms/m^3 at a certain temperature and pressure. When an electric field of 4 kV/m is applied, each dipole formed by the electron and positive nucleus has an effective length of 7.1×10^{-19} m.

a) Find P: With all identical dipoles, we have

$$P = Nqd = (5.5 \times 10^{25})(1.602 \times 10^{-19})(7.1 \times 10^{-19}) = 6.26 \times 10^{-12}\ \text{C/m}^2 = \underline{6.26\ \text{pC/m}^2}$$

b) Find ϵ_r: We use $P = \epsilon_0 \chi_e E$, and so

$$\chi_e = \frac{P}{\epsilon_0 E} = \frac{6.26 \times 10^{-12}}{(8.85 \times 10^{-12})(4 \times 10^3)} = 1.76 \times 10^{-4}$$

Then $\epsilon_r = 1 + \chi_e = \underline{1.000176}$.

5.28. Find the dielectric constant of a material in which the electric flux density is four times the polarization.

First we use $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + (1/4)\mathbf{D}$. Therefore $\mathbf{D} = (4/3)\epsilon_0 \mathbf{E}$, so we identify $\epsilon_r = \underline{4/3}$.

5.29. A coaxial conductor has radii $a = 0.8$ mm and $b = 3$ mm and a polystyrene dielectric for which $\epsilon_r = 2.56$. If $\mathbf{P} = (2/\rho)\mathbf{a}_\rho$ nC/m^2 in the dielectric, find:

a) $\mathbf{D}$ and $\mathbf{E}$ as functions of ρ: Use

$$\mathbf{E} = \frac{\mathbf{P}}{\epsilon_0(\epsilon_r - 1)} = \frac{(2/\rho) \times 10^{-9} \mathbf{a}_\rho}{(8.85 \times 10^{-12})(1.56)} = \underline{\frac{144.9}{\rho} \mathbf{a}_\rho\ \text{V/m}}$$

Then

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \frac{2 \times 10^{-9} \mathbf{a}_\rho}{\rho} \left[\frac{1}{1.56} + 1 \right] = \frac{3.28 \times 10^{-9} \mathbf{a}_\rho}{\rho}\ \text{C/m}^2 = \underline{\frac{3.28 \mathbf{a}_\rho}{\rho}\ \text{nC/m}^2}$$

b) Find V_{ab} and χ_e: Use

$$V_{ab} = -\int_3^{0.8} \frac{144.9}{\rho}\, d\rho = 144.9 \ln\left(\frac{3}{0.8}\right) = \underline{192\,\text{V}}$$

$\chi_e = \epsilon_r - 1 = \underline{1.56}$, as found in part a.

c) If there are 4×10^{19} molecules per cubic meter in the dielectric, find $\mathbf{p}(\rho)$: Use

$$\mathbf{p} = \frac{\mathbf{P}}{N} = \frac{(2 \times 10^{-9}/\rho)}{4 \times 10^{19}} \mathbf{a}_\rho = \underline{\frac{5.0 \times 10^{-29}}{\rho} \mathbf{a}_\rho\ \text{C} \cdot \text{m}}$$

5.30. Consider a composite material made up of two species, having number densities N_1 and N_2 molecules/m^3 respectively. The two materials are uniformly mixed, yielding a total number density of $N = N_1 + N_2$. The presence of an electric field $\mathbf{E}$, induces molecular dipole moments $\mathbf{p}_1$ and $\mathbf{p}_2$ within the individual species, whether mixed or not. Show that the dielectric constant of the composite material is given by $\epsilon_r = f\epsilon_{r1} + (1-f)\epsilon_{r2}$, where f is the number fraction of species 1 dipoles in the composite, and where ϵ_{r1} and ϵ_{r2} are the dielectric constants that the unmixed species would have if each had number density N.

We may write the total polarization vector as

$$\mathbf{P}_{tot} = N_1\mathbf{p}_1 + N_2\mathbf{p}_2 = N\left(\frac{N_1}{N}\mathbf{p}_1 + \frac{N_2}{N}\mathbf{p}_2\right) = N\left[f\mathbf{p}_1 + (1-f)\mathbf{p}_2\right] = f\mathbf{P}_1 + (1-f)\mathbf{P}_2$$

In terms of the susceptibilities, this becomes $\mathbf{P}_{tot} = \epsilon_0\left[f\chi_{e1} + (1-f)\chi_{e2}\right]\mathbf{E}$, where χ_{e1} and χ_{e2} are evaluated at the composite number density, N. Now

$$\mathbf{D} = \epsilon_r\epsilon_0\mathbf{E} = \epsilon_0\mathbf{E} + \mathbf{P}_{tot} = \epsilon_0\underbrace{\left[1 + f\chi_{e1} + (1-f)\chi_{e2}\right]}_{\epsilon_r}\mathbf{E}$$

Identifying ϵ_r as shown, we may rewrite it by adding and subracting f:

$$\begin{aligned}\epsilon_r &= \left[1 + f - f + f\chi_{e1} + (1-f)\chi_{e2}\right] = \left[f(1+\chi_{e1}) + (1-f)(1+\chi_{e2})\right]\\ &= \left[f\epsilon_{r1} + (1-f)\epsilon_{r2}\right] \quad \text{Q.E.D.}\end{aligned}$$

5.31. The surface $x = 0$ separates two perfect dielectrics. For $x > 0$, let $\epsilon_r = \epsilon_{r1} = 3$, while $\epsilon_{r2} = 5$ where $x < 0$. If $\mathbf{E}_1 = 80\mathbf{a}_x - 60\mathbf{a}_y - 30\mathbf{a}_z$ V/m, find:

a) E_{N1}: This will be $\mathbf{E}_1 \cdot \mathbf{a}_x = \underline{80\,\text{V/m}}$.

b) $\mathbf{E}_{T1}$. This has components of $\mathbf{E}_1$ *not* normal to the surface, or $\mathbf{E}_{T1} = \underline{-60\mathbf{a}_y - 30\mathbf{a}_z\ \text{V/m}}$.

c) $E_{T1} = \sqrt{(60)^2 + (30)^2} = \underline{67.1\,\text{V/m}}$.

d) $E_1 = \sqrt{(80)^2 + (60)^2 + (30)^2} = \underline{104.4\,\text{V/m}}$.

e) The angle θ_1 between $\mathbf{E}_1$ and a normal to the surface: Use

$$\cos\theta_1 = \frac{\mathbf{E}_1 \cdot \mathbf{a}_x}{E_1} = \frac{80}{104.4} \;\Rightarrow\; \theta_1 = \underline{40.0^\circ}$$

f) $D_{N2} = D_{N1} = \epsilon_{r1}\epsilon_0 E_{N1} = 3(8.85 \times 10^{-12})(80) = \underline{2.12\,\text{nC/m}^2}$.

g) $D_{T2} = \epsilon_{r2}\epsilon_0 E_{T1} = 5(8.85 \times 10^{-12})(67.1) = \underline{2.97\,\text{nC/m}^2}$.

h) $\mathbf{D}_2 = \epsilon_{r1}\epsilon_0 E_{N1}\mathbf{a}_x + \epsilon_{r2}\epsilon_0\mathbf{E}_{T1} = \underline{2.12\mathbf{a}_x - 2.66\mathbf{a}_y - 1.33\mathbf{a}_z\ \text{nC/m}^2}$.

i) $\mathbf{P}_2 = \mathbf{D}_2 - \epsilon_0\mathbf{E}_2 = \mathbf{D}_2\left[1 - (1/\epsilon_{r2})\right] = (4/5)\mathbf{D}_2 = \underline{1.70\mathbf{a}_x - 2.13\mathbf{a}_y - 1.06\mathbf{a}_z\ \text{nC/m}^2}$.

j) the angle θ_2 between $\mathbf{E}_2$ and a normal to the surface: Use

$$\cos\theta_2 = \frac{\mathbf{E}_2 \cdot \mathbf{a}_x}{E_2} = \frac{\mathbf{D}_2 \cdot \mathbf{a}_x}{D_2} = \frac{2.12}{\sqrt{(2.12)^2 = (2.66)^2 + (1.33)^2}} = .581$$

Thus $\theta_2 = \cos^{-1}(.581) = \underline{54.5^\circ}$.

5.32. Two equal but opposite-sign point charges of 3μC are held x meters apart by a spring that provides a repulsive force given by $F_{sp} = 12(0.5 - x)$ N. Without any force of attraction, the spring would be fully-extended to 0.5m.

a) Determine the charge separation: The Coulomb and spring forces must be equal in magnitude. We set up

$$\frac{(3 \times 10^{-6})^2}{4\pi\epsilon_0 x^2} = \frac{9 \times 10^{-12}}{4\pi(8.85 \times 10^{-12})x^2} = 12(0.5 - x)$$

which leads to the cubic equation:

$$x^3 - 0.5x^2 + 6.74 \times 10^{-3}$$

whose solution, found using a calculator, is $x \doteq \underline{0.136}$ m.

b) what is the dipole moment?

Dipole moment magnitude will be $p = qd = (3 \times 10^{-6})(0.136) = \underline{4.08 \times 10^{-7}}$ C-m.

5.33. Two perfect dielectrics have relative permittivities $\epsilon_{r1} = 2$ and $\epsilon_{r2} = 8$. The planar interface between them is the surface $x - y + 2z = 5$. The origin lies in region 1. If $\mathbf{E}_1 = 100\mathbf{a}_x + 200\mathbf{a}_y - 50\mathbf{a}_z$ V/m, find $\mathbf{E}_2$: We need to find the components of $\mathbf{E}_1$ that are normal and tangent to the boundary, and then apply the appropriate boundary conditions. The normal component will be $E_{N1} = \mathbf{E}_1 \cdot \mathbf{n}$. Taking $f = x - y + 2z$, the unit vector that is normal to the surface is

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{6}}\left[\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z\right]$$

This normal will point in the direction of increasing f, which will be away from the origin, or into region 2 (you can visualize a portion of the surface as a triangle whose vertices are on the three coordinate axes at $x = 5$, $y = -5$, and $z = 2.5$). So $E_{N1} = (1/\sqrt{6})[100 - 200 - 100] = -81.7\,$V/m. Since the magnitude is negative, the normal component points into region 1 from the surface. Then

$$\mathbf{E}_{N1} = -81.65\left(\frac{1}{\sqrt{6}}\right)\left[\mathbf{a}_x - \mathbf{a}_y + 2\mathbf{a}_z\right] = -33.33\mathbf{a}_x + 33.33\mathbf{a}_y - 66.67\mathbf{a}_z \text{ V/m}$$

Now, the tangential component will be $\mathbf{E}_{T1} = \mathbf{E}_1 - \mathbf{E}_{N1} = 133.3\mathbf{a}_x + 166.7\mathbf{a}_y + 16.67\mathbf{a}_z$. Our boundary conditions state that $\mathbf{E}_{T2} = \mathbf{E}_{T1}$ and $\mathbf{E}_{N2} = (\epsilon_{r1}/\epsilon_{r2})\mathbf{E}_{N1} = (1/4)\mathbf{E}_{N1}$. Thus

$$\mathbf{E}_2 = \mathbf{E}_{T2} + \mathbf{E}_{N2} = \mathbf{E}_{T1} + \frac{1}{4}\mathbf{E}_{N1} = 133.3\mathbf{a}_x + 166.7\mathbf{a}_y + 16.67\mathbf{a}_z - 8.3\mathbf{a}_x + 8.3\mathbf{a}_y - 16.67\mathbf{a}_z$$
$$= \underline{125\mathbf{a}_x + 175\mathbf{a}_y \text{ V/m}}$$

5.34. Region 1 ($x \geq 0$) is a dielectric with $\epsilon_{r1} = 2$, while region 2 ($x < 0$) has $\epsilon_{r2} = 5$. Let $\mathbf{E}_1 = 20\mathbf{a}_x - 10\mathbf{a}_y + 50\mathbf{a}_z$ V/m.

a) Find $\mathbf{D}_2$: One approach is to first find $\mathbf{E}_2$. This will have the same y and z (tangential) components as $\mathbf{E}_1$, but the normal component, E_x, will differ by the ratio $\epsilon_{r1}/\epsilon_{r2}$; this arises from $D_{x1} = D_{x2}$ (normal component of $\mathbf{D}$ is continuous across a non-charged interface). Therefore $\mathbf{E}_2 = 20(\epsilon_{r1}/\epsilon_{r2})\,\mathbf{a}_x - 10\,\mathbf{a}_y + 50\,\mathbf{a}_z = 8\,\mathbf{a}_x - 10\,\mathbf{a}_y + 50\,\mathbf{a}_z$. The flux density is then

$$\mathbf{D}_2 = \epsilon_{r2}\epsilon_0\mathbf{E}_2 = 40\epsilon_0\,\mathbf{a}_x - 50\epsilon_0\,\mathbf{a}_y + 250\epsilon_0\,\mathbf{a}_z = \underline{0.35\,\mathbf{a}_x - 0.44\,\mathbf{a}_y + 2.21\,\mathbf{a}_z \text{ nC/m}^2}$$

b) Find the energy density in both regions: These will be

$$w_{e1} = \frac{1}{2}\epsilon_{r1}\epsilon_0\mathbf{E}_1 \cdot \mathbf{E}_1 = \frac{1}{2}(2)\epsilon_0\left[(20)^2 + (10)^2 + (50)^2\right] = 3000\epsilon_0 = \underline{26.6 \text{ nJ/m}^3}$$

$$w_{e2} = \frac{1}{2}\epsilon_{r2}\epsilon_0\mathbf{E}_2 \cdot \mathbf{E}_2 = \frac{1}{2}(5)\epsilon_0\left[(8)^2 + (10)^2 + (50)^2\right] = 6660\epsilon_0 = \underline{59.0 \text{ nJ/m}^3}$$

5.35. Let the cylindrical surfaces $\rho = 4$ cm and $\rho = 9$ cm enclose two wedges of perfect dielectrics, $\epsilon_{r1} = 2$ for $0 < \phi < \pi/2$, and $\epsilon_{r2} = 5$ for $\pi/2 < \phi < 2\pi$. If $\mathbf{E}_1 = (2000/\rho)\mathbf{a}_\rho$ V/m, find:

a) $\mathbf{E}_2$: The interfaces between the two media will lie on planes of constant ϕ, to which $\mathbf{E}_1$ is parallel. Thus the field is the same on either side of the boundaries, and so $\underline{\mathbf{E}_2 = \mathbf{E}_1}$.

b) the total electrostatic energy stored in a 1m length of each region: In general we have $w_E = (1/2)\epsilon_r\epsilon_0 E^2$. So in region 1:

$$W_{E1} = \int_0^1 \int_0^{\pi/2} \int_4^9 \frac{1}{2}(2)\epsilon_0 \frac{(2000)^2}{\rho_2}\,\rho\,d\rho\,d\phi\,dz = \frac{\pi}{2}\epsilon_0(2000)^2 \ln\left(\frac{9}{4}\right) = \underline{45.1\,\mu\text{J}}$$

In region 2, we have

$$W_{E2} = \int_0^1 \int_{\pi/2}^{2\pi} \int_4^9 \frac{1}{2}(5)\epsilon_0 \frac{(2000)^2}{\rho_2}\,\rho\,d\rho\,d\phi\,dz = \frac{15\pi}{4}\epsilon_0(2000)^2 \ln\left(\frac{9}{4}\right) = \underline{338\,\mu\text{J}}$$

CHAPTER 6.

6.1. Consider a coaxial capacitor having inner radius a, outer radius b, unit length, and filled with a material with dielectric constant, ϵ_r. Compare this to a parallel-plate capacitor having plate width, w, plate separation d, filled with the same dielectric, and having unit length. Express the ratio b/a in terms of the ratio d/w, such that the two structures will store the same energy for a given applied voltage.

Storing the same energy for a given applied voltage means that the capacitances will be equal. With both structures having unit length and containing the same dielectric (permittivity ϵ), we equate the two capacitances:

$$\frac{2\pi\epsilon}{\ln(b/a)} = \frac{\epsilon w}{d} \;\Rightarrow\; \underline{\frac{b}{a} = \exp\left(2\pi\frac{d}{w}\right)}$$

6.2. Let $S = 100\,\text{mm}^2$, $d = 3$ mm, and $\epsilon_r = 12$ for a parallel-plate capacitor.

a) Calculate the capacitance:

$$C = \frac{\epsilon_r\epsilon_0 A}{d} = \frac{12\epsilon_0(100\times10^{-6})}{3\times10^{-3}} = 0.4\epsilon_0 = \underline{3.54 \text{ pf}}$$

b) After connecting a 6 V battery across the capacitor, calculate E, D, Q, and the total stored electrostatic energy: First,

$$E = V_0/d = 6/(3\times10^{-3}) = \underline{2000 \text{ V/m}}, \quad \text{then} \quad D = \epsilon_r\epsilon_0 E = 2.4\times10^4\epsilon_0 = \underline{0.21\ \mu\text{C/m}^2}$$

The charge in this case is

$$Q = \mathbf{D}\cdot\mathbf{n}|_s = DA = 0.21\times(100\times10^{-6}) = 0.21\times10^{-4}\ \mu\text{C} = \underline{21 \text{ pC}}$$

Finally, $W_e = (1/2)QV_0 = 0.5(21)(6) = \underline{63 \text{ pJ}}$.

c) With the source still connected, the dielectric is carefully withdrawn from between the plates. With the dielectric gone, re-calculate E, D, Q, and the energy stored in the capacitor.

$$E = V_0/d = 6/(3\times10^{-3}) = \underline{2000 \text{ V/m}}, \quad \text{as before.} \quad D = \epsilon_0 E = 2000\epsilon_0 = \underline{17.7 \text{ nC/m}^2}$$

The charge is now $Q = DA = 17.7\times(100\times10^{-6})$ nC $= \underline{1.8 \text{ pC}}$.

Finally, $W_e = (1/2)QV_0 = 0.5(1.8)(6) = \underline{5.4 \text{ pJ}}$.

d) If the charge and energy found in (c) are less than that found in (b) (which you should have discovered), what became of the missing charge and energy? In the absence of friction in removing the dielectric, the charge and energy have returned to the battery that gave it.

6.3. Capacitors tend to be more expensive as their capacitance and maximum voltage, V_{max}, increase. The voltage V_{max} is limited by the field strength at which the dielectric breaks down, E_{BD}. Which of these dielectrics will give the largest CV_{max} product for equal plate areas: (a) air: $\epsilon_r = 1$, $E_{BD} = 3$ MV/m; (b) barium titanate: $\epsilon_r = 1200$, $E_{BD} = 3$ MV/m; (c) silicon dioxide: $\epsilon_r = 3.78$, $E_{BD} = 16$ MV/m; (d) polyethylene: $\epsilon_r = 2.26$, $E_{BD} = 4.7$ MV/m? Note that $V_{max} = EBDd$, where d is the plate separation. Also, $C = \epsilon_r \epsilon_0 A/d$, and so $V_{max}C = \epsilon_r \epsilon_0 A E_{BD}$, where A is the plate area. The maximum CV_{max} product is found through the maximum $\epsilon_r E_{BD}$ product. Trying this with the given materials yields the winner, which is barium titanate.

6.4. An air-filled parallel-plate capacitor with plate separation d and plate area A is connected to a battery which applies a voltage V_0 between plates. With the battery left connected, the plates are moved apart to a distance of $10d$. Determine by what factor each of the following quantities changes:

a) V_0: Remains the same, since the battery is left connected.

b) C: As $C = \epsilon_0 A/d$, increasing d by a factor of ten decreases C by a factor of 0.1.

c) E: We require $E \times d = V_0$, where V_0 has not changed. Therefore, E has decreased by a factor of 0.1.

d) D: As $D = \epsilon_0 E$, and since E has decreased by 0.1, D decreases by 0.1.

e) Q: Since $Q = CV_0$, and as C is down by 0.1, Q also decreases by 0.1.

f) ρ_S: As Q is reduced by 0.1, ρ_S reduces by 0.1. This is also consistent with D having been reduced by 0.1.

g) W_e: Use $W_e = 1/2\, CV_0^2$, to observe its reduction by 0.1, since C is reduced by that factor.

6.5. A parallel plate capacitor is filled with a nonuniform dielectric characterized by $\epsilon_r = 2 + 2 \times 10^6 x^2$, where x is the distance from one plate. If $S = 0.02\,\text{m}^2$, and $d = 1$ mm, find C: Start by assuming charge density ρ_s on the top plate. **D** will, as usual, be x-directed, originating at the top plate and terminating on the bottom plate. The key here is that **D** *will be constant over the distance between plates*. This can be understood by considering the x-varying dielectric as constructed of many thin layers, each having constant permittivity. The permittivity changes from layer to layer to approximate the given function of x. The approximation becomes exact as the layer thicknesses approach zero. We know that **D**, which is normal to the layers, will be continuous across each boundary, and so **D** is constant over the plate separation distance, and will be given in magnitude by ρ_s. The electric field magnitude is now

$$E = \frac{D}{\epsilon_0 \epsilon_r} = \frac{\rho_s}{\epsilon_0 (2 + 2 \times 10^6 x^2)}$$

The voltage beween plates is then

$$V_0 = \int_0^{10^{-3}} \frac{\rho_s\, dx}{\epsilon_0 (2 + 2 \times 10^6 x^2)} = \frac{\rho_s}{\epsilon_0} \frac{1}{\sqrt{4 \times 10^6}} \tan^{-1}\left(\frac{x\sqrt{4 \times 10^6}}{2} \right) \Big|_0^{10^{-3}} = \frac{\rho_s}{\epsilon_0} \frac{1}{2 \times 10^3} \left(\frac{\pi}{4} \right)$$

Now $Q = \rho_s(.02)$, and so

$$C = \frac{Q}{V_0} = \frac{\rho_s(.02)\epsilon_0 (2 \times 10^3)(4)}{\rho_s \pi} = 4.51 \times 10^{-10}\,\text{F} = \underline{451\,\text{pF}}$$

6.6. Repeat Problem 6.4 assuming the battery is disconnected before the plate separation is increased: The ordering of parameters is changed over that in Problem 6.4, as the progression of thought on the matter is different.

a) Q: Remains the <u>same</u>, since with the battery disconnected, the charge has nowhere to go.

b) ρ_S: As Q is unchanged, ρ_S is also <u>unchanged</u>, since the plate area is the same.

c) D: As $D = \rho_S$, it will remain the <u>same</u> also.

d) E: Since $E = D/\epsilon_0$, and as D is not changed, E will also remain the <u>same</u>.

e) V_0: We require $E \times d = V_0$, where E has not changed. Therefore, V_0 has increased by a factor of <u>10</u>.

f) C: As $C = \epsilon_0 A/d$, increasing d by a factor of ten decreases C by a factor of <u>0.1</u>. The same result occurs because $C = Q/V_0$, where V_0 is increased by 10, whereas Q has not changed.

g) W_e: Use $W_e = 1/2\, CV_0^2 = 1/2\, QV_0$, to observe its increase by a factor of <u>10</u>.

6.7. Let $\epsilon_{r1} = 2.5$ for $0 < y < 1$ mm, $\epsilon_{r2} = 4$ for $1 < y < 3$ mm, and ϵ_{r3} for $3 < y < 5$ mm. Conducting surfaces are present at $y = 0$ and $y = 5$ mm. Calculate the capacitance per square meter of surface area if: a) ϵ_{r3} is that of air; b) $\epsilon_{r3} = \epsilon_{r1}$; c) $\epsilon_{r3} = \epsilon_{r2}$; d) region 3 is silver: The combination will be three capacitors in series, for which

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} = \frac{d_1}{\epsilon_{r1}\epsilon_0(1)} + \frac{d_2}{\epsilon_{r2}\epsilon_0(1)} + \frac{d_3}{\epsilon_{r3}\epsilon_0(1)} = \frac{10^{-3}}{\epsilon_0}\left[\frac{1}{2.5} + \frac{2}{4} + \frac{2}{\epsilon_{r3}}\right]$$

So that

$$C = \frac{(5 \times 10^{-3})\epsilon_0\epsilon_{r3}}{10 + 4.5\epsilon_{r3}}$$

Evaluating this for the four cases, we find a) $C = \underline{3.05\,\text{nF}}$ for $\epsilon_{r3} = 1$, b) $C = \underline{5.21\,\text{nF}}$ for $\epsilon_{r3} = 2.5$, c) $C = \underline{6.32\,\text{nF}}$ for $\epsilon_{r3} = 4$, and d) $C = \underline{9.83\,\text{nF}}$ if silver (taken as a perfect conductor) forms region 3; this has the effect of removing the term involving ϵ_{r3} from the original formula (first equation line), or equivalently, allowing ϵ_{r3} to approach infinity.

6.8. A parallel-plate capacitor is made using two circular plates of radius a, with the bottom plate on the xy plane, centered at the origin. The top plate is located at $z = d$, with its center on the z axis. Potential V_0 is on the top plate; the bottom plate is grounded. Dielectric having *radially-dependent* permittivity fills the region between plates. The permittivity is given by $\epsilon(\rho) = \epsilon_0(1 + \rho^2/a^2)$. Find:

a) $\mathbf{E}$: Since ϵ does not vary in the z direction, and since we must always obtain V_0 when integrating $\mathbf{E}$ between plates, it must follow that $\mathbf{E} = \underline{-V_0/d\,\mathbf{a}_z \text{ V/m}}$.

b) $\mathbf{D}$: $\mathbf{D} = \epsilon\mathbf{E} = \underline{-[\epsilon_0(1 + \rho^2/a^2)V_0/d]\ \mathbf{a}_z \text{ C/m}^2}$.

c) Q: Here we find the integral of the surface charge density over the top plate:

$$Q = \int_S \mathbf{D} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^a \frac{-\epsilon_0(1 + \rho^2/a^2)V_0}{d}\, \mathbf{a}_z \cdot (-\mathbf{a}_z)\, \rho\, d\rho\, d\phi$$

$$= \frac{2\pi\epsilon_0 V_0}{d} \int_0^a (\rho + \rho^3/a^2)\, d\rho = \underline{\frac{3\pi\epsilon_0 a^2}{2d} V_0}$$

d) C: We use $C = Q/V_0$ and our previous result to find $C = \underline{3\epsilon_0(\pi a^2)/(2d) \text{ F}}$.

6.9. Two coaxial conducting cylinders of radius 2 cm and 4 cm have a length of 1m. The region between the cylinders contains a layer of dielectric from $\rho = c$ to $\rho = d$ with $\epsilon_r = 4$. Find the capacitance if

a) $c = 2$ cm, $d = 3$ cm: This is two capacitors in series, and so

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{2\pi\epsilon_0}\left[\frac{1}{4}\ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right)\right] \Rightarrow C = \underline{143\,\text{pF}}$$

b) $d = 4$ cm, and the volume of the dielectric is the same as in part a: Having equal volumes requires that $3^2 - 2^2 = 4^2 - c^2$, from which $c = 3.32$ cm. Now

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{2\pi\epsilon_0}\left[\ln\left(\frac{3.32}{2}\right) + \frac{1}{4}\ln\left(\frac{4}{3.32}\right)\right] \Rightarrow C = \underline{101\,\text{pF}}$$

6.10. A coaxial cable has conductor dimensions of $a = 1.0$ mm and $b = 2.7$ mm. The inner conductor is supported by dielectric spacers ($\epsilon_r = 5$) in the form of washers with a hole radius of 1 mm and an outer radius of 2.7 mm, and with a thickness of 3.0 mm. The spacers are located every 2 cm down the cable.

a) By what factor do the spacers increase the capacitance per unit length? The net capacitance can be constructed as a composite quantity, composed of weighted contributions from the air-filled and dielectric-filled regions:

$$C_{net} = \frac{2\pi\epsilon_0}{\ln(b/a)} f_1 + \frac{2\pi\epsilon_r\epsilon_0}{\ln(b/a)} f_2$$

where $f_1 = (2 - 0.3)/2$ and $f_2 = 0.3/2$ are the filling factors for air and dielectric. Substituting these gives

$$C_{net} = \frac{2\pi\epsilon_0}{\ln(b/a)} \underbrace{\left[\frac{1.7}{2} + 0.15\epsilon_r\right]}_{f}$$

where the bracketed term, f, is the capactance increase factor that we seek. Substituting $\epsilon_r = 5$ gives $\underline{f = 1.6}$.

b) If 100V is maintained across the cable, find **E** at all points:

Method 1: We recall the expression for electric field in a coaxial line from Gauss' Law:

$$\mathbf{E} = \frac{a\rho_s}{\epsilon\rho}\,\mathbf{a}_\rho$$

where ρ_s is the surface charge density on the inner conductor. We also note that electric field will be the same both inside and outside the dielectric rings because the integral of **E** between conductors must always give 100 V. Another way to justify this is through the dielectric boundary condition that requires tangential electric field to be continuous across an interface between two dielectrics. This again means that **E** in our case will be the same inside and outside the rings. We can therefore set up the integral for the voltage:

$$V_0 = -\int_b^a \mathbf{E}\cdot d\mathbf{L} = -\int_b^a \frac{a\rho_s}{\epsilon\rho}\,d\rho = \frac{a\rho_s}{\epsilon}\ln(b/a) = 100$$

Then

$$\frac{a\rho_s}{\epsilon} = \frac{100}{\ln(2.7)} = 101 \;\Rightarrow\; \underline{\mathbf{E} = \frac{101}{\rho}\mathbf{a}_\rho}\ \text{V/m}$$

Method 2: Solve Laplace's equation. This was done for the cylindrical geometry in Example 6.3, which gave the potential distribution between conductors, Eq. (35):

$$V(\rho) = V_0\frac{\ln(b/\rho)}{\ln(b/a)}$$

from which

$$\mathbf{E} = -\nabla V = -\frac{dV}{d\rho}\,\mathbf{a}_\rho = \frac{V_0}{\rho\ln(b/a)}\,\mathbf{a}_\rho = \frac{100}{\rho\ln(2.7)}\,\mathbf{a}_\rho = \underline{\frac{101}{\rho}\mathbf{a}_\rho}\ \text{V/m}$$

as before.

6.11. Two conducting spherical shells have radii $a = 3$ cm and $b = 6$ cm. The interior is a perfect dielectric for which $\epsilon_r = 8$.

a) Find C: For a spherical capacitor, we know that:

$$C = \frac{4\pi\epsilon_r\epsilon_0}{\frac{1}{a} - \frac{1}{b}} = \frac{4\pi(8)\epsilon_0}{\left(\frac{1}{3} - \frac{1}{6}\right)(100)} = 1.92\pi\epsilon_0 = \underline{53.3\,\text{pF}}$$

b) A portion of the dielectric is now removed so that $\epsilon_r = 1.0$, $0 < \phi < \pi/2$, and $\epsilon_r = 8$, $\pi/2 < \phi < 2\pi$. Again, find C: We recognize here that removing that portion leaves us with two capacitors in parallel (whose C's will add). We use the fact that with the dielectric *completely* removed, the capacitance would be $C(\epsilon_r = 1) = 53.3/8 = 6.67\,\text{pF}$. With one-fourth the dielectric removed, the total capacitance will be

$$C = \frac{1}{4}(6.67) + \frac{3}{4}(53.4) = \underline{41.7\,\text{pF}}$$

6.12. a) Determine the capacitance of an isolated conducting sphere of radius a in free space (consider an outer conductor existing at $r \to \infty$). If we assume charge Q on the sphere, the electric field will be

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\,\mathbf{a}_r \text{ V/m} \quad (a < r < \infty)$$

The potential on the sphere surface will then be

$$V_0 = -\int_\infty^a \mathbf{E}\cdot d\mathbf{L} = -\int_\infty^a \frac{Q\,dr}{4\pi\epsilon_0 r^2} = \frac{Q}{4\pi\epsilon_0 a}$$

Capacitance is then

$$C = \frac{Q}{V_0} = \underline{4\pi\epsilon_0 a}\text{ F}$$

b) The sphere is to be covered with a dielectric layer of thickness d and dielectric contant ϵ_r. If $\epsilon_r = 3$, find d in terms of a such that the capacitance is twice that of part a: Let the dielectric radius be b, where $b = d + a$. The potential at the conductor surface is then

$$V_0 = -\int_\infty^b \frac{Q}{4\pi\epsilon_0 r^2}\,dr - \int_b^a \frac{Q}{4\pi\epsilon_r\epsilon_0 r^2}\,dr = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{b} + \frac{1}{\epsilon_r}\left(\frac{1}{a} - \frac{1}{b}\right)\right]$$

Then, substituting capacitance, $C = Q/V_0$, and solving for b, we find, after a little algebra:

$$b = \frac{a(\epsilon_r - 1)}{a\epsilon_r\,(4\pi\epsilon_0/C) - 1}$$

Then, substitute $\epsilon_r = 3$ and $C = 8\pi\epsilon_0 a$ (twice the part a result) to obtain:

$$b = 4a \;\Rightarrow\; d = b - a = \underline{3a}$$

6.13. With reference to Fig. 6.5, let $b = 6$ m, $h = 15$ m, and the conductor potential be 250 V. Take $\epsilon = \epsilon_0$. Find values for K_1, ρ_L, a, and C: We have

$$K_1 = \left[\frac{h + \sqrt{h^2 + b^2}}{b}\right]^2 = \left[\frac{15 + \sqrt{(15)^2 + (6)^2}}{6}\right]^2 = \underline{23.0}$$

We then have

$$\rho_L = \frac{4\pi\epsilon_0 V_0}{\ln K_1} = \frac{4\pi\epsilon_0 (250)}{\ln(23)} = \underline{8.87\,\text{nC/m}}$$

Next, $a = \sqrt{h^2 - b^2} = \sqrt{(15)^2 - (6)^2} = \underline{13.8\,\text{m}}$. Finally,

$$C = \frac{2\pi\epsilon}{\cosh^{-1}(h/b)} = \frac{2\pi\epsilon_0}{\cosh^{-1}(15/6)} = \underline{35.5\,\text{pF}}$$

6.14. Two #16 copper conductors (1.29-mm diameter) are parallel with a separation d between axes. Determine d so that the capacitance between wires in air is 30 pF/m.

We use

$$\frac{C}{L} = 60 \text{ pF/m} = \frac{2\pi\epsilon_0}{\cosh^{-1}(h/b)}$$

The above expression evaluates the capacitance of one of the wires suspended over a plane at mid-span, $h = d/2$. Therefore the capacitance of that structure is doubled over that required (from 30 to 60 pF/m). Using this,

$$\frac{h}{b} = \cosh\left(\frac{2\pi\epsilon_0}{C/L}\right) = \cosh\left(\frac{2\pi \times 8.854}{60}\right) = 1.46$$

Therefore, $d = 2h = 2b(1.46) = 2(1.29/2)(1.46) = \underline{1.88 \text{ mm}}$.

6.15. A 2 cm diameter conductor is suspended in air with its axis 5 cm from a conducting plane. Let the potential of the cylinder be 100 V and that of the plane be 0 V. Find the surface charge density on the:

a) cylinder at a point nearest the plane: The cylinder will image across the plane, producing an equivalent two-cylinder problem, with the second one at location 5 cm below the plane. We will take the plane as the zy plane, with the cylinder positions at $x = \pm 5$. Now $b = 1$ cm, $h = 5$ cm, and $V_0 = 100$ V. Thus $a = \sqrt{h^2 - b^2} = 4.90$ cm. Then $K_1 = [(h + a)/b]^2 = 98.0$, and $\rho_L = (4\pi\epsilon_0 V_0)/\ln K_1 = 2.43\,\text{nC/m}$. Now

$$\mathbf{D} = \epsilon_0 \mathbf{E} = -\frac{\rho_L}{2\pi}\left[\frac{(x+a)\mathbf{a}_x + y\mathbf{a}_y}{(x+a)^2 + y^2} - \frac{(x-a)\mathbf{a}_x + y\mathbf{a}_y}{(x-a)^2 + y^2}\right]$$

and

$$\rho_{s,\,max} = \mathbf{D} \cdot (-\mathbf{a}_x)\Big|_{x=h-b,y=0} = \frac{\rho_L}{2\pi}\left[\frac{h-b+a}{(h-b+a)^2} - \frac{h-b-a}{(h-b-a)^2}\right] = \underline{473\,\text{nC/m}^2}$$

6.15b) plane at a point nearest the cylinder: At $x = y = 0$,

$$\mathbf{D}(0,0) = -\frac{\rho_L}{2\pi}\left[\frac{a\mathbf{a}_x}{a^2} - \frac{-a\mathbf{a}_x}{a^2}\right] = -\frac{\rho_L}{2\pi}\frac{2}{a}\mathbf{a}_x$$

from which

$$\rho_s = \mathbf{D}(0,0)\cdot\mathbf{a}_x = -\frac{\rho_L}{\pi a} = \underline{-15.8\,\mathrm{nC/m^2}}$$

c) Find the capacitance per unit length. This will be $C = \rho_L/V_0 = 2.43$ [nC/m]/100 = $\underline{24.3\text{ pF/m}}$.

6.16. Consider an arrangement of two isolated conducting surfaces of any shape that form a capacitor. Use the definitions of capacitance (Eq. (2) in this chapter) and resistance (Eq. (14) in Chapter 5) to show that when the region between the conductors is filled with either conductive material (conductivity σ) or with a perfect dielectric (permittivity ϵ), the resulting resistance and capacitance of the structures are related through the simple formula $RC = \epsilon/\sigma$. What basic properties must be true about both the dielectric and the conducting medium for this condition to hold for certain?

Considering the two surfaces, with location a one surface, and location b on the other, the definitions are written as:

$$C = \frac{\oint_s \epsilon\mathbf{E}\cdot d\mathbf{S}}{-\int_b^a \mathbf{E}\cdot d\mathbf{L}} \quad \text{and} \quad R = \frac{-\int_b^a \mathbf{E}\cdot d\mathbf{L}}{\oint_s \sigma\mathbf{E}\cdot d\mathbf{S}}$$

Note that the integration surfaces in the two definitions are (in this case) closed, because each must completely surround one of the two conductors. The surface integrals would thus yield the total charge on – or the total current flowing out of – the conductor inside.

The two formulas are muliplied together to give

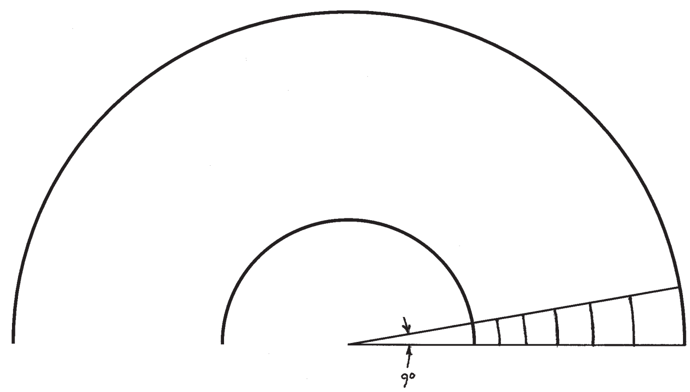
$$RC = \frac{\oint_s \epsilon\mathbf{E}\cdot d\mathbf{S}}{\oint_s \sigma\mathbf{E}\cdot d\mathbf{S}} = \frac{\epsilon}{\sigma}$$

The far-right result is valid provided that ϵ and σ are constant-valued over the integration surfaces. Since in prinicple *any* surface can be chosen over which to integrate, the safest (and correct) conclusion is that ϵ and σ must be constants over the capacitor (or resistor) volume; i.e., the medium must be homogeneous. A little more subtle points are that ϵ and σ generally cannot vary with field orientation (isotropic medium), and cannot vary with field intensity (linear medium), for the simple relation $RC = \epsilon/\sigma$ to work.

6.17 Construct a curvilinear square map for a coaxial capacitor of 3-cm inner radius and 8-cm outer radius. These dimensions are suitable for the drawing.

a) Use your sketch to calculate the capacitance per meter length, assuming $\epsilon_R = 1$: The sketch is shown below. Note that only a 9° sector was drawn, since this would then be duplicated 40 times around the circumference to complete the drawing. The capacitance is thus

$$C \doteq \epsilon_0 \frac{N_Q}{N_V} = \epsilon_0 \frac{40}{6} = \underline{59 \text{ pF/m}}$$



b) Calculate an exact value for the capacitance per unit length: This will be

$$C = \frac{2\pi\epsilon_0}{\ln(8/3)} = \underline{57 \text{ pF/m}}$$

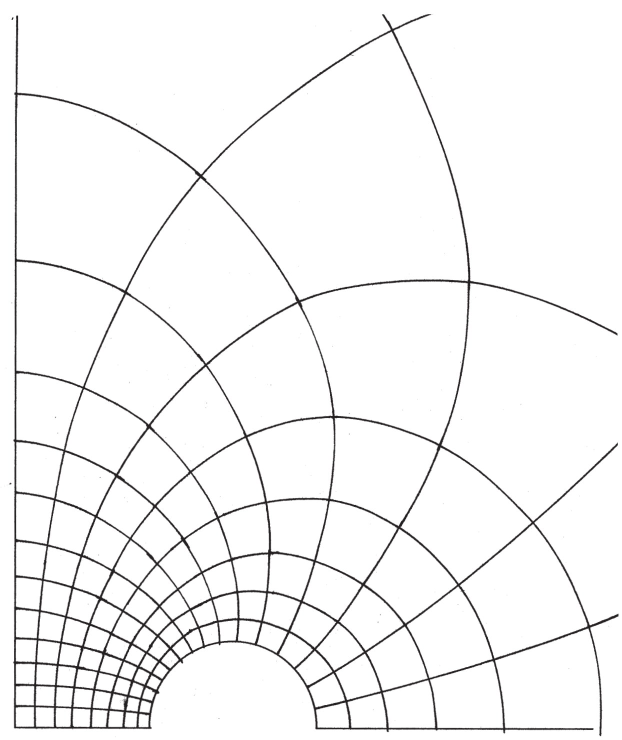
6.18 Construct a curvilinear-square map of the potential field about two parallel circular cylinders, each of 2.5 cm radius, separated by a center-to-center distance of 13cm. These dimensions are suitable for the actual sketch if symmetry is considered. As a check, compute the capacitance per meter both from your sketch and from the exact formula. Assume $\epsilon_R = 1$.

Symmetry allows us to plot the field lines and equipotentials over just the first quadrant, as is done in the sketch below (shown to one-half scale). The capacitance is found from the formula $C = (N_Q/N_V)\epsilon_0$, where N_Q is twice the number of squares around the perimeter of the half-circle and N_V is twice the number of squares between the half-circle and the left vertical plane. The result is

$$C = \frac{N_Q}{N_V}\epsilon_0 = \frac{32}{16}\epsilon_0 = 2\epsilon_0 = \underline{17.7 \text{ pF/m}}$$

We check this result with that using the exact formula:

$$C = \frac{\pi\epsilon_0}{\cosh^{-1}(d/2a)} = \frac{\pi\epsilon_0}{\cosh^{-1}(13/5)} = 1.95\epsilon_0 = \underline{17.3 \text{ pF/m}}$$



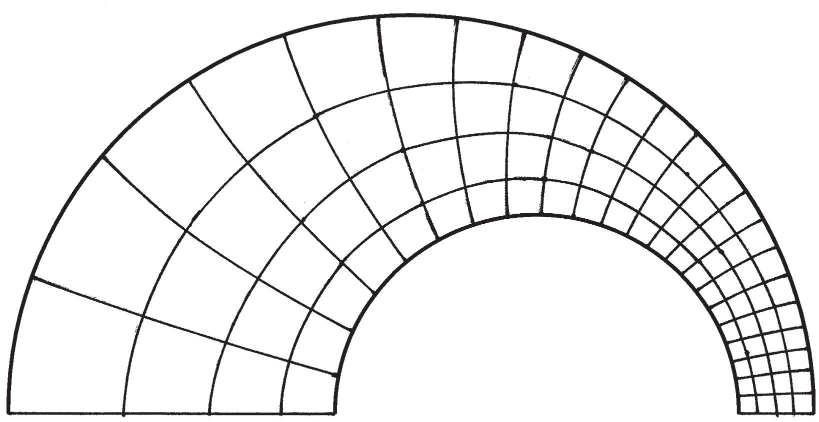
6.19. Construct a curvilinear square map of the potential field between two parallel circular cylinders, one of 4-cm radius inside one of 8-cm radius. The two axes are displaced by 2.5 cm. These dimensions are suitable for the drawing. As a check on the accuracy, compute the capacitance per meter from the sketch and from the exact expression:

$$C = \frac{2\pi\epsilon}{\cosh^{-1}\left[(a^2+b^2-D^2)/(2ab)\right]}$$

where a and b are the conductor radii and D is the axis separation.

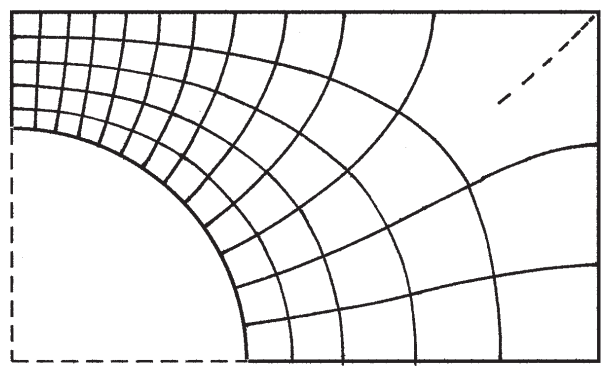
The drawing is shown below. Use of the exact expression above yields a capacitance value of $C = \underline{11.5\epsilon_0 \text{ F/m}}$. Use of the drawing produces:

$$C \doteq \frac{22 \times 2}{4}\epsilon_0 = \underline{11\epsilon_0 \text{ F/m}}$$



6.20. A solid conducting cylinder of 4-cm radius is centered within a rectangular conducting cylinder with a 12-cm by 20-cm cross-section.

a) Make a full-size sketch of one quadrant of this configuration and construct a curvilinear-square map for its interior: The result below could still be improved a little, but is nevertheless sufficient for a reasonable capacitance estimate. Note that the five-sided region in the upper right corner has been partially subdivided (dashed line) in anticipation of how it would look when the next-level subdivision is done (doubling the number of field lines and equipotentials).

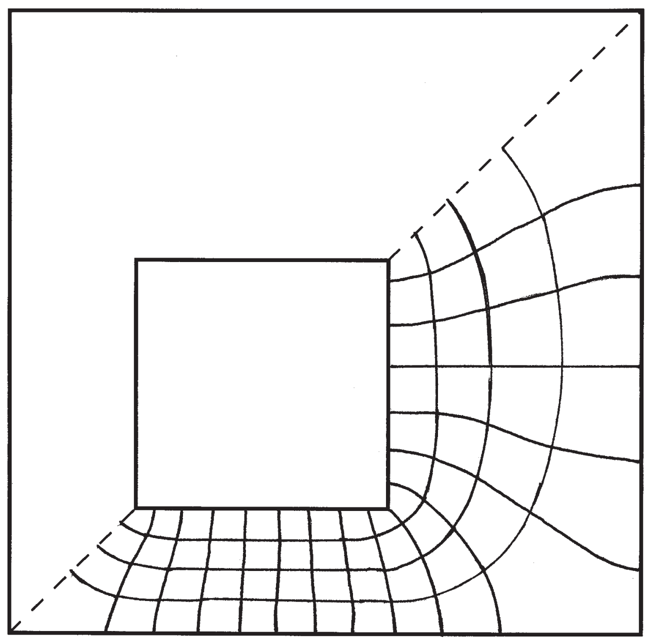


b) Assume $\epsilon = \epsilon_0$ and estimate C per meter length: In this case N_Q is the number of squares around the full perimeter of the circular conductor, or four times the number of squares shown in the drawing. N_V is the number of squares between the circle and the rectangle, or 5. The capacitance is estimated to be

$$C = \frac{N_Q}{N_V}\epsilon_0 = \frac{4 \times 13}{5}\epsilon_0 = 10.4\epsilon_0 \doteq \underline{90 \text{ pF/m}}$$

6.21. The inner conductor of the transmission line shown in Fig. 6.14 has a square cross-section $2a \times 2a$, while the outer square is $5a \times 5a$. The axes are displaced as shown. (a) Construct a good-sized drawing of the transmission line, say with $a = 2.5$ cm, and then prepare a curvilinear-square plot of the electrostatic field between the conductors. (b) Use the map to calculate the capacitance per meter length if $\epsilon = 1.6\epsilon_0$. (c) How would your result to part b change if $a = 0.6$ cm?

a) The plot is shown below. Some improvement is possible, depending on how much time one wishes to spend.



b) From the plot, the capacitance is found to be

$$C \doteq \frac{16 \times 2}{4}(1.6)\epsilon_0 = 12.8\epsilon_0 \doteq \underline{110 \text{ pF/m}}$$

c) If a is changed, the result of part b would not change, since all dimensions retain the same relative scale.

6.22. Two conducting plates, each 3 by 6 cm, and three slabs of dielectric, each 1 by 3 by 6 cm, and having dielectric constants of 1, 2, and 3 are assembled into a capacitor with $d = 3$ cm. Determine the two values of capacitance obtained by the two possible methods of assembling the capactor.

The two possible configurations are 1) all slabs positioned vertically, side-by-side; 2) all slabs positioned horizontally, stacked on top of one another. For vertical positioning, the 1x3 surfaces of each slab are in contact with the plates, and we have three capacitors in parallel. The individual capacitances will thus add to give:

$$C_{vert} = C_1 + C_2 + C_3 = \frac{\epsilon_0(1 \times 3)}{3}(1+2+3) = \underline{6\epsilon_0}$$

With the slabs positioned horizontally, the configuration becomes three capacitors in series, with the total capacitance found through:

$$\frac{1}{C_{horiz}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} = \frac{3}{\epsilon_0(3)}\left(1 + \frac{1}{2} + \frac{1}{3}\right) = \frac{11}{6\epsilon_0} \;\Rightarrow\; C_{horiz} = \underline{\frac{6\epsilon_0}{11}}$$

6.23. A two-wire transmission line consists of two parallel perfectly-conducting cylinders, each having a radius of 0.2 mm, separated by center-to-center distance of 2 mm. The medium surrounding the wires has $\epsilon_r = 3$ and $\sigma = 1.5$ mS/m. A 100-V battery is connected between the wires. Calculate:

a) the magnitude of the charge per meter length on each wire: Use

$$C = \frac{\pi\epsilon}{\cosh^{-1}(h/b)} = \frac{\pi \times 3 \times 8.85 \times 10^{-12}}{\cosh^{-1}(1/0.2)} = 3.64 \times 10^{-9} \text{ C/m}$$

Then the charge per unit length will be

$$Q = CV_0 = (3.64 \times 10^{-11})(100) = 3.64 \times 10^{-9} \text{ C/m} = \underline{3.64\,\text{nC/m}}$$

b) the battery current: Use

$$RC = \frac{\epsilon}{\sigma} \quad\Rightarrow\quad R = \frac{3 \times 8.85 \times 10^{-12}}{(1.5 \times 10^{-3})(3.64 \times 10^{-11})} = 486\,\Omega$$

Then

$$I = \frac{V_0}{R} = \frac{100}{486} = 0.206\,\text{A} = \underline{206\,\text{mA}}$$

6.24. A potential field in free space is given in spherical coordinates as:

$$V(r) = \begin{cases} [\rho_0/(6\epsilon_0)]\,[3a^2 - r^2] & (r \le a) \\ (a^3\rho_0)/(3\epsilon_0 r) & (r \ge a) \end{cases}$$

where ρ_0 and a are constants.

a) Use Poisson's equation to find the volume charge density everywhere: Inside $r = a$, we apply Poisson's equation to the potential there:

$$\nabla^2 V_1 = -\frac{\rho_v}{\epsilon_0} = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV_1}{dr}\right) = \frac{\rho_0}{6\epsilon_0}\frac{1}{r^2}\frac{d}{dr}\left(r^2(-2r)\right) = -\frac{\rho_0}{\epsilon_0}$$

from which we identify $\rho_v = \underline{\rho_0} \quad (r \le a)$.

Outside $r = a$, we use

$$\nabla^2 V_2 = -\frac{\rho_v}{\epsilon_0} = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV_2}{dr}\right) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\left(\frac{-a^3\rho_0}{3\epsilon_0 r^2}\right)\right) = 0$$

from which $\rho_v = \underline{0} \quad (r \ge a)$.

b) find the total charge present: We have a constant charge density confined within a spherical volume of radius a. The total charge is therefore $Q = \underline{(4/3)\pi a^3 \rho_0}$ C.

6.25. Let $V = 2xy^2z^3$ and $\epsilon = \epsilon_0$. Given point $P(1, 2, -1)$, find:

a) V at P: Substituting the coordinates into V, find $V_P = \underline{-8\,\text{V}}$.

b) $\mathbf{E}$ at P: We use $\mathbf{E} = -\nabla V = -2y^2z^3\mathbf{a}_x - 4xyz^3\mathbf{a}_y - 6xy^2z^2\mathbf{a}_z$, which, when evaluated at P, becomes $\mathbf{E}_P = \underline{8\mathbf{a}_x + 8\mathbf{a}_y - 24\mathbf{a}_z \text{ V/m}}$

c) ρ_v at P: This is $\rho_v = \nabla \cdot \mathbf{D} = -\epsilon_0 \nabla^2 V = \underline{-4xz(z^2 + 3y^2) \text{ C/m}^3}$

d) the equation of the equipotential surface passing through P: At P, we know $V = -8\,\text{V}$, so the equation will be $\underline{xy^2z^3 = -4}$.

e) the equation of the streamline passing through P: First,

$$\frac{E_y}{E_x} = \frac{dy}{dx} = \frac{4xyz^3}{2y^2z^3} = \frac{2x}{y}$$

Thus

$$ydy = 2xdx, \text{ and so } \frac{1}{2}y^2 = x^2 + C_1$$

Evaluating at P, we find $C_1 = 1$. Next,

$$\frac{E_z}{E_x} = \frac{dz}{dx} = \frac{6xy^2z^2}{2y^2z^3} = \frac{3x}{z}$$

Thus

$$3xdx = zdz, \text{ and so } \frac{3}{2}x^2 = \frac{1}{2}z^2 + C_2$$

Evaluating at P, we find $C_2 = 1$. The streamline is now specified by the equations: $\underline{y^2 - 2x^2 = 2}$ and $\underline{3x^2 - z^2 = 2}$.

f) Does V satisfy Laplace's equation? No, since the charge density is not zero.

6.26. Given the spherically-symmetric potential field in free space, $V = V_0 e^{-r/a}$, find:

a) ρ_v at $r = a$; Use Poisson's equation, $\nabla^2 V = -\rho_v/\epsilon$, which in this case becomes

$$-\frac{\rho_v}{\epsilon_0} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = \frac{-V_0}{ar^2}\frac{d}{dr}\left(r^2 e^{-r/a}\right) = \frac{-V_0}{ar}\left(2 - \frac{r}{a}\right)e^{-r/a}$$

from which

$$\rho_v(r) = \frac{\epsilon_0 V_0}{ar}\left(2 - \frac{r}{a}\right)e^{-r/a} \quad \Rightarrow \quad \rho_v(a) = \frac{\epsilon_0 V_0}{a^2}e^{-1} \text{ C/m}^3$$

b) the electric field at $r = a$; this we find through the negative gradient:

$$\mathbf{E}(r) = -\nabla V = -\frac{dV}{dr}\,\mathbf{a}_r = \frac{V_0}{a}e^{-r/a}\,\mathbf{a}_r \quad \Rightarrow \quad \mathbf{E}(a) = \frac{V_0}{a}e^{-1}\mathbf{a}_r \text{ V/m}$$

c) the total charge: The easiest way is to first find the electric flux density, which from part b is $\mathbf{D} = \epsilon_0 \mathbf{E} = (\epsilon_0 V_0/a)e^{-r/a}\,\mathbf{a}_r$. Then the net outward flux of $\mathbf{D}$ through a sphere of radius r would be

$$\Phi(r) = Q_{encl}(r) = 4\pi r^2 D = 4\pi\epsilon_0 V_0 r^2 e^{-r/a} \text{ C}$$

As $r \to \infty$, this result approaches zero, so the total charge is therefore $Q_{net} = \underline{0}$.

6.27. Let $V(x, y) = 4e^{2x} + f(x) - 3y^2$ in a region of free space where $\rho_v = 0$. It is known that both E_x and V are zero at the origin. Find $f(x)$ and $V(x, y)$: Since $\rho_v = 0$, we know that $\nabla^2 V = 0$, and so

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 16e^{2x} + \frac{d^2 f}{dx^2} - 6 = 0$$

Therefore

$$\frac{d^2 f}{dx^2} = -16e^{2x} + 6 \quad \Rightarrow \quad \frac{df}{dx} = -8e^{2x} + 6x + C_1$$

Now

$$E_x = \frac{\partial V}{\partial x} = 8e^{2x} + \frac{df}{dx}$$

and at the origin, this becomes

$$E_x(0) = 8 + \left.\frac{df}{dx}\right|_{x=0} = 0 \text{(as given)}$$

Thus $df/dx\,|_{x=0} = -8$, and so it follows that $C_1 = 0$. Integrating again, we find

$$f(x, y) = -4e^{2x} + 3x^2 + C_2$$

which at the origin becomes $f(0, 0) = -4 + C_2$. However, $V(0, 0) = 0 = 4 + f(0, 0)$. So $f(0, 0) = -4$ and $C_2 = 0$. Finally, $f(x, y) = \underline{-4e^{2x} + 3x^2}$, and $V(x, y) = 4e^{2x} - 4e^{2x} + 3x^2 - 3y^2 = \underline{3(x^2 - y^2)}$.

6.28. Show that in a homogeneous medium of conductivity σ, the potential field V satisfies Laplace's equation if any volume charge density present does not vary with time: We begin with the continuity equation, Eq. (5), Chapter 5:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$$

where $\mathbf{J} = \sigma \mathbf{E}$, and where, in our homogeneous medium, σ is constant with position. Now write

$$\nabla \cdot \mathbf{J} = \nabla \cdot (\sigma \mathbf{E}) = \sigma \nabla \cdot (-\nabla V) = -\sigma \nabla^2 V = -\frac{\partial \rho_v}{\partial t}$$

This becomes

$$\nabla^2 V = 0 \text{ (Laplace's equation)}$$

when ρ_v is time-independent. Q.E.D.

6.29. Given the potential field $V = (A\rho^4 + B\rho^{-4}) \sin 4\phi$:

a) Show that $\nabla^2 V = 0$: In cylindrical coordinates,

$$\begin{aligned}\nabla^2 V &= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} \\ &= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho(4A\rho^3 - 4B\rho^{-5})\right)\sin 4\phi - \frac{1}{\rho^2} 16(A\rho^4 + B\rho^{-4})\sin 4\phi \\ &= \frac{16}{\rho}(A\rho^3 + B\rho^{-5})\sin 4\phi - \frac{16}{\rho^2}(A\rho^4 + B\rho^{-4})\sin 4\phi = 0\end{aligned}$$

b) Select A and B so that $V = 100$ V and $|\mathbf{E}| = 500$ V/m at $P(\rho = 1, \phi = 22.5^\circ, z = 2)$: First,

$$\begin{aligned}\mathbf{E} = -\nabla V &= -\frac{\partial V}{\partial \rho}\,\mathbf{a}_\rho - \frac{1}{\rho}\frac{\partial V}{\partial \phi}\,\mathbf{a}_\phi \\ &= -4\left[(A\rho^3 - B\rho^{-5})\sin 4\phi\,\mathbf{a}_\rho + (A\rho^3 + B\rho^{-5})\cos 4\phi\,\mathbf{a}_\phi\right]\end{aligned}$$

and at P, $\mathbf{E}_P = -4(A - B)\,\mathbf{a}_\rho$. Thus $|\mathbf{E}_P| = \pm 4(A - B)$. Also, $V_P = A + B$. Our two equations are:

$$4(A - B) = \pm 500$$

and

$$A + B = 100$$

We thus have two pairs of values for A and B:

$$\underline{A = 112.5,\ B = -12.5} \quad \text{or} \quad \underline{A = -12.5,\ B = 112.5}$$

6.30. A parallel-plate capacitor has plates located at $z = 0$ and $z = d$. The region between plates is filled with a material containing volume charge of uniform density ρ_0 C/m^3, and which has permittivity ϵ. Both plates are held at ground potential.

a) Determine the potential field between plates: We solve Poisson's equation, under the assumption that V varies only with z:

$$\nabla^2 V = \frac{d^2 V}{dz^2} = -\frac{\rho_0}{\epsilon} \;\Rightarrow\; V = \frac{-\rho_0 z^2}{2\epsilon} + C_1 z + C_2$$

At $z = 0$, $V = 0$, and so $C_2 = 0$. Then, at $z = d$, $V = 0$ as well, so we find $C_1 = \rho_0 d/2\epsilon$. Finally, $V(z) = \underline{(\rho_0 z/2\epsilon)[d - z]}$.

b) Determine the electric field intensity, **E** between plates: Taking the answer to part *a*, we find **E** through

$$\mathbf{E} = -\nabla V = -\frac{dV}{dz}\,\mathbf{a}_z = -\frac{d}{dz}\left[\frac{\rho_0 z}{2\epsilon}(d - z)\right] = \frac{\rho_0}{2\epsilon}(2z - d)\,\mathbf{a}_z \text{ V/m}$$

c) Repeat *a* and *b* for the case of the plate at $z = d$ raised to potential V_0, with the $z = 0$ plate grounded: Begin with

$$V(z) = \frac{-\rho_0 z^2}{2\epsilon} + C_1 z + C_2$$

with $C_2 = 0$ as before, since $V(z = 0) = 0$. Then

$$V(z = d) = V_0 = \frac{-\rho_0 d^2}{2\epsilon} + C_1 d \;\Rightarrow\; C_1 = \frac{V_0}{d} + \frac{\rho_0 d}{2\epsilon}$$

So that

$$V(z) = \frac{V_0}{d} z + \frac{\rho_0 z}{2\epsilon}(d - z)$$

We recognize this as the simple superposition of the voltage as found in part *a* and the voltage of a capacitor carrying voltage V_0, but without the charged dielectric. The electric field is now

$$\mathbf{E} = -\frac{dV}{dz}\,\mathbf{a}_z = \frac{-V_0}{d}\,\mathbf{a}_z + \frac{\rho_0}{2\epsilon}(2z - d)\,\mathbf{a}_z \text{ V/m}$$

6.31. Let $V = (\cos 2\phi)/\rho$ in free space.

a) Find the volume charge density at point $A(0.5, 60^\circ, 1)$: Use Poisson's equation:

$$\rho_v = -\epsilon_0 \nabla^2 V = -\epsilon_0 \left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} \right)$$

$$= -\epsilon_0 \left(\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{-\cos 2\phi}{\rho}\right) - \frac{4}{\rho^2}\frac{\cos 2\phi}{\rho} \right) = \frac{3\epsilon_0 \cos 2\phi}{\rho^3}$$

So at A we find:

$$\rho_{vA} = \frac{3\epsilon_0 \cos(120^\circ)}{0.5^3} = -12\epsilon_0 = \underline{-106\,\text{pC/m}^3}$$

b) Find the surface charge density on a conductor surface passing through $B(2, 30^\circ, 1)$: First, we find $\mathbf{E}$:

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial \rho}\,\mathbf{a}_\rho - \frac{1}{\rho}\frac{\partial V}{\partial \phi}\,\mathbf{a}_\phi$$

$$= \frac{\cos 2\phi}{\rho^2}\,\mathbf{a}_\rho + \frac{2\sin 2\phi}{\rho^2}\,\mathbf{a}_\phi$$

At point B the field becomes

$$\mathbf{E}_B = \frac{\cos 60^\circ}{4}\,\mathbf{a}_\rho + \frac{2\sin 60^\circ}{4}\,\mathbf{a}_\phi = 0.125\,\mathbf{a}_\rho + 0.433\,\mathbf{a}_\phi$$

The surface charge density will now be

$$\rho_{sB} = \pm|\mathbf{D}_B| = \pm\epsilon_0|\mathbf{E}_B| = \pm 0.451\epsilon_0 = \underline{\pm 0.399\,\text{pC/m}^2}$$

The charge is positive or negative depending on which side of the surface we are considering. The problem did not provide information necessary to determine this.

6.32. A uniform volume charge has constant density $\rho_v = \rho_0$ C/m^3, and fills the region $r < a$, in which permittivity ϵ as assumed. A conducting spherical shell is located at $r = a$, and is held at ground potential. Find:

a) the potential everywhere: Inside the sphere, we solve Poisson's equation, assuming radial variation only:

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = \frac{-\rho_0}{\epsilon} \quad \Rightarrow \quad V(r) = \frac{-\rho_0 r^2}{6\epsilon_0} + \frac{C_1}{r} + C_2$$

We require that V is finite at the orgin (or as $r \to 0$), and so therefore $C_1 = 0$. Next, $V = 0$ at $r = a$, which gives $C_2 = \rho_0 a^2/6\epsilon$. Outside, $r > a$, we know the potential must be zero, since the sphere is grounded. To show this, solve Laplace's equation:

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0 \quad \Rightarrow \quad V(r) = \frac{C_1}{r} + C_2$$

Requiring $V = 0$ at both $r = a$ and at infinity leads to $C_1 = C_2 = 0$. To summarize

$$V(r) = \begin{cases} \frac{\rho_0}{6\epsilon}(a^2 - r^2) & r < a \\ 0 & r > a \end{cases}$$

b) the electric field intensity, **E**, everywhere: Use

$$\mathbf{E} = -\nabla V = \frac{-dV}{dr}\,\mathbf{a}_r = \frac{\rho_0 r}{3\epsilon}\,\mathbf{a}_r \quad r < a$$

Outside ($r > a$), the potential is zero, and so $\mathbf{E} = 0$ there as well.

6.33. The functions $V_1(\rho, \phi, z)$ and $V_2(\rho, \phi, z)$ both satisfy Laplace's equation in the region $a < \rho < b$, $0 \leq \phi < 2\pi$, $-L < z < L$; each is zero on the surfaces $\rho = b$ for $-L < z < L$; $z = -L$ for $a < \rho < b$; and $z = L$ for $a < \rho < b$; and each is 100 V on the surface $\rho = a$ for $-L < z < L$.

a) In the region specified above, is Laplace's equation satisfied by the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$? Yes for the first three, since Laplace's equation is linear. No for $V_1 V_2$.

b) On the boundary surfaces specified, are the potential values given above obtained from the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$? At the 100 V surface ($\rho = a$), No for all. At the 0 V surfaces, yes, except for $V_1 + 3$.

c) Are the functions $V_1 + V_2$, $V_1 - V_2$, $V_1 + 3$, and $V_1 V_2$ identical with V_1? Only V_2 is, since it is given as satisfying all the boundary conditions that V_1 does. Therefore, by the uniqueness theorem, $V_2 = V_1$. The others, not satisfying the boundary conditions, are not the same as V_1.

6.34. Consider the parallel-plate capacitor of Problem 6.30, but this time the charged dielectric exists only between $z = 0$ and $z = b$, where $b < d$. Free space fills the region $b < z < d$. Both plates are at ground potential. No surface charge exists at $z = b$, so that both V and $\mathbf{D}$ are continuous there. By solving Laplace's *and* Poisson's equations, find:

a) $V(z)$ for $0 < z < d$: In Region 1 ($z < b$), we solve Poisson's equation, assuming z variation only:

$$\frac{d^2V_1}{dz^2} = \frac{-\rho_0}{\epsilon} \quad \Rightarrow \quad \frac{dV_1}{dz} = \frac{-\rho_0 z}{\epsilon} + C_1 \quad (z < b)$$

In Region 2 ($z > b$), we solve Laplace's equation, assuming z variation only:

$$\frac{d^2V_2}{dz^2} = 0 \quad \Rightarrow \quad \frac{dV_2}{dz} = C_1' \quad (z > b)$$

At this stage we apply the first boundary condition, which is continuity of $\mathbf{D}$ across the interface at $z = b$. Knowing that the electric field magnitude is given by dV/dz, we write

$$\epsilon \frac{dV_1}{dz}\bigg|_{z=b} = \epsilon_0 \frac{dV_2}{dz}\bigg|_{z=b} \quad \Rightarrow \quad -\rho_0 b + \epsilon C_1 = \epsilon_0 C_1' \quad \Rightarrow \quad C_1' = \frac{-\rho_0 b}{\epsilon_0} + \frac{\epsilon}{\epsilon_0} C_1$$

Substituting the above expression for C_1', and performing a second integration on the Poisson and Laplace equations, we find

$$V_1(z) = -\frac{\rho_0 z^2}{2\epsilon} + C_1 z + C_2 \quad (z < b)$$

and

$$V_2(z) = -\frac{\rho_0 bz}{2\epsilon_0} + \frac{\epsilon}{\epsilon_0} C_1 z + C_2' \quad (z > b)$$

Next, requiring $V_1 = 0$ at $z = 0$ leads to $C_2 = 0$. Then, the requirement that $V_2 = 0$ at $z = d$ leads to

$$0 = -\frac{\rho_0 bd}{\epsilon_0} + \frac{\epsilon}{\epsilon_0} C_1 d + C_2' \quad \Rightarrow \quad C_2' = \frac{\rho_0 bd}{\epsilon_0} - \frac{\epsilon}{\epsilon_0} C_1 d$$

With C_2 and C_2' known, the voltages now become

$$V_1(z) = -\frac{\rho_0 z^2}{2\epsilon} + C_1 z \quad \text{and} \quad V_2(z) = \frac{\rho_0 b}{\epsilon_0}(d - z) - \frac{\epsilon}{\epsilon_0} C_1 (d - z)$$

Finally, to evaluate C_1, we equate the two voltage expressions at $z = b$:

$$V_1|_{z=b} = V_2|_{z=b} \quad \Rightarrow \quad C_1 = \frac{\rho_0 b}{2\epsilon}\left[\frac{b + 2\epsilon_r(d - b)}{b + \epsilon_r(d - b)}\right]$$

where $\epsilon_r = \epsilon/\epsilon_0$. Substituting C_1 as found above into V_1 and V_2 leads to the final expressions for the voltages:

$$V_1(z) = \frac{\rho_0 bz}{2\epsilon}\left[\left(\frac{b + 2\epsilon_r(d - b)}{b + \epsilon_r(d - b)}\right) - \frac{z}{b}\right] \quad (z < b)$$

$$V_2(z) = \frac{\rho_0 b^2}{2\epsilon_0}\left[\frac{d - z}{b + \epsilon_r(d - b)}\right] \quad (z > b)$$

6.34b) the electric field intensity for $0 < z < d$: This involves taking the negative gradient of the final voltage expressions of part *a*. We find

$$\mathbf{E}_1 = -\frac{dV_1}{dz}\,\mathbf{a}_z = \frac{\rho_0}{\epsilon}\left[z - \frac{b}{2}\left(\frac{b + 2\epsilon_r(d-b)}{b + \epsilon_r(d-b)}\right)\right]\mathbf{a}_z \ \ \text{V/m} \quad (z < b)$$

$$\mathbf{E}_2 = -\frac{dV_2}{dz}\,\mathbf{a}_z = \frac{\rho_0 b^2}{2\epsilon_0}\left[\frac{1}{b + \epsilon_r(d-b)}\right]\mathbf{a}_z \ \ \text{V/m} \quad (z > b)$$

6.35. The conducting planes $2x + 3y = 12$ and $2x + 3y = 18$ are at potentials of 100 V and 0, respectively. Let $\epsilon = \epsilon_0$ and find:

a) V at $P(5, 2, 6)$: The planes are parallel, and so we expect variation in potential in the direction normal to them. Using the two boundary condtions, our general potential function can be written:

$$V(x, y) = A(2x + 3y - 12) + 100 = A(2x + 3y - 18) + 0$$

and so $A = -100/6$. We then write

$$V(x, y) = -\frac{100}{6}(2x + 3y - 18) = -\frac{100}{3}x - 50y + 300$$

and $V_P = -\frac{100}{3}(5) - 100 + 300 = \underline{33.33 \text{ V}}$.

b) Find $\mathbf{E}$ at P: Use

$$\mathbf{E} = -\nabla V = \underline{\frac{100}{3}\,\mathbf{a}_x + 50\,\mathbf{a}_y \ \text{V/m}}$$

6.36. The derivation of Laplace's and Poisson's equations assumed constant permittivity, but there are cases of spatially-varying permittivity in which the equations will still apply. Consider the vector identity, $\nabla \cdot (\psi \mathbf{G}) = \mathbf{G} \cdot \nabla\psi + \psi\nabla \cdot \mathbf{G}$, where ψ and $\mathbf{G}$ are scalar and vector functions, respectively. Determine a general rule on the allowed *directions* in which ϵ may vary with respect to the electric field.

In the original derivation of Poisson's equation, we started with $\nabla \cdot \mathbf{D} = \rho_v$, where $\mathbf{D} = \epsilon\mathbf{E}$. Therefore

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon\mathbf{E}) = -\nabla \cdot (\epsilon\nabla V) = -\nabla V \cdot \nabla\epsilon - \epsilon\nabla^2 V = \rho_v$$

We see from this that Poisson's equation, $\nabla^2 V = -\rho_v/\epsilon$, results when $\nabla V \cdot \nabla\epsilon = 0$. In words, ϵ is allowed to vary, provided it does so in directions that are normal to the local electric field.

6.37. Coaxial conducting cylinders are located at $\rho = 0.5$ cm and $\rho = 1.2$ cm. The region between the cylinders is filled with a homogeneous perfect dielectric. If the inner cylinder is at 100V and the outer at 0V, find:

a) the location of the 20V equipotential surface: From Eq. (35) we have

$$V(\rho) = 100\frac{\ln(.012/\rho)}{\ln(.012/.005)} \text{ V}$$

We seek ρ at which $V = 20$ V, and thus we need to solve:

$$20 = 100\frac{\ln(.012/\rho)}{\ln(2.4)} \quad \Rightarrow \quad \rho = \frac{.012}{(2.4)^{0.2}} = \underline{1.01 \text{ cm}}$$

b) $E_{\rho\,max}$: We have

$$E_\rho = -\frac{\partial V}{\partial \rho} = -\frac{dV}{d\rho} = \frac{100}{\rho \ln(2.4)}$$

whose maximum value will occur at the inner cylinder, or at $\rho = .5$ cm:

$$E_{\rho\,max} = \frac{100}{.005\ln(2.4)} = 2.28 \times 10^4 \text{ V/m} = \underline{22.8 \text{ kV/m}}$$

c) ϵ_r if the charge per meter length on the inner cylinder is 20 nC/m: The capacitance per meter length is

$$C = \frac{2\pi\epsilon_0\epsilon_r}{\ln(2.4)} = \frac{Q}{V_0}$$

We solve for ϵ_r:

$$\epsilon_r = \frac{(20 \times 10^{-9})\ln(2.4)}{2\pi\epsilon_0(100)} = \underline{3.15}$$

6.38. Repeat Problem 6.37, but with the dielectric only partially filling the volume, within $0 < \phi < \pi$, and with free space in the remaining volume.

We note that the dielectric changes with ϕ, and not with ρ. Also, since **E** is radially-directed and varies only with radius, Laplace's equation for this case is valid (see Problem 6.36) and is the same as that which led to the potential and field in Problem 6.37. Therefore, the solutions to parts a and b are unchanged from Problem 6.37. Part c, however, is different. We write the charge per unit length as the sum of the charges along each half of the center conductor (of radius a)

$$Q = \epsilon_r\epsilon_0 E_{\rho,max}(\pi a) + \epsilon_0 E_{\rho,max}(\pi a) = \epsilon_0 E_{\rho,max}(\pi a)(1+\epsilon_r) \text{ C/m}$$

Using the numbers given or found in Problem 6.37, we obtain

$$1 + \epsilon_r = \frac{20 \times 10^{-9} \text{ C/m}}{(8.852 \times 10^{-12})(22.8 \times 10^3 \text{ V/m})(0.5 \times 10^{-2} \text{ m})\pi} = 6.31 \quad \Rightarrow \quad \epsilon_r = \underline{5.31}$$

We may also note that the *average* dielectric constant in this problem, $(\epsilon_r + 1)/2$, is the same as that of the uniform dielectric constant found in Problem 6.37.

6.39. The two conducting planes illustrated in Fig. 6.14 are defined by $0.001 < \rho < 0.120$ m, $0 < z < 0.1$ m, $\phi = 0.179$ and 0.188 rad. The medium surrounding the planes is air. For region 1, $0.179 < \phi < 0.188$, neglect fringing and find:

a) $V(\phi)$: The general solution to Laplace's equation will be $V = C_1\phi + C_2$, and so

$$20 = C_1(.188) + C_2 \quad \text{and} \quad 200 = C_1(.179) + C_2$$

Subtracting one equation from the other, we find

$$-180 = C_1(.188 - .179) \quad \Rightarrow \quad C_1 = -2.00 \times 10^4$$

Then

$$20 = -2.00 \times 10^4(.188) + C_2 \quad \Rightarrow \quad C_2 = 3.78 \times 10^3$$

Finally, $V(\phi) = \underline{(-2.00 \times 10^4)\phi + 3.78 \times 10^3 \text{ V}}$.

b) $\mathbf{E}(\rho)$: Use

$$\mathbf{E}(\rho) = -\nabla V = -\frac{1}{\rho}\frac{dV}{d\phi} = \underline{\frac{2.00 \times 10^4}{\rho}\,\mathbf{a}_\phi \text{ V/m}}$$

c) $\mathbf{D}(\rho) = \epsilon_0\mathbf{E}(\rho) = \underline{(2.00 \times 10^4\epsilon_0/\rho)\,\mathbf{a}_\phi \text{ C/m}^2}$.

d) ρ_s on the upper surface of the lower plane: We use

$$\rho_s = \mathbf{D}\cdot\mathbf{n}\Big|_{surface} = \frac{2.00 \times 10^4}{\rho}\,\mathbf{a}_\phi \cdot \mathbf{a}_\phi = \underline{\frac{2.00 \times 10^4}{\rho} \text{ C/m}^2}$$

e) Q on the upper surface of the lower plane: This will be

$$Q_t = \int_0^{.1}\int_{.001}^{.120} \frac{2.00 \times 10^4\epsilon_0}{\rho}\,d\rho\,dz = 2.00 \times 10^4\epsilon_0(.1)\ln(120) = 8.47 \times 10^{-8} \text{ C} = \underline{84.7 \text{ nC}}$$

f) Repeat *a*) to *c*) for region 2 by letting the location of the upper plane be $\phi = .188 - 2\pi$, and then find ρ_s and Q on the lower surface of the lower plane. Back to the beginning, we use

$$20 = C_1'(.188 - 2\pi) + C_2' \quad \text{and} \quad 200 = C_1'(.179) + C_2'$$

Subtracting one from the other, we find

$$-180 = C_1'(.009 - 2\pi) \quad \Rightarrow \quad C_1' = 28.7$$

Then $200 = 28.7(.179) + C_2' \;\Rightarrow\; C_2' = 194.9$. Thus $V(\phi) = \underline{28.7\phi + 194.9}$ in region 2. Then

$$\underline{\mathbf{E} = -\frac{28.7}{\rho}\,\mathbf{a}_\phi \text{ V/m}} \quad \text{and} \quad \underline{\mathbf{D} = -\frac{28.7\epsilon_0}{\rho}\,\mathbf{a}_\phi \text{ C/m}^2}$$

ρ_s on the lower surface of the lower plane will now be

$$\rho_s = -\frac{28.7\epsilon_0}{\rho}\,\mathbf{a}_\phi \cdot (-\mathbf{a}_\phi) = \underline{\frac{28.7\epsilon_0}{\rho} \text{ C/m}^2}$$

The charge on that surface will then be $Q_b = 28.7\epsilon_0(.1)\ln(120) = \underline{122 \text{ pC}}$.

6.39g) Find the total charge on the lower plane and the capacitance between the planes: Total charge will be $Q_{net} = Q_t + Q_b = 84.7 \text{ nC} + 0.122 \text{ nC} = \underline{84.8 \text{ nC}}$. The capacitance will be

$$C = \frac{Q_{net}}{\Delta V} = \frac{84.8}{200 - 20} = 0.471 \text{ nF} = \underline{471 \text{ pF}}$$

6.40. A parallel-plate capacitor is made using two circular plates of radius a, with the bottom plate on the xy plane, centered at the origin. The top plate is located at $z = d$, with its center on the z axis. Potential V_0 is on the top plate; the bottom plate is grounded. Dielectric having *radially-dependent* permittivity fills the region between plates. The permittivity is given by $\epsilon(\rho) = \epsilon_0(1 + \rho^2/a^2)$. Find:

a) $V(z)$: Since ϵ varies in the direction normal to $\mathbf{E}$, Laplace's equation applies, and we write

$$\nabla^2 V = \frac{d^2V}{dz^2} = 0 \quad \Rightarrow \quad V(z) = C_1 z + C_2$$

With the given boundary conditions, $C_2 = 0$, and $C_1 = V_0/d$. Therefore $\underline{V(z) = V_0 z/d \text{ V}}$.

b) $\mathbf{E}$: This will be $\mathbf{E} = -\nabla V = -dV/dz\,\mathbf{a}_z = \underline{-(V_0/d)\,\mathbf{a}_z \text{ V/m}}$.

c) Q: First we find the electric flux density: $\mathbf{D} = \epsilon\mathbf{E} = -\epsilon_0(1 + \rho^2/a^2)(V_0/d)\,\mathbf{a}_z$ C/m^2. The charge density on the top plate is then $\rho_s = \mathbf{D} \cdot -\mathbf{a}_z = \epsilon_0(1 + \rho^2/a^2)(V_0/d)$ C/m^2. From this we find the charge on the top plate:

$$Q = \int_0^{2\pi}\int_0^a \epsilon_0(1 + \rho^2/a^2)(V_0/d)\,\rho\,d\rho\,d\phi = \frac{3\pi a^2 \epsilon_0 V_0}{2d} \text{ C}$$

d) C. The capacitance is $C = Q/V_0 = \underline{3\pi a^2 \epsilon_0/(2d) \text{ F}}$.

6.41. Concentric conducting spheres are located at $r = 5$ mm and $r = 20$ mm. The region between the spheres is filled with a perfect dielectric. If the inner sphere is at 100 V and the outer sphere at 0 V:

a) Find the location of the 20 V equipotential surface: Solving Laplace's equation gives us

$$V(r) = V_0 \frac{\frac{1}{r} - \frac{1}{b}}{\frac{1}{a} - \frac{1}{b}}$$

where $V_0 = 100$, $a = 5$ and $b = 20$. Setting $V(r) = 20$, and solving for r produces $r = \underline{12.5\,\text{mm}}$.

b) Find $E_{r,max}$: Use

$$\mathbf{E} = -\nabla V = -\frac{dV}{dr}\mathbf{a}_r = \frac{V_0\,\mathbf{a}_r}{r^2\left(\frac{1}{a} - \frac{1}{b}\right)}$$

$$E_{r,max} = E(r = a) = \frac{V_0}{a(1 - (a/b))} = \frac{100}{5(1 - (5/20))} = 26.7 \text{ V/mm} = \underline{26.7 \text{ kV/m}}$$

c) Find ϵ_r if the surface charge density on the inner sphere is $1.0\,\mu$C/m^2: ρ_s will be equal in magnitude to the electric flux density at $r = a$. So $\rho_s = (2.67 \times 10^4 \text{ V/m})\epsilon_r\epsilon_0 = 10^{-6}$ C/m^2. Thus $\epsilon_r = \underline{4.23}$. Note, in the first printing, the given charge density was $100\,\mu$C/m^2, leading to a ridiculous answer of $\epsilon_r = 423$.

6.42. The hemisphere $0 < r < a$, $0 < \theta < \pi/2$, is composed of homogeneous conducting material of conductivity σ. The flat side of the hemisphere rests on a perfectly-conducting plane. Now, the material within the conical region $0 < \theta < \alpha$, $0 < r < a$, is drilled out, and replaced with material that is perfectly-conducting. An air gap is maintained between the $r = 0$ tip of this new material and the plane. What resistance is measured between the two perfect conductors? Neglect fringing fields.

With no fringing fields, we have θ-variation only in the potential. Laplace's equation is therefore:

$$\nabla^2 V = \frac{1}{r^2 \sin\theta} \frac{d}{d\theta} \left(\sin\theta \frac{dV}{d\theta} \right) = 0$$

This reduces to

$$\frac{dV}{d\theta} = \frac{C_1}{\sin\theta} \quad \Rightarrow \quad V(\theta) = C_1 \ln\tan(\theta/2) + C_2$$

We assume zero potential on the plane (at $\theta = \pi/2$), which means that $C_2 = 0$. On the cone (at $\theta = \alpha$), we assume potential V_0, and so $V_0 = C_1 \ln\tan(\alpha/2)$ $\Rightarrow\ C_1 = V_0 / \ln\tan(\alpha/2)$ The potential function is now

$$V(\theta) = V_0 \frac{\ln\tan(\theta/2)}{\ln\tan(\alpha/2)} \qquad \alpha < \theta < \pi/2$$

The electric field is then

$$\mathbf{E} = -\nabla V = -\frac{1}{r}\frac{dV}{d\theta}\,\mathbf{a}_\theta = -\frac{V_0}{r\sin\theta\ln\tan(\alpha/2)}\,\mathbf{a}_\theta \ \ \text{V/m}$$

The total current can now be found by integrating the current density, $\mathbf{J} = \sigma\mathbf{E}$, over any cross-section. Choosing the lower plane at $\theta = \pi/2$, this becomes

$$I = \int_0^{2\pi}\int_0^a -\frac{\sigma V_0}{r\sin(\pi/2)\ln\tan(\alpha/2)}\,\mathbf{a}_\theta \cdot \mathbf{a}_\theta\, r\,dr\,d\phi = -\frac{2\pi a\sigma V_0}{\ln\tan(\alpha/2)} \ \ \text{A}$$

The resistance is finally

$$R = \frac{V_0}{I} = -\frac{\ln\tan(\alpha/2)}{2\pi a\sigma} \ \text{ohms}$$

Note that R is in fact positive (despite the minus sign) since $\ln\tan(\alpha/2)$ is negative when $\alpha < \pi/2$ (which it must be).

6.43. Two coaxial conducting cones have their vertices at the origin and the z axis as their axis. Cone A has the point $A(1,0,2)$ on its surface, while cone B has the point $B(0,3,2)$ on its surface. Let $V_A = 100$ V and $V_B = 20$ V. Find:

a) α for each cone: Have $\alpha_A = \tan^{-1}(1/2) = \underline{26.57^\circ}$ and $\alpha_B = \tan^{-1}(3/2) = \underline{56.31^\circ}$.

b) V at $P(1,1,1)$: The potential function between cones can be written as

$$V(\theta) = C_1 \ln \tan(\theta/2) + C_2$$

Then

$$20 = C_1 \ln \tan(56.31/2) + C_2 \quad \text{and} \quad 100 = C_1 \ln \tan(26.57/2) + C_2$$

Solving these two equations, we find $C_1 = -97.7$ and $C_2 = -41.1$. Now at P, $\theta = \tan^{-1}(\sqrt{2}) = 54.7^\circ$. Thus

$$V_P = -97.7 \ln \tan(54.7/2) - 41.1 = \underline{23.3\,\text{V}}$$

6.44. A potential field in free space is given as $V = 100 \ln \tan(\theta/2) + 50$ V.

a) Find the maximum value of $|\mathbf{E}_\theta|$ on the surface $\theta = 40^\circ$ for $0.1 < r < 0.8$ m, $60^\circ < \phi < 90^\circ$. First

$$\mathbf{E} = -\frac{1}{r}\frac{dV}{d\theta}\mathbf{a}_\theta = -\frac{100}{2r\tan(\theta/2)\cos^2(\theta/2)}\mathbf{a}_\theta = -\frac{100}{2r\sin(\theta/2)\cos(\theta/2)}\mathbf{a}_\theta = -\frac{100}{r\sin\theta}\mathbf{a}_\theta$$

This will maximize at the smallest value of r, or 0.1:

$$\mathbf{E}_{max}(\theta = 40^\circ) = \mathbf{E}(r = 0.1, \theta = 40^\circ) = -\frac{100}{0.1\sin(40)}\mathbf{a}_\theta = \underline{1.56\,\mathbf{a}_\theta \text{ kV/m}}$$

b) Describe the surface $V = 80$ V: Set $100 \ln \tan\theta/2 + 50 = 80$ and solve for θ: Obtain $\ln \tan\theta/2 = 0.3 \quad \Rightarrow \quad \tan\theta/2 = e^{.3} = 1.35 \quad \Rightarrow \quad \theta = \underline{107^\circ}$ (the cone surface at $\theta = 107$ degrees).

6.45. In free space, let $\rho_v = 200\epsilon_0/r^{2.4}$.

a) Use Poisson's equation to find $V(r)$ if it is assumed that $r^2E_r \rightarrow 0$ when $r \rightarrow 0$, and also that $V \rightarrow 0$ as $r \rightarrow \infty$: With r variation only, we have

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = -\frac{\rho_v}{\epsilon} = -200r^{-2.4}$$

or

$$\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = -200r^{-.4}$$

Integrate once:

$$\left(r^2\frac{dV}{dr}\right) = -\frac{200}{.6}r^{.6} + C_1 = -333.3r^{.6} + C_1$$

or

$$\frac{dV}{dr} = -333.3r^{-1.4} + \frac{C_1}{r^2} = \nabla V \text{ (in this case)} = -E_r$$

Our first boundary condition states that $r^2E_r \rightarrow 0$ when $r \rightarrow 0$ Therefore $C_1 = 0$. Integrate again to find:

$$V(r) = \frac{333.3}{.4}r^{-.4} + C_2$$

From our second boundary condition, $V \rightarrow 0$ as $r \rightarrow \infty$, we see that $C_2 = 0$. Finally,

$$V(r) = \underline{833.3r^{-.4}\ \text{V}}$$

b) Now find $V(r)$ by using Gauss' Law and a line integral: Gauss' law applied to a spherical surface of radius r gives:

$$4\pi r^2 D_r = 4\pi\int_0^r \frac{200\epsilon_0}{(r')^{2.4}}(r')^2 dr = 800\pi\epsilon_0\frac{r^{.6}}{.6}$$

Thus

$$E_r = \frac{D_r}{\epsilon_0} = \frac{800\pi\epsilon_0 r^{.6}}{.6(4\pi)\epsilon_0 r^2} = 333.3r^{-1.4}\ \text{V/m}$$

Now

$$V(r) = -\int_\infty^r 333.3(r')^{-1.4}dr' = \underline{833.3r^{-.4}\ \text{V}}$$

6.46. By appropriate solution of Laplace's *and* Poisson's equations, determine the absolute potential at the center of a sphere of radius a, containing uniform volume charge of density ρ_0. Assume permittivity ϵ_0 everywhere. HINT: What must be true about the potential and the electric field at $r = 0$ and at $r = a$?

With radial dependence only, Poisson's equation (applicable to $r \leq a$) becomes

$$\nabla^2 V_1 = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV_1}{dr}\right) = -\frac{\rho_0}{\epsilon_0} \quad \Rightarrow \quad V_1(r) = -\frac{\rho_0 r^2}{6\epsilon_0} + \frac{C_1}{r} + C_2 \quad (r \leq a)$$

For region 2 ($r \geq a$) there is no charge and so Laplace's equation becomes

$$\nabla^2 V_2 = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV_2}{dr}\right) = 0 \quad \Rightarrow \quad V_2(r) = \frac{C_3}{r} + C_4 \quad (r \geq a)$$

Now, as $r \rightarrow \infty$, $V_2 \rightarrow 0$, so therefore $C_4 = 0$. Also, as $r \rightarrow 0$, V_1 must be finite, so therefore $C_1 = 0$. Then, V must be continuous across the boundary, $r = a$:

$$V_1\Big|_{r=a} = V_2\Big|_{r=a} \quad \Rightarrow \quad -\frac{\rho_0 a^2}{6\epsilon_0} + C_2 = \frac{C_3}{a} \quad \Rightarrow \quad C_2 = \frac{C_3}{a} + \frac{\rho_0 a^2}{6\epsilon_0}$$

So now

$$V_1(r) = \frac{\rho_0}{6\epsilon_0}(a^2 - r^2) + \frac{C_3}{a} \quad \text{and} \quad V_2(r) = \frac{C_3}{r}$$

Finally, since the permittivity is ϵ_0 everywhere, the electric field will be continuous at $r = a$. This is equivalent to the continuity of the voltage derivatives:

$$\frac{dV_1}{dr}\Big|_{r=a} = \frac{dV_2}{dr}\Big|_{r=a} \quad \Rightarrow \quad -\frac{\rho_0 a}{3\epsilon_0} = -\frac{C_3}{a^2} \quad \Rightarrow \quad C_3 = \frac{\rho_0 a^3}{3\epsilon_0}$$

So the potentials in their final forms are

$$V_1(r) = \frac{\rho_0}{6\epsilon_0}(3a^2 - r^2) \quad \text{and} \quad V_2(r) = \frac{\rho_0 a^3}{3\epsilon_0 r}$$

The requested absolute potential at the origin is now $V_1(r = 0) = \underline{\rho_0 a^2/(2\epsilon_0)\ \text{V}}$.

CHAPTER 7

7.1a. Find $\mathbf{H}$ in cartesian components at $P(2,3,4)$ if there is a current filament on the z axis carrying 8 mA in the $\mathbf{a}_z$ direction:
Applying the Biot-Savart Law, we obtain

$$\mathbf{H}_a = \int_{-\infty}^{\infty} \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \int_{-\infty}^{\infty} \frac{Idz\,\mathbf{a}_z \times [2\mathbf{a}_x + 3\mathbf{a}_y + (4-z)\mathbf{a}_z]}{4\pi(z^2 - 8z + 29)^{3/2}} = \int_{-\infty}^{\infty} \frac{Idz[2\mathbf{a}_y - 3\mathbf{a}_x]}{4\pi(z^2 - 8z + 29)^{3/2}}$$

Using integral tables, this evaluates as

$$\mathbf{H}_a = \frac{I}{4\pi}\left[\frac{2(2z-8)(2\mathbf{a}_y - 3\mathbf{a}_x)}{52(z^2 - 8z + 29)^{1/2}}\right]_{-\infty}^{\infty} = \frac{I}{26\pi}(2\mathbf{a}_y - 3\mathbf{a}_x)$$

Then with $I = 8$ mA, we finally obtain $\mathbf{H}_a = \underline{-294\mathbf{a}_x + 196\mathbf{a}_y\ \mu\text{A/m}}$

b. Repeat if the filament is located at $x = -1$, $y = 2$: In this case the Biot-Savart integral becomes

$$\mathbf{H}_b = \int_{-\infty}^{\infty} \frac{Idz\,\mathbf{a}_z \times [(2+1)\mathbf{a}_x + (3-2)\mathbf{a}_y + (4-z)\mathbf{a}_z]}{4\pi(z^2 - 8z + 26)^{3/2}} = \int_{-\infty}^{\infty} \frac{Idz[3\mathbf{a}_y - \mathbf{a}_x]}{4\pi(z^2 - 8z + 26)^{3/2}}$$

Evaluating as before, we obtain with $I = 8$ mA:

$$\mathbf{H}_b = \frac{I}{4\pi}\left[\frac{2(2z-8)(3\mathbf{a}_y - \mathbf{a}_x)}{40(z^2 - 8z + 26)^{1/2}}\right]_{-\infty}^{\infty} = \frac{I}{20\pi}(3\mathbf{a}_y - \mathbf{a}_x) = \underline{-127\mathbf{a}_x + 382\mathbf{a}_y\ \mu\text{A/m}}$$

c. Find $\mathbf{H}$ if both filaments are present: This will be just the sum of the results of parts a and b, or

$$\mathbf{H}_T = \mathbf{H}_a + \mathbf{H}_b = \underline{-421\mathbf{a}_x + 578\mathbf{a}_y\ \mu\text{A/m}}$$

This problem can also be done (somewhat more simply) by using the known result for $\mathbf{H}$ from an infinitely-long wire in cylindrical components, and transforming to cartesian components. The Biot-Savart method was used here for the sake of illustration.

7.2. A filamentary conductor is formed into an equilateral triangle with sides of length ℓ carrying current I. Find the magnetic field intensity at the center of the triangle.

I will work this one from scratch, using the Biot-Savart law. Consider one side of the triangle, oriented along the z axis, with its end points at $z = \pm\ell/2$. Then consider a point, x_0, on the x axis, which would correspond to the center of the triangle, and at which we want to find $\mathbf{H}$ associated with the wire segment. We thus have $Id\mathbf{L} = Idz\,\mathbf{a}_z$, $R = \sqrt{z^2 + x_0^2}$, and $\mathbf{a}_R = [x_0\,\mathbf{a}_x - z\,\mathbf{a}_z]/R$. The differential magnetic field at x_0 is now

$$d\mathbf{H} = \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \frac{Idz\,\mathbf{a}_z \times (x_0\mathbf{a}_x - z\,\mathbf{a}_z)}{4\pi(x_0^2 + z^2)^{3/2}} = \frac{I\,dz\,x_0\,\mathbf{a}_y}{4\pi(x_0^2 + z^2)^{3/2}}$$

where $\mathbf{a}_y$ would be normal to the plane of the triangle. The magnetic field at x_0 is then

$$\mathbf{H} = \int_{-\ell/2}^{\ell/2} \frac{I\,dz\,x_0\,\mathbf{a}_y}{4\pi(x_0^2 + z^2)^{3/2}} = \left.\frac{I\,z\,\mathbf{a}_y}{4\pi x_0\sqrt{x_0^2 + z^2}}\right|_{-\ell/2}^{\ell/2} = \frac{I\ell\,\mathbf{a}_y}{2\pi x_0\sqrt{\ell^2 + 4x_0^2}}$$

7.2. (continued). Now, x_0 lies at the center of the equilateral triangle, and from the geometry of the triangle, we find that $x_0 = (\ell/2)\tan(30°) = \ell/(2\sqrt{3})$. Substituting this result into the just-found expression for $\mathbf{H}$ leads to $\mathbf{H} = 3I/(2\pi\ell)\,\mathbf{a}_y$. The contributions from the other two sides of the triangle effectively multiply the above result by three. The final answer is therefore $\underline{\mathbf{H}_{net} = 9I/(2\pi\ell)\,\mathbf{a}_y \text{ A/m}}$. It is also possible to work this problem (somewhat more easily) by using Eq. (9), applied to the triangle geometry.

7.3. Two semi-infinite filaments on the z axis lie in the regions $-\infty < z < -a$ (note typographical error in problem statement) and $a < z < \infty$. Each carries a current I in the $\mathbf{a}_z$ direction.

a) Calculate $\mathbf{H}$ as a function of ρ and ϕ at $z = 0$: One way to do this is to use the field from an infinite line and subtract from it that portion of the field that would arise from the current segment at $-a < z < a$, found from the Biot-Savart law. Thus,

$$\mathbf{H} = \frac{I}{2\pi\rho}\,\mathbf{a}_\phi - \int_{-a}^{a} \frac{I\,dz\,\mathbf{a}_z \times [\rho\,\mathbf{a}_\rho - z\,\mathbf{a}_z]}{4\pi[\rho^2 + z^2]^{3/2}}$$

The integral part simplifies and is evaluated:

$$\int_{-a}^{a} \frac{I\,dz\,\rho\,\mathbf{a}_\phi}{4\pi[\rho^2 + z^2]^{3/2}} = \frac{I\rho}{4\pi}\,\mathbf{a}_\phi\,\frac{z}{\rho^2\sqrt{\rho^2 + z^2}}\bigg|_{-a}^{a} = \frac{Ia}{2\pi\rho\sqrt{\rho^2 + a^2}}\,\mathbf{a}_\phi$$

Finally,

$$\mathbf{H} = \frac{I}{2\pi\rho}\left[1 - \frac{a}{\sqrt{\rho^2 + a^2}}\right]\mathbf{a}_\phi \;\; \text{A/m}$$

b) What value of a will cause the magnitude of $\mathbf{H}$ at $\rho = 1$, $z = 0$, to be one-half the value obtained for an infinite filament? We require

$$\left[1 - \frac{a}{\sqrt{\rho^2 + a^2}}\right]_{\rho=1} = \frac{1}{2} \;\;\Rightarrow\;\; \frac{a}{\sqrt{1 + a^2}} = \frac{1}{2} \;\;\Rightarrow\;\; a = \underline{1/\sqrt{3}}$$

7.4. Two circular current loops are centered on the z axis at $z = \pm h$. Each loop has radius a and carries current I in the $\mathbf{a}_\phi$ direction.

a) Find $\mathbf{H}$ on the z axis over the range $-h < z < h$: As a first step, we find the magnetic field on the z axis arising from a current loop of radius a, centered at the origin in the plane $z = 0$. It carries a current I in the $\mathbf{a}_\phi$ direction. Using the Biot-Savart law, we have $Id\mathbf{L} = Iad\phi\,\mathbf{a}_\phi$, $R = \sqrt{a^2 + z^2}$, and $\mathbf{a}_R = (z\mathbf{a}_z - a\mathbf{a}_\rho)/\sqrt{a^2 + z^2}$. The field on the z axis is then

$$\mathbf{H} = \int_0^{2\pi} \frac{Iad\phi\,\mathbf{a}_\phi \times (z\mathbf{a}_z - a\mathbf{a}_\rho)}{4\pi(a^2 + z^2)^{3/2}} = \int_0^{2\pi} \frac{Ia^2 d\phi\,\mathbf{a}_z}{4\pi(a^2 + z^2)^{3/2}} = \frac{a^2 I}{2(a^2 + z^2)^{3/2}}\,\mathbf{a}_z \text{ A/m}$$

In obtaining this result, the term involving $\mathbf{a}_\phi \times z\mathbf{a}_z = z\mathbf{a}_\rho$ has integrated to zero, when taken over the range $0 < \phi < 2\pi$. Substitute $\mathbf{a}_\rho = \mathbf{a}_x\cos\phi + \mathbf{a}_y\sin\phi$ to show this.

We now have two loops, displaced from the x-y plane to $z = \pm h$. The field is now the superposition of the two loop fields, which we can construct using displaced versions of the $\mathbf{H}$ field we just found:

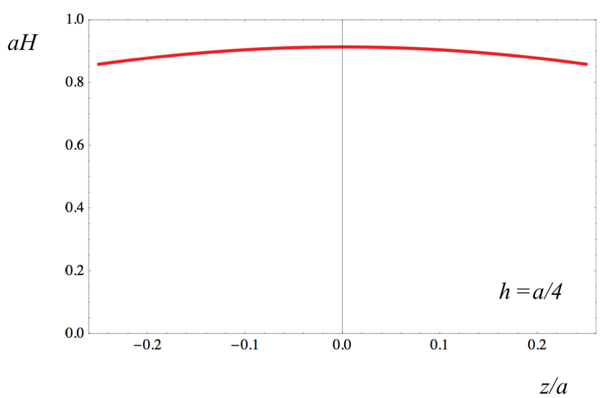
$$\mathbf{H} = \frac{a^2 I}{2}\left[\frac{1}{[(z-h)^2 + a^2]^{3/2}} + \frac{1}{[(z+h)^2 + a^2]^{3/2}}\right]\mathbf{a}_z \;\; \text{A/m}$$

7.4 (continued) We can rewrite this in terms of normalized distances, z/a, and h/a:

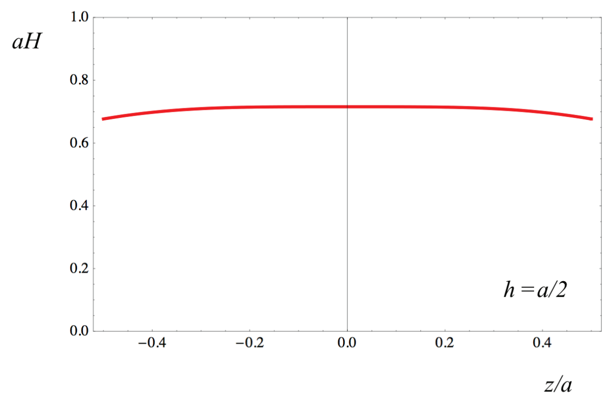
$$a\mathbf{H} = \frac{I}{2}\left[\left(\left(\frac{z}{a} - \frac{h}{a}\right)^2 + 1\right)^{-3/2} + \left(\left(\frac{z}{a} + \frac{h}{a}\right)^2 + 1\right)^{-3/2}\right]\mathbf{a}_z \text{ A}$$

Take $I = 1$ A and plot $|\mathbf{H}|$ as a function of z/a if:

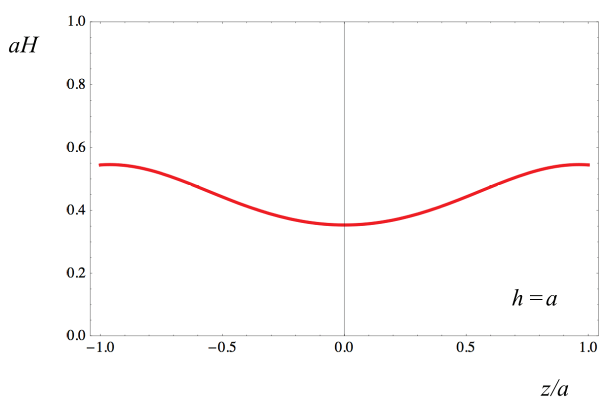
b) $h = a/4$,



c) $h = a/2$:



d) $h = a$.



Which choice for h gives the most uniform field? From the results, $h = a/2$ is evidently the best. This is the Helmholtz coil configuration – in which the spacing is equal to the coil radius.

7.5. The parallel filamentary conductors shown in Fig. 8.21 lie in free space. Plot $|\mathbf{H}|$ versus y, $-4 < y < 4$, along the line $x = 0$, $z = 2$: We need an expression for $\mathbf{H}$ in cartesian coordinates. We can start with the known $\mathbf{H}$ in cylindrical for an infinite filament along the z axis: $\mathbf{H} = I/(2\pi\rho)\,\mathbf{a}_\phi$, which we transform to cartesian to obtain:

$$\mathbf{H} = \frac{-Iy}{2\pi(x^2+y^2)}\,\mathbf{a}_x + \frac{Ix}{2\pi(x^2+y^2)}\,\mathbf{a}_y$$

If we now rotate the filament so that it lies along the x axis, with current flowing in positive x, we obtain the field from the above expression by replacing x with y and y with z:

$$\mathbf{H} = \frac{-Iz}{2\pi(y^2+z^2)}\,\mathbf{a}_y + \frac{Iy}{2\pi(y^2+z^2)}\,\mathbf{a}_z$$

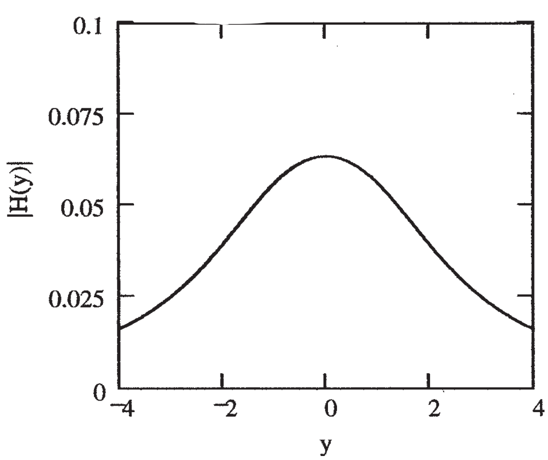
Now, with two filaments, displaced from the x axis to lie at $y = \pm 1$, and with the current directions as shown in the figure, we use the previous expression to write

$$\mathbf{H} = \left[\frac{Iz}{2\pi[(y+1)^2+z^2]} - \frac{Iz}{2\pi[(y-1)^2+z^2]}\right]\mathbf{a}_y + \left[\frac{I(y-1)}{2\pi[(y-1)^2+z^2]} - \frac{I(y+1)}{2\pi[(y+1)^2+z^2]}\right]\mathbf{a}_z$$

We now evaluate this at $z = 2$, and find the magnitude ($\sqrt{\mathbf{H}\cdot\mathbf{H}}$), resulting in

$$|\mathbf{H}| = \frac{I}{2\pi}\left[\left(\frac{2}{y^2+2y+5} - \frac{2}{y^2-2y+5}\right)^2 + \left(\frac{(y-1)}{y^2-2y+5} - \frac{(y+1)}{y^2+2y+5}\right)^2\right]^{1/2}$$

This function is plotted below



7.6. A disk of radius a lies in the xy plane, with the z axis through its center. Surface charge of uniform density ρ_s lies on the disk, which rotates about the z axis at angular velocity Ω rad/s. Find $\mathbf{H}$ at any point on the z axis.

We use the Biot-Savart law in the form of Eq. (6), with the following parameters: $\mathbf{K} = \rho_s\mathbf{v} = \rho_s\rho\Omega\,\mathbf{a}_\phi$, $R = \sqrt{z^2+\rho^2}$, and $\mathbf{a}_R = (z\,\mathbf{a}_z - \rho\,\mathbf{a}_\rho)/R$. The differential field at point z is

$$d\mathbf{H} = \frac{\mathbf{K}da \times \mathbf{a}_R}{4\pi R^2} = \frac{\rho_s\rho\,\Omega\,\mathbf{a}_\phi \times (z\,\mathbf{a}_z - \rho\,\mathbf{a}_\rho)}{4\pi(z^2+\rho^2)^{3/2}}\,\rho\,d\rho\,d\phi = \frac{\rho_s\rho\,\Omega\,(z\,\mathbf{a}_\rho + \rho\,\mathbf{a}_z)}{4\pi(z^2+\rho^2)^{3/2}}\,\rho\,d\rho\,d\phi$$

7.6. (continued). On integrating the above over ϕ around a complete circle, the $\mathbf{a}_\rho$ components cancel from symmetry, leaving us with

$$\mathbf{H}(z) = \int_0^{2\pi}\int_0^a \frac{\rho_s \rho\, \Omega\, \rho\, \mathbf{a}_z}{4\pi(z^2+\rho^2)^{3/2}}\,\rho\, d\rho\, d\phi = \int_0^a \frac{\rho_s\, \Omega\, \rho^3\, \mathbf{a}_z}{2(z^2+\rho^2)^{3/2}}\,d\rho$$

$$= \frac{\rho_s \Omega}{2}\left[\sqrt{z^2+\rho^2} + \frac{z^2}{\sqrt{z^2+\rho^2}}\right]_0^a \mathbf{a}_z = \frac{\rho_s\Omega}{2z}\left[\frac{a^2+2z^2\left(1-\sqrt{1+a^2/z^2}\right)}{\sqrt{1+a^2/z^2}}\right]\mathbf{a}_z \text{ A/m}$$

7.7. A filamentary conductor carrying current I in the $\mathbf{a}_z$ direction extends along the entire negative z axis. At $z = 0$ it connects to a copper sheet that fills the $x > 0$, $y > 0$ quadrant of the xy plane.

a) Set up the Biot-Savart law and find $\mathbf{H}$ everywhere on the z axis (Hint: express $\mathbf{a}_\phi$ in terms of $\mathbf{a}_x$ and $\mathbf{a}_y$ and angle ϕ in the integral): First, the contribution to the field at z from the current on the negative z axis will be zero, because the cross product, $Id\mathbf{L} \times \mathbf{a}_R = 0$ for all current elements on the z axis. This leaves the contribution of the current sheet in the first quadrant. On exiting the origin, current fans out over the first quadrant in the $\mathbf{a}_\rho$ direction and is uniform at a given radius. The surface current density can therefore be written as $\mathbf{K}(\rho) = 2I/(\pi\rho)\,\mathbf{a}_\rho$ A/m^2 over the region $(0 < \phi < \pi/2)$. The Biot-Savart law applicable to surface current is written as

$$\mathbf{H} = \int_s \frac{\mathbf{K}\times\mathbf{a}_R}{4\pi R^2}\,dA$$

where $R = \sqrt{z^2+\rho^2}$ and $\mathbf{a}_R = (z\mathbf{a}_z - \rho\mathbf{a}_\rho)/\sqrt{z^2+\rho^2}$ Substituting these and integrating over the first quadrant yields the setup:

$$\mathbf{H} = \int_0^{\pi/2}\int_0^\infty \frac{2I\,\mathbf{a}_\rho \times (z\mathbf{a}_z - \rho\mathbf{a}_\rho)}{4\pi^2\rho(z^2+\rho^2)^{3/2}}\,\rho\, d\rho\, d\phi = \int_0^{\pi/2}\int_0^\infty \frac{-Iz\,\mathbf{a}_\phi}{2\pi^2(z^2+\rho^2)^{3/2}}\,d\rho\, d\phi$$

Following the hint, we now substitute $\mathbf{a}_\phi = \mathbf{a}_y\cos\phi - \mathbf{a}_x\sin\phi$, and write:

$$\mathbf{H} = \frac{-Iz}{2\pi^2}\int_0^{\pi/2}\int_0^\infty \frac{(\mathbf{a}_y\cos\phi - \mathbf{a}_x\sin\phi)}{(z^2+\rho^2)^{3/2}}\,d\rho\, d\phi = \frac{Iz}{2\pi^2}(\mathbf{a}_x - \mathbf{a}_y)\int_0^\infty \frac{d\rho}{(z^2+\rho^2)^{3/2}}$$

$$= \frac{Iz}{2\pi^2}(\mathbf{a}_x - \mathbf{a}_y)\left.\frac{\rho}{z^2\sqrt{z^2+\rho^2}}\right|_0^\infty = \underline{\frac{I}{2\pi^2 z}(\mathbf{a}_x - \mathbf{a}_y)} \text{ A/m}$$

b) repeat part a, but with the copper sheet occupying the *entire* xy plane. In this case, the ϕ limits are $(0 < \phi < 2\pi)$. The $\cos\phi$ and $\sin\phi$ terms would then integrate to zero, so the answer is just that: $\mathbf{H} = \underline{0}$.

7.8. For the finite-length current element on the z axis, as shown in Fig. 8.5, use the Biot-Savart law to derive Eq. (9) of Sec. 8.1: The Biot-Savart law reads:

$$\mathbf{H} = \int_{z_1}^{z_2} \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \int_{\rho \tan\alpha_1}^{\rho\tan\alpha_2} \frac{Idz\mathbf{a}_z \times (\rho\mathbf{a}_\rho - z\mathbf{a}_z)}{4\pi(\rho^2+z^2)^{3/2}} = \int_{\rho\tan\alpha_1}^{\rho\tan\alpha_2} \frac{I\rho\mathbf{a}_\phi\, dz}{4\pi(\rho^2+z^2)^{3/2}}$$

The integral is evaluated (using tables) and gives the desired result:

$$\mathbf{H} = \frac{Iz\mathbf{a}_\phi}{4\pi\rho\sqrt{\rho^2+z^2}}\bigg|_{\rho\tan\alpha_1}^{\rho\tan\alpha_2} = \frac{I}{4\pi\rho}\left[\frac{\tan\alpha_2}{\sqrt{1+\tan^2\alpha_2}} - \frac{\tan\alpha_1}{\sqrt{1+\tan^2\alpha_1}}\right]\mathbf{a}_\phi$$

$$= \frac{I}{4\pi\rho}(\sin\alpha_2 - \sin\alpha_1)\mathbf{a}_\phi$$

7.9. A current sheet $\mathbf{K} = 8\mathbf{a}_x$ A/m flows in the region $-2 < y < 2$ in the plane $z = 0$. Calculate H at $P(0,0,3)$: Using the Biot-Savart law, we write

$$\mathbf{H}_P = \int\int \frac{\mathbf{K}\times\mathbf{a}_R\, dx\, dy}{4\pi R^2} = \int_{-2}^{2}\int_{-\infty}^{\infty} \frac{8\mathbf{a}_x \times (-x\mathbf{a}_x - y\mathbf{a}_y + 3\mathbf{a}_z)}{4\pi(x^2+y^2+9)^{3/2}}\, dx\, dy$$

Taking the cross product gives:

$$\mathbf{H}_P = \int_{-2}^{2}\int_{-\infty}^{\infty} \frac{8(-y\mathbf{a}_z - 3\mathbf{a}_y)\, dx\, dy}{4\pi(x^2+y^2+9)^{3/2}}$$

We note that the z component is anti-symmetric in y about the origin (odd parity). Since the limits are symmetric, the integral of the z component over y is zero. We are left with

$$\mathbf{H}_P = \int_{-2}^{2}\int_{-\infty}^{\infty} \frac{-24\,\mathbf{a}_y\, dx\, dy}{4\pi(x^2+y^2+9)^{3/2}} = -\frac{6}{\pi}\mathbf{a}_y \int_{-2}^{2} \frac{x}{(y^2+9)\sqrt{x^2+y^2+9}}\bigg|_{-\infty}^{\infty} dy$$

$$= -\frac{6}{\pi}\mathbf{a}_y\int_{-2}^{2}\frac{2}{y^2+9}\,dy = -\frac{12}{\pi}\mathbf{a}_y\,\frac{1}{3}\tan^{-1}\left(\frac{y}{3}\right)\bigg|_{-2}^{2} = -\frac{4}{\pi}(2)(0.59)\,\mathbf{a}_y = \underline{-1.50\,\mathbf{a}_y \text{ A/m}}$$

7.10. A hollow spherical conducting shell of radius a has filamentary connections made at the top ($r = a$, $\theta = 0$) and bottom ($r = a$, $\theta = \pi$). A direct current I flows down the upper filament, down the spherical surface, and out the lower filament. Find $\mathbf{H}$ in spherical coordinates (a) inside and (b) outside the sphere.

Applying Ampere's circuital law, we use a circular contour, centered on the z axis, and find that within the sphere, no current is enclosed, and so $\underline{\mathbf{H} = 0 \text{ when } r < a}$. The same contour drawn outside the sphere at any z position will always enclose I amps, flowing in the negative z direction, and so

$$\mathbf{H} = -\frac{I}{2\pi\rho}\,\mathbf{a}_\phi = -\frac{I}{2\pi r\sin\theta}\,\mathbf{a}_\phi \text{ A/m} \quad (r > a)$$

7.11. An infinite filament on the z axis carries 20π mA in the $\mathbf{a}_z$ direction. Three uniform cylindrical current sheets are also present: 400 mA/m at $\rho = 1$ cm, -250 mA/m at $\rho = 2$ cm, and -300 mA/m at $\rho = 3$ cm. Calculate H_ϕ at $\rho = 0.5$, 1.5, 2.5, and 3.5 cm: We find H_ϕ at each of the required radii by applying Ampere's circuital law to circular paths of those radii; the paths are centered on the z axis. So, at $\rho_1 = 0.5$ cm:

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho_1 H_{\phi 1} = I_{encl} = 20\pi \times 10^{-3} \text{ A}$$

Thus

$$H_{\phi 1} = \frac{10 \times 10^{-3}}{\rho_1} = \frac{10 \times 10^{-3}}{0.5 \times 10^{-2}} = \underline{2.0 \text{ A/m}}$$

At $\rho = \rho_2 = 1.5$ cm, we enclose the first of the current cylinders at $\rho = 1$ cm. Ampere's law becomes:

$$2\pi\rho_2 H_{\phi 2} = 20\pi + 2\pi(10^{-2})(400) \text{ mA} \quad \Rightarrow \quad H_{\phi 2} = \frac{10 + 4.00}{1.5 \times 10^{-2}} = \underline{933 \text{ mA/m}}$$

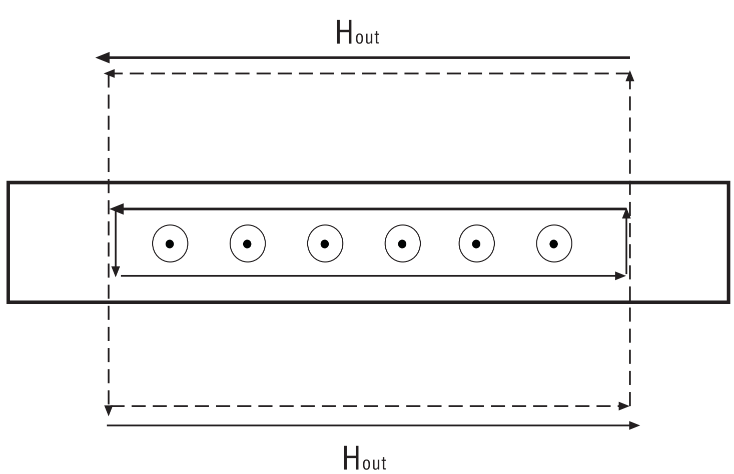
Following this method, at 2.5 cm:

$$H_{\phi 3} = \frac{10 + 4.00 - (2 \times 10^{-2})(250)}{2.5 \times 10^{-2}} = \underline{360 \text{ mA/m}}$$

and at 3.5 cm,

$$H_{\phi 4} = \frac{10 + 4.00 - 5.00 - (3 \times 10^{-2})(300)}{3.5 \times 10^{-2}} = \underline{0}$$

7.12. In Fig. 8.22, let the regions $0 < z < 0.3$ m and $0.7 < z < 1.0$ m be conducting slabs carrying uniform current densities of 10 A/m^2 in opposite directions as shown. The problem asks you to find $\mathbf{H}$ at various positions. Before continuing, we need to know how to find $\mathbf{H}$ for this type of current configuration. The sketch below shows one of the slabs (of thickness D) oriented with the current coming out of the page. The problem statement implies that both slabs are of infinite length and width. To find the magnetic field *inside* a slab, we apply Ampere's circuital law to the rectangular path of height d and width w, as shown, since by symmetry, $\mathbf{H}$ should be oriented horizontally. For example, if the sketch below shows the upper slab in Fig. 8.22, current will be in the positive y direction. Thus $\mathbf{H}$ will be in the positive x direction above the slab midpoint, and will be in the negative x direction below the midpoint.



7.12. (continued). In taking the line integral in Ampere's law, the two vertical path segments will cancel each other. Ampere's circuital law for the interior loop becomes

$$\oint \mathbf{H}\cdot d\mathbf{L} = 2H_{in}\times w = I_{encl} = J\times w\times d \quad\Rightarrow\quad H_{in} = \frac{Jd}{2}$$

The field outside the slab is found similarly, but with the enclosed current now bounded by the slab thickness, rather than the integration path height:

$$2H_{out}\times w = J\times w\times D \quad\Rightarrow\quad H_{out} = \frac{JD}{2}$$

where H_{out} is directed from right to left below the slab and from left to right above the slab (right hand rule). Reverse the current, and the fields, of course, reverse direction. We are now in a position to solve the problem. Find **H** at:

a) $z = -0.2$m: Here the fields from the top and bottom slabs (carrying opposite currents) will cancel, and so $\mathbf{H} = \underline{0}$.

b) $z = 0.2$m. This point lies within the lower slab above its midpoint. Thus the field will be oriented in the negative x direction. Referring to Fig. 8.22 and to the sketch on the previous page, we find that $d = 0.1$. The total field will be this field plus the contribution from the upper slab current:

$$\mathbf{H} = \underbrace{\frac{-10(0.1)}{2}\mathbf{a}_x}_{\text{lower slab}} - \underbrace{\frac{10(0.3)}{2}\mathbf{a}_x}_{\text{upper slab}} = \underline{-2\mathbf{a}_x\ \text{A/m}}$$

c) $z = 0.4$m: Here the fields from both slabs will add constructively in the negative x direction:

$$\mathbf{H} = -2\frac{10(0.3)}{2}\mathbf{a}_x = \underline{-3\mathbf{a}_x\ \text{A/m}}$$

d) $z = 0.75$m: This is in the interior of the upper slab, whose midpoint lies at $z = 0.85$. Therefore $d = 0.2$. Since 0.75 lies below the midpoint, magnetic field from the upper slab will lie in the negative x direction. The field from the lower slab will be negative x-directed as well, leading to:

$$\mathbf{H} = \underbrace{\frac{-10(0.2)}{2}\mathbf{a}_x}_{\text{upper slab}} - \underbrace{\frac{10(0.3)}{2}\mathbf{a}_x}_{\text{lower slab}} = \underline{-2.5\mathbf{a}_x\ \text{A/m}}$$

e) $z = 1.2$m: This point lies above both slabs, where again fields cancel completely: Thus $\mathbf{H} = \underline{0}$.

7.13. A hollow cylindrical shell of radius a is centered on the z axis and carries a uniform surface current density of $K_a\mathbf{a}_\phi$.

a) Show that H is not a function of ϕ or z: Consider this situation as illustrated in Fig. 8.11. There (sec. 8.2) it was stated that the field will be entirely z-directed. We can see this by applying Ampere's circuital law to a closed loop path whose orientation we choose such that current is enclosed by the path. The only way to enclose current is to set up the loop (which we choose to be rectangular) such that it is oriented with two parallel opposing segments lying in the z direction; one of these lies inside the cylinder, the other outside. The other two parallel segments lie in the ρ direction. The loop is now cut by the current sheet, and if we assume a length of the loop in z of d, then the enclosed current will be given by Kd A. There will be no ϕ variation in the field because where we position the loop around the circumference of the cylinder does not affect the result of Ampere's law. If we assume an infinite cylinder length, there will be no z dependence in the field, since as we lengthen the loop in the z direction, the path length (over which the integral is taken) increases, but then so does the enclosed current – by the same factor. Thus H would not change with z. There would also be no change if the loop was simply moved along the z direction.

b) Show that H_ϕ and H_ρ are everywhere zero. First, if H_ϕ were to exist, then we should be able to find a closed loop path *that encloses current*, in which all or or portion of the path lies in the ϕ direction. This we cannot do, and so H_ϕ must be zero. Another argument is that when applying the Biot-Savart law, there is no current element that would produce a ϕ component. Again, using the Biot-Savart law, we note that radial field components will be produced by individual current elements, but such components will cancel from two elements that lie at symmetric distances in z on either side of the observation point.

c) Show that $H_z = 0$ for $\rho > a$: Suppose the rectangular loop was drawn such that the outside z-directed segment is moved further and further away from the cylinder. We would expect H_z outside to decrease (as the Biot-Savart law would imply) but the same amount of current is always enclosed no matter how far away the outer segment is. We therefore must conclude that the field outside is zero.

d) Show that $H_z = K_a$ for $\rho < a$: With our rectangular path set up as in part a, we have no path integral contributions from the two radial segments, and no contribution from the outside z-directed segment. Therefore, Ampere's circuital law would state that

$$\oint \mathbf{H} \cdot d\mathbf{L} = H_z d = I_{encl} = K_a d \quad \Rightarrow \quad H_z = K_a$$

where d is the length of the loop in the z direction.

e) A second shell, $\rho = b$, carries a current $K_b\mathbf{a}_\phi$. Find $\mathbf{H}$ everywhere: For $\rho < a$ we would have both cylinders contributing, or $H_z(\rho < a) = K_a + K_b$. Between the cylinders, we are outside the inner one, so its field will not contribute. Thus $H_z(a < \rho < b) = K_b$. Outside $(\rho > b)$ the field will be zero.

7.14. A toroid having a cross section of rectangular shape is defined by the following surfaces: the cylinders $\rho = 2$ and $\rho = 3$ cm, and the planes $z = 1$ and $z = 2.5$ cm. The toroid carries a surface current density of $-50\mathbf{a}_z$ A/m on the surface $\rho = 3$ cm. Find $\mathbf{H}$ at the point $P(\rho, \phi, z)$: The construction is similar to that of the toroid of round cross section as done on p.239. Again, magnetic field exists only inside the toroid cross section, and is given by

$$\mathbf{H} = \frac{I_{encl}}{2\pi\rho}\mathbf{a}_\phi \quad (2 < \rho < 3) \text{ cm}, \quad (1 < z < 2.5) \text{ cm}$$

where I_{encl} is found from the given current density: On the outer radius, the current is

$$I_{outer} = -50(2\pi \times 3 \times 10^{-2}) = -3\pi \text{ A}$$

This current is directed along negative z, which means that the current on the *inner* radius ($\rho = 2$) is directed along *positive* z. Inner and outer currents have the same magnitude. It is the inner current that is enclosed by the circular integration path in $\mathbf{a}_\phi$ within the toroid that is used in Ampere's law. So $I_{encl} = +3\pi$ A. We can now proceed with what is requested:

a) $P_A(1.5\text{cm}, 0, 2\text{cm})$: The radius, $\rho = 1.5$ cm, lies outside the cross section, and so $\mathbf{H}_A = \underline{0}$.

b) $P_B(2.1\text{cm}, 0, 2\text{cm})$: This point does lie inside the cross section, and the ϕ and z values do not matter. We find

$$\mathbf{H}_B = \frac{I_{encl}}{2\pi\rho}\mathbf{a}_\phi = \frac{3\mathbf{a}_\phi}{2(2.1 \times 10^{-2})} = \underline{71.4\,\mathbf{a}_\phi \text{ A/m}}$$

c) $P_C(2.7\text{cm}, \pi/2, 2\text{cm})$: again, ϕ and z values make no difference, so

$$\mathbf{H}_C = \frac{3\mathbf{a}_\phi}{2(2.7 \times 10^{-2})} = \underline{55.6\,\mathbf{a}_\phi \text{ A/m}}$$

d) $P_D(3.5\text{cm}, \pi/2, 2\text{cm})$. This point lies outside the cross section, and so $\mathbf{H}_D = \underline{0}$.

7.15. Assume that there is a region with cylindrical symmetry in which the conductivity is given by $\sigma = 1.5e^{-150\rho}$ kS/m. An electric field of $30\,\mathbf{a}_z$ V/m is present.

a) Find $\mathbf{J}$: Use

$$\mathbf{J} = \sigma\mathbf{E} = \underline{45e^{-150\rho}\,\mathbf{a}_z \;\; \text{kA/m}^2}$$

b) Find the total current crossing the surface $\rho < \rho_0$, $z = 0$, all ϕ:

$$I = \int\!\!\int \mathbf{J} \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\rho_0} 45e^{-150\rho}\,\rho\,d\rho\,d\phi = \frac{2\pi(45)}{(150)^2}e^{-150\rho}\left[-150\rho - 1\right]\Big|_0^{\rho_0} \text{ kA}$$
$$= \underline{12.6\left[1 - (1 + 150\rho_0)e^{-150\rho_0}\right] \text{ A}}$$

c) Make use of Ampere's circuital law to find $\mathbf{H}$: Symmetry suggests that $\mathbf{H}$ will be ϕ-directed only, and so we consider a circular path of integration, centered on and perpendicular to the z axis. Ampere's law becomes: $2\pi\rho H_\phi = I_{encl}$, where I_{encl} is the current found in part *b*, except with ρ_0 replaced by the variable, ρ. We obtain

$$H_\phi = \underline{\frac{2.00}{\rho}\left[1 - (1 + 150\rho)e^{-150\rho}\right] \text{ A/m}}$$

7.16. A current filament carrying I in the $-\mathbf{a}_z$ direction lies along the entire positive z axis. At the origin, it connects to a conducting sheet that forms the xy plane.

a) Find $\mathbf{K}$ in the conducting sheet: The current fans outward radially with uniform surface current density at a fixed radius. The current density at radius ρ will be the total current, I, divided by the circumference at radius ρ:

$$\mathbf{K} = \frac{I}{2\pi\rho}\,\mathbf{a}_\rho \text{ A/m}$$

b) Use Ampere's circuital law to find $\mathbf{H}$ everywhere for $z > 0$: Circular lines of $\mathbf{H}$ are expected, centered on the z axis – in the $-\mathbf{a}_\phi$ direction. Ampere's law is set up by considering a circular path integral taken around the wire at fixed z. The enclosed current is that which passes through *any* surface that is bounded by the line integration path:

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho H_\phi = I_{encl}$$

If the surface is that of the disk whose perimeter is the integration path, then the enclosed current is just I, and the magnetic field becomes

$$H_\phi = -\frac{I}{2\pi\rho} \quad \Rightarrow \quad \mathbf{H} = -\frac{I}{2\pi\rho}\,\mathbf{a}_\phi \text{ A/m}$$

But the disk surface can be "stretched" so that it forms a balloon shape. Suppose the "balloon" is a right circular cylinder, with its open top circumference at the path integral location. The cylinder extends downward, intersecting the surface current in the x-y plane, with the bottom of the cylinder below the x-y plane. Now, the path integral is unchanged from before, and the enclosed current is the radial current in the x-y plane that passes through the side of the cylinder. This current will be $I = 2\pi\rho[I/(2\pi\rho)] = I$, as before. So the answer given above for $\mathbf{H}$ applies to anywhere in the region $z > 0$.

c) Find $\mathbf{H}$ for $z < 0$: Consider the same cylinder as in part *b*, except take the path integral of $\mathbf{H}$ around the *bottom* circumference (below the x-y plane). The enclosed current now consists of the filament current that enters through the top, plus the radial current that exits though the side. The two currents are equal magnitude but opposite in sign. Therefore, the net enclosed current is *zero*, and thus $\underline{\mathbf{H} = 0}$ $(z < 0)$.

7.17. A current filament on the z axis carries a current of 7 mA in the $\mathbf{a}_z$ direction, and current sheets of $0.5\,\mathbf{a}_z$ A/m and $-0.2\,\mathbf{a}_z$ A/m are located at $\rho = 1$ cm and $\rho = 0.5$ cm, respectively. Calculate $\mathbf{H}$ at:

a) $\rho = 0.5$ cm: Here, we are either just inside or just outside the first current sheet, so both we will calculate $\mathbf{H}$ for both cases. Just inside, applying Ampere's circuital law to a circular path centered on the z axis produces:

$$2\pi\rho H_\phi = 7\times 10^{-3} \quad\Rightarrow\quad \mathbf{H}(\text{just inside}) = \frac{7\times 10^{-3}}{2\pi(0.5\times 10^{-2}}\mathbf{a}_\phi = \underline{2.2\times 10^{-1}\mathbf{a}_\phi \text{ A/m}}$$

Just outside the current sheet at .5 cm, Ampere's law becomes

$$2\pi\rho H_\phi = 7\times 10^{-3} - 2\pi(0.5\times 10^{-2})(0.2)$$
$$\Rightarrow\quad \mathbf{H}(\text{just outside}) = \frac{7.2\times 10^{-4}}{2\pi(0.5\times 10^{-2})}\mathbf{a}_\phi = \underline{2.3\times 10^{-2}\mathbf{a}_\phi \text{ A/m}}$$

b) $\rho = 1.5$ cm: Here, all three currents are enclosed, so Ampere's law becomes

$$2\pi(1.5\times 10^{-2})H_\phi = 7\times 10^{-3} - 6.28\times 10^{-3} + 2\pi(10^{-2})(0.5)$$
$$\Rightarrow\quad \mathbf{H}(\rho = 1.5) = \underline{3.4\times 10^{-1}\mathbf{a}_\phi \text{ A/m}}$$

c) $\rho = 4$ cm: Ampere's law as used in part *b* applies here, except we replace $\rho = 1.5$ cm with $\rho = 4$ cm on the left hand side. The result is $\mathbf{H}(\rho = 4) = \underline{1.3\times 10^{-1}\mathbf{a}_\phi \text{ A/m}}$.

d) What current sheet should be located at $\rho = 4$ cm so that $\mathbf{H} = 0$ for all $\rho > 4$ cm? We require that the total enclosed current be zero, and so the net current in the proposed cylinder at 4 cm must be negative the right hand side of the first equation in part *b*. This will be -3.2×10^{-2}, so that the surface current density at 4 cm must be

$$\mathbf{K} = \frac{-3.2\times 10^{-2}}{2\pi(4\times 10^{-2})}\mathbf{a}_z = \underline{-1.3\times 10^{-1}\,\mathbf{a}_z \text{ A/m}}$$

7.18. A wire of 3-mm radius is made up of an inner material ($0 < \rho < 2$ mm) for which $\sigma = 10^7$ S/m, and an outer material (2mm $< \rho <$ 3mm) for which $\sigma = 4\times 10^7$ S/m. If the wire carries a total current of 100 mA dc, determine $\mathbf{H}$ everywhere as a function of ρ.

Since the materials have different conductivities, the current densities within them will differ. Electric field, however is constant throughout. The current can be expressed as

$$I = \pi(.002)^2 J_1 + \pi[(.003)^2 - (.002)^2]J_2 = \pi\left[(.002)^2\sigma_1 + [(.003)^2 - (.002)^2]\sigma_2\right]E$$

Solve for E to obtain

$$E = \frac{0.1}{\pi[(4\times 10^{-6})(10^7) + (9\times 10^{-6} - 4\times 10^{-6})(4\times 10^7)]} = 1.33\times 10^{-4} \text{ V/m}$$

We next apply Ampere's circuital law to a circular path of radius ρ, where $\rho <$ 2mm:

$$2\pi\rho H_{\phi 1} = \pi\rho^2 J_1 = \pi\rho^2\sigma_1 E \quad\Rightarrow\quad H_{\phi 1} = \frac{\sigma_1 E\rho}{2} = \underline{663 \text{ A/m}}$$

7.18 (continued) . Next, for the region 2mm $< \rho <$ 3mm, Ampere's law becomes

$$2\pi\rho H_{\phi 2} = \pi[(4 \times 10^{-6})(10^7) + (\rho^2 - 4 \times 10^{-6})(4 \times 10^7)]E$$

$$\Rightarrow \quad H_{\phi 2} = 2.7 \times 10^3 \rho \; - \; \frac{8.0 \times 10^{-3}}{\rho} \text{ A/m}$$

Finally, for $\rho >$ 3mm, the field outside is that for a long wire:

$$H_{\phi 3} = \frac{I}{2\pi\rho} = \frac{0.1}{2\pi\rho} = \frac{1.6 \times 10^{-2}}{\rho} \text{ A/m}$$

7.19. In spherical coordinates, the surface of a solid conducting cone is described by $\theta = \pi/4$ and a conducting plane by $\theta = \pi/2$. Each carries a total current I. The current flows as a surface current radially inward on the plane to the vertex of the cone, and then flows radially-outward throughout the cross-section of the conical conductor.

a) Express the surface current density as a function of r: This will be the total current divided by the circumference of a circle of radius r in the plane, directed toward the origin:

$$\mathbf{K}(r) = -\frac{I}{2\pi r}\,\mathbf{a}_r \text{ A/m}^2 \quad (\theta = \pi/2)$$

b) Express the volume current density inside the cone as a function of r: This will be the total current divided by the area of the spherical cap subtending angle $\theta = \pi/4$:

$$\mathbf{J}(r) = I\left[\int_0^{2\pi}\int_0^{\pi/4} r^2 \sin\theta'\, d\theta'\, d\phi\right]^{-1} \mathbf{a}_r \;=\; \underline{\frac{I\,\mathbf{a}_r}{2\pi r^2(1-1/\sqrt{2})}} \text{ A/m}^2 \quad (0 < \theta < \pi/4)$$

c) Determine $\mathbf{H}$ as a function of r and θ in the region between the cone and the plane: From symmetry, we expect $\mathbf{H}$ to be ϕ-directed and uniform at constant r and θ. Ampere's circuital law can therefore be stated as:

$$\oint \mathbf{H}\cdot d\mathbf{L} = 2\pi r \sin\theta\, H_\phi = I \;\Rightarrow\; \mathbf{H} = \frac{I}{2\pi r \sin\theta}\,\mathbf{a}_\phi \text{ A/m} \quad (\pi/4 < \theta < \pi/2)$$

d) Determine $\mathbf{H}$ as a function of r and θ inside the cone: Again, ϕ-directed $\mathbf{H}$ is anticipated, so we apply Ampere's law in the following way:

$$\oint \mathbf{H}\cdot d\mathbf{L} = 2\pi r \sin\theta\, H_\phi = \int_s \mathbf{J}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\theta} \frac{I\,\mathbf{a}_r}{2\pi r^2(1-1/\sqrt{2})}\cdot \mathbf{a}_r\, r^2 \sin\theta'\, d\theta'\, d\phi$$

This becomes

$$2\pi r \sin\theta\, H_\phi = -2\pi \frac{I}{2\pi(1-1/\sqrt{2})}\cos\theta'\Big|_0^\theta$$

or

$$\mathbf{H} = \underline{\frac{I}{2\pi r(1-1/\sqrt{2})}\left[\frac{(1-\cos\theta)}{\sin\theta}\right]\mathbf{a}_\phi} \text{ A/m} \quad (0 < \theta < \pi/4)$$

As a test of this, note that the inside and outside fields (results of parts *c* and *d*) are equal at the cone surface ($\theta = \pi/4$) as they must be.

7.20. A solid conductor of circular cross-section with a radius of 5 mm has a conductivity that varies with radius. The conductor is 20 m long and there is a potential difference of 0.1 V dc between its two ends. Within the conductor, $\mathbf{H} = 10^5 \rho^2 \mathbf{a}_\phi$ A/m.

a) Find σ as a function of ρ: Start by finding $\mathbf{J}$ from $\mathbf{H}$ by taking the curl. With $\mathbf{H}$ ϕ-directed, and varying with radius only, the curl becomes:

$$\mathbf{J} = \nabla \times \mathbf{H} = \frac{1}{\rho}\frac{d}{d\rho}(\rho H_\phi)\,\mathbf{a}_z = \frac{1}{\rho}\frac{d}{d\rho}\left(10^5\rho^3\right)\mathbf{a}_z = 3 \times 10^5 \rho\,\mathbf{a}_z \text{ A/m}^2$$

Then $\mathbf{E} = 0.1/20 = 0.005\,\mathbf{a}_z$ V/m, which we then use with $\mathbf{J} = \sigma\mathbf{E}$ to find

$$\sigma = \frac{J}{E} = \frac{3 \times 10^5 \rho}{0.005} = \underline{6 \times 10^7 \rho \text{ S/m}}$$

b) What is the resistance between the two ends? The current in the wire is

$$I = \int_s \mathbf{J} \cdot d\mathbf{S} = 2\pi \int_0^a (3 \times 10^5 \rho)\,\rho\,d\rho = 6\pi \times 10^5 \left(\frac{1}{3}a^3\right) = 2\pi \times 10^5 (0.005)^3 = 0.079 \text{ A}$$

Finally, $R = V_0/I = 0.1/0.079 = \underline{1.3\ \Omega}$

7.21. A cylindrical wire of radius a is oriented with the z axis down its center line. The wire carries a non-uniform current down its length of density $\mathbf{J} = b\rho\,\mathbf{a}_z$ A/m^2, where b is a constant.

a) What total current flows in the wire? We integrate the current density over the wire cross-section:

$$I_{tot} = \int_s \mathbf{J}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^a b\rho\,\mathbf{a}_z\cdot\mathbf{a}_z\,\rho\,d\rho\,d\phi = \underline{\frac{2\pi ba^3}{3}}\ \text{A}$$

b) find $\mathbf{H}_{in}$ $(0<\rho<a)$, as a function of ρ: From the symmetry, ϕ-directed $\mathbf{H}$ $(= H_\phi\,\mathbf{a}_\phi)$ is expected in the interior; this will be constant at a fixed radius, ρ. Apply Ampere's circuital law to a circular path of radius ρ inside:

$$\oint \mathbf{H}_{in}\cdot d\mathbf{L} = 2\pi\rho H_{\phi,in} = \int_s \mathbf{J}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\rho} b\rho'\,\mathbf{a}_z\cdot\mathbf{a}_z\,\rho'\,d\rho'\,d\phi = \frac{2\pi b\rho^3}{3}$$

So that

$$\mathbf{H}_{in} = \underline{\frac{b\rho^2}{3}\,\mathbf{a}_\phi}\ \ \text{A/m}\ \ (0<\rho<a)$$

c) find $\mathbf{H}_{out}(\rho>a)$, as a function of ρ Same as part b, except the path integral is taken at a radius outside the wire:

$$\oint \mathbf{H}_{out}\cdot d\mathbf{L} = 2\pi\rho H_{\phi,out} = \int_s \mathbf{J}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{a} b\rho\,\mathbf{a}_z\cdot\mathbf{a}_z\,\rho\,d\rho\,d\phi = \frac{2\pi ba^3}{3}$$

So that

$$\mathbf{H}_{out} = \underline{\frac{ba^3}{3\rho}\,\mathbf{a}_\phi}\ \ \text{A/m}\ \ (\rho>a)$$

d) verify your results of parts b and c by using $\nabla\times\mathbf{H} = \mathbf{J}$: With a ϕ component of $\mathbf{H}$ only, varying only with ρ, the curl in cylindrical coordinates reduces to

$$\mathbf{J} = \nabla\times\mathbf{H} = \frac{1}{\rho}\frac{d}{d\rho}(\rho H_\phi)\,\mathbf{a}_z$$

Apply this to the inside field to get

$$\mathbf{J}_{in} = \frac{1}{\rho}\frac{d}{d\rho}\left(\frac{b\rho^3}{3}\right)\mathbf{a}_z = b\rho\,\mathbf{a}_z$$

For the outside field, we find

$$\mathbf{J}_{out} = \frac{1}{\rho}\frac{d}{d\rho}\left(\frac{\rho ba^3}{3\rho}\right)\mathbf{a}_z = 0$$

as expected.

7.22. A solid cylinder of radius a and length L, where $L >> a$, contains volume charge of uniform density ρ_0 C/m^3. The cylinder rotates about its axis (the z axis) at angular velocity Ω rad/s.

a) Determine the current density $\mathbf{J}$, as a function of position within the rotating cylinder: Use $\mathbf{J} = \rho_0 \mathbf{v} = \underline{\rho_0 \rho \Omega \, \mathbf{a}_\phi \text{ A/m}^2}$.

b) Determine the magnetic field intensity $\mathbf{H}$ inside and outside: It helps initially to obtain the field on-axis. To do this, we use the result of Problem 8.6, but give the rotating charged disk in that problem a differential thickness, dz. We can then evaluate the on-axis field in the rotating cylinder as the superposition of fields from a stack of disks which exist between $\pm L/2$. Here, we make the problem easier by letting $L \to \infty$ (since $L >> a$) thereby specializing our evaluation to positions near the half-length. The on-axis field is therefore:

$$H_z(\rho = 0) = \int_{-\infty}^{\infty} \frac{\rho_0 \Omega}{2z} \left[\frac{a^2 + 2z^2 \left(1 - \sqrt{1 + a^2/z^2}\right)}{\sqrt{1 + a^2/z^2}} \right] dz$$

$$= 2 \int_0^{\infty} \frac{\rho_0 \Omega}{2} \left[\frac{a^2}{\sqrt{z^2 + a^2}} + \frac{2z^2}{\sqrt{z^2 + a^2}} - 2z \right] dz$$

$$= 2\rho_0 \Omega \left[\frac{a^2}{2} \ln(z + \sqrt{z^2 + a^2}) + \frac{z}{2} \sqrt{z^2 + a^2} - \frac{a^2}{2} \ln(z + \sqrt{z^2 + a^2}) - \frac{z^2}{2} \right]_0^{\infty}$$

$$= \rho_0 \Omega \left[z\sqrt{z^2 + a^2} - z^2 \right]_0^{\infty} = \rho_0 \Omega \left[z\sqrt{z^2 + a^2} - z^2 \right]_{z \to \infty}$$

Using the large z approximation in the radical, we obtain

$$H_z(\rho = 0) = \rho_0 \Omega \left[z^2 \left(1 + \frac{a^2}{2z^2}\right) - z^2 \right] = \frac{\rho_0 \Omega a^2}{2}$$

To find the field as a function of radius, we apply Ampere's circuital law to a rectangular loop, drawn in two locations described as follows: First, construct the rectangle with one side along the z axis, and with the opposite side lying at any radius *outside* the cylinder. In taking the line integral of $\mathbf{H}$ around the rectangle, we note that the two segments that are perpendicular to the cylinder axis will have their path integrals exactly cancel, since the two path segments are oppositely-directed, while from symmetry the field should not be different along each segment. This leaves only the path segment that coindides with the axis, and that lying parallel to the axis, but outside. Choosing the length of these segments to be ℓ, Ampere's circuital law becomes:

$$\oint \mathbf{H} \cdot d\mathbf{L} = H_z(\rho = 0)\ell + H_z(\rho > a)\ell = I_{encl} = \int_s \mathbf{J} \cdot d\mathbf{S} = \int_0^{\ell} \int_0^a \rho_0 \rho \Omega \, \mathbf{a}_\phi \cdot \mathbf{a}_\phi \, d\rho \, dz$$

$$= \ell \frac{\rho_0 \Omega a^2}{2}$$

But we found earlier that $H_z(\rho = 0) = \rho_0 \Omega a^2/2$. Therefore, we identify the outside field, $H_z(\rho > a) = 0$. Next, change the rectangular path only by displacing the central path component off-axis by distance ρ, but still lying within the cylinder. The enclosed current is now somewhat less, and Ampere's law becomes

$$\oint \mathbf{H} \cdot d\mathbf{L} = H_z(\rho)\ell + H_z(\rho > a)\ell = I_{encl} = \int_s \mathbf{J} \cdot d\mathbf{S} = \int_0^{\ell} \int_{\rho}^a \rho_0 \rho' \Omega \, \mathbf{a}_\phi \cdot \mathbf{a}_\phi \, d\rho \, dz$$

$$= \ell \frac{\rho_0 \Omega}{2}(a^2 - \rho^2) \quad \Rightarrow \quad \mathbf{H}(\rho) = \frac{\rho_0 \Omega}{2}(a^2 - \rho^2) \, \mathbf{a}_z \text{ A/m}$$

7.22c) Check your result of part *b* by taking the curl of **H**. With **H** *z*-directed, and varying only with ρ, the curl in cylindrical coordinates becomes

$$\nabla \times \mathbf{H} = -\frac{dH_z}{d\rho}\,\mathbf{a}_\phi = \rho_0 \Omega \rho\,\mathbf{a}_\phi \text{ A/m}^2 = \mathbf{J}$$

as expected.

7.23. Given the field $\mathbf{H} = 20\rho^2\,\mathbf{a}_\phi$ A/m:

a) Determine the current density **J**: This is found through the curl of **H**, which simplifies to a single term, since **H** varies only with ρ and has only a ϕ component:

$$\mathbf{J} = \nabla \times \mathbf{H} = \frac{1}{\rho}\frac{d(\rho H_\phi)}{d\rho}\,\mathbf{a}_z = \frac{1}{\rho}\frac{d}{d\rho}\left(20\rho^3\right)\,\mathbf{a}_z = \underline{60\rho\,\mathbf{a}_z \text{ A/m}^2}$$

b) Integrate **J** over the circular surface $\rho = 1$, $0 < \phi < 2\pi$, $z = 0$, to determine the total current passing through that surface in the $\mathbf{a}_z$ direction: The integral is:

$$I = \int\!\!\int \mathbf{J}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^1 60\rho\mathbf{a}_z \cdot \rho\,d\rho\,d\phi\mathbf{a}_z = \underline{40\pi \text{ A}}$$

c) Find the total current once more, this time by a line integral around the circular path $\rho = 1$, $0 < \phi < 2\pi$, $z = 0$:

$$I = \oint \mathbf{H}\cdot d\mathbf{L} = \int_0^{2\pi} 20\rho^2\,\mathbf{a}_\phi\Big|_{\rho=1}\cdot(1)d\phi\mathbf{a}_\phi = \int_0^{2\pi} 20\,d\phi = \underline{40\pi \text{ A}}$$

7.24. Infinitely-long filamentary conductors are located in the $y = 0$ plane at $x = n$ meters where $n = 0, \pm 1, \pm 2, \ldots$ Each carries 1 A in the $\mathbf{a}_z$ direction.

a) Find $\mathbf{H}$ on the y axis. As a help,

$$\sum_{n=1}^{\infty} \frac{y}{y^2 + n^2} = \frac{\pi}{2} - \frac{1}{2y} + \frac{\pi}{e^{2\pi y} - 1}$$

We can begin by determining the field on the y axis arising from two wires only, located at $x = \pm n$. We start from basics, using the Biot-Savart law:

$$\mathbf{H} = \int \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

where $R = \left(n^2 + y^2 + z^2\right)^2$ for both wires, and where, for the wire at $x = +n$

$$\mathbf{a}_R^+ = \frac{-n\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z}{\left(n^2 + y^2 + z^2\right)^{1/2}} \qquad \text{and} \qquad \mathbf{a}_R^- = \frac{n\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z}{\left(n^2 + y^2 + z^2\right)^{1/2}}$$

for the wire located at $x = -n$. Then with $Id\mathbf{L} = dz\,\mathbf{a}_z$ ($I = 1$) for both wires, the Biot-Savart construction becomes

$$\mathbf{H} = \int_{-\infty}^{\infty} \frac{dz\,\mathbf{a}_z \times \left[(-n\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z) + (n\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z)\right]}{4\pi\left(n^2 + y^2 + z^2\right)^{3/2}} = \frac{-y\mathbf{a}_x}{2\pi} \int_{-\infty}^{\infty} \frac{dz}{\left(n^2 + y^2 + z^2\right)^{3/2}}$$

This evaluates as

$$\mathbf{H} = -\frac{1}{\pi}\left(\frac{y}{n^2 + y^2}\right) \mathbf{a}_x \text{ A/m}$$

Now if we include *all* wire pairs, the result is the superposition of an infinite number of fields of the above form. Specifically,

$$\mathbf{H}_{net} = -\frac{1}{\pi} \sum_{n=1}^{\infty} \left(\frac{y}{n^2 + y^2}\right) \mathbf{a}_x \text{ A/m}$$

Using the given closed form of the series expansion, our final answer is

$$\underline{\mathbf{H}_{net} = -\left[\frac{1}{2} - \frac{1}{2\pi y} + \frac{1}{e^{2\pi y} - 1}\right]} \mathbf{a}_x \text{ A/m}$$

b) Compare your result of part *a* to that obtained if the filaments are replaced by a current sheet in the $y = 0$ plane that carries surface current density $\mathbf{K} = 1\mathbf{a}_z$ A/m: This is found through Eq. (11):

$$\mathbf{H} = \frac{1}{2}\mathbf{K} \times \mathbf{a}_N = \frac{1}{2}\mathbf{a}_z \times \mathbf{a}_y = -\frac{1}{2}\mathbf{a}_x$$

Our answer of part *a* approaches this value as $y \to \infty$, demonstrating that at large distances, the parallel wires act like a current sheet. Interestingly, at close-in locations, such that $2\pi y << 1$, we may expand $e^{2\pi y} \doteq 1 + 2\pi y$, leading to the cancellation of the last two terms in the part *a* result, and again, $\mathbf{H}_{net} \doteq -1/2\,\mathbf{a}_x$.

7.25. When x, y, and z are positive and less than 5, a certain magnetic field intensity may be expressed as $\mathbf{H} = [x^2yz/(y+1)]\mathbf{a}_x + 3x^2z^2\mathbf{a}_y - [xyz^2/(y+1)]\mathbf{a}_z$. Find the total current in the $\mathbf{a}_x$ direction that crosses the strip, $x = 2$, $1 \le y \le 4$, $3 \le z \le 4$, by a method utilizing:

a) a surface integral: We need to find the current density by taking the curl of the given $\mathbf{H}$. Actually, since the strip lies parallel to the yz plane, we need only find the x component of the current density, as only this component will contribute to the requested current. This is

$$J_x = (\nabla \times \mathbf{H})_x = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) = -\left(\frac{xz^2}{(y+1)^2} + 6x^2z\right)\mathbf{a}_x$$

The current through the strip is then

$$I = \int_s \mathbf{J}\cdot\mathbf{a}_x\,da = -\int_3^4\int_1^4 \left(\frac{2z^2}{(y+1)^2} + 24z\right)dy\,dz = -\int_3^4 \left(\frac{-2z^2}{(y+1)} + 24zy\right)_1^4 dz$$
$$= -\int_3^4 \left(\frac{3}{5}z^2 + 72z\right)dz = -\left(\frac{1}{5}z^3 + 36z^2\right)_3^4 = \underline{-259}$$

b) a closed line integral: We integrate counter-clockwise around the strip boundary (using the right-hand convention), where the path normal is positive $\mathbf{a}_x$. The current is then

$$I = \oint \mathbf{H}\cdot d\mathbf{L} = \int_1^4 3(2)^2(3)^2\,dy + \int_3^4 -\frac{2(4)z^2}{(4+1)}\,dz + \int_4^1 3(2)^2(4)^2\,dy + \int_4^3 -\frac{2(1)z^2}{(1+1)}\,dz$$
$$= 108(3) - \frac{8}{15}(4^3 - 3^3) + 192(1-4) - \frac{1}{3}(3^3 - 4^3) = -259$$

7.26. Consider a sphere of radius $r = 4$ centered at (0,0,3). Let S_1 be that portion of the spherical surface that lies above the xy plane. Find $\int_{S_1}(\nabla \times \mathbf{H})\cdot d\mathbf{S}$ if $\mathbf{H} = 3\rho\,\mathbf{a}_\phi$ in cylindrical coordinates: First, the intersection of the sphere with the x-y plane is a disk in the plane of radius $\rho_d = \sqrt{4^2 - 3^2} = \sqrt{7}$. The curl of the given field (having a ϕ component that varies only with ρ) is

$$\nabla \times \mathbf{H} = \frac{1}{\rho}\frac{d}{d\rho}(\rho H_\phi)\,\mathbf{a}_z = \frac{1}{\rho}\frac{d}{d\rho}\left(3\rho^2\right)\mathbf{a}_z = 6\,\mathbf{a}_z$$

Since this is a constant field, its flux through S_1 will simply be the flux through the disk, which in turn is simply the product of the field with the disk area:

$$\Phi = 6 \times \pi(\sqrt{7})^2 = \underline{42\pi}$$

Another way to solve the problem is to use Stokes' theorem and write the flux as

$$\Phi = \oint \mathbf{H}\cdot d\mathbf{L}$$

where the path integral is taken around the disk perimeter. Doing this gives:

$$\Phi = 2\pi\rho_d H_\phi|_{\rho_d} = 2\pi\sqrt{7} \times 3\sqrt{7} = \underline{42\pi}$$

7.27. The magnetic field intensity is given in a certain region of space as

$$\mathbf{H} = \frac{x+2y}{z^2}\,\mathbf{a}_y + \frac{2}{z}\,\mathbf{a}_z \text{ A/m}$$

a) Find $\nabla\times\mathbf{H}$: For this field, the general curl expression in rectangular coordinates simplifies to

$$\nabla\times\mathbf{H} = -\frac{\partial H_y}{\partial z}\,\mathbf{a}_x + \frac{\partial H_y}{\partial x}\,\mathbf{a}_z = \underline{\frac{2(x+2y)}{z^3}\,\mathbf{a}_x + \frac{1}{z^2}\mathbf{a}_z \text{ A/m}}$$

b) Find $\mathbf{J}$: This will be the answer of part a, since $\nabla\times\mathbf{H} = \mathbf{J}$.

c) Use $\mathbf{J}$ to find the total current passing through the surface $z = 4$, $1 < x < 2$, $3 < y < 5$, in the $\mathbf{a}_z$ direction: This will be

$$I = \int\int \mathbf{J}\big|_{z=4}\cdot\mathbf{a}_z\,dx\,dy = \int_3^5\int_1^2 \frac{1}{4^2}dx\,dy = \underline{1/8 \text{ A}}$$

d) Show that the same result is obtained using the other side of Stokes' theorem: We take $\oint\mathbf{H}\cdot d\mathbf{L}$ over the square path at $z = 4$ as defined in part c. This involves two integrals of the y component of $\mathbf{H}$ over the range $3 < y < 5$. Integrals over x, to complete the loop, do not exist since there is no x component of $\mathbf{H}$. We have

$$I = \oint\mathbf{H}\big|_{z=4}\cdot d\mathbf{L} = \int_3^5 \frac{2+2y}{16}\,dy + \int_5^3 \frac{1+2y}{16}\,dy = \frac{1}{8}(2) - \frac{1}{16}(2) = \underline{1/8 \text{ A}}$$

7.28. Given $\mathbf{H} = (3r^2/\sin\theta)\mathbf{a}_\theta + 54r\cos\theta\mathbf{a}_\phi$ A/m in free space:

a) find the total current in the $\mathbf{a}_\theta$ direction through the conical surface $\theta = 20^\circ$, $0 \le \phi \le 2\pi$, $0 \le r \le 5$, by whatever side of Stokes' theorem you like best. I chose the line integral side, where the integration path is the circular path in ϕ around the top edge of the cone, at $r = 5$. The path direction is chosen to be *clockwise* looking down on the xy plane. This, by convention, leads to the normal from the cone surface that points in the positive $\mathbf{a}_\theta$ direction (right hand rule). We find

$$\oint \mathbf{H}\cdot d\mathbf{L} = \int_0^{2\pi} \left[(3r^2/\sin\theta)\mathbf{a}_\theta + 54r\cos\theta\mathbf{a}_\phi\right]_{r=5,\theta=20} \cdot 5\sin(20^\circ)\,d\phi\,(-\mathbf{a}_\phi)$$
$$= -2\pi(54)(25)\cos(20^\circ)\sin(20^\circ) = \underline{-2.73\times 10^3\text{ A}}$$

This result means that there is a component of current that enters the cone surface in the $-\mathbf{a}_\theta$ direction, to which is associated a component of $\mathbf{H}$ in the positive $\mathbf{a}_\phi$ direction.

b) Check the result by using the other side of Stokes' theorem: We first find the current density through the curl of the magnetic field, where three of the six terms in the spherical coordinate formula survive:

$$\nabla\times\mathbf{H} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(54r\cos\theta\sin\theta\right)\,\mathbf{a}_r - \frac{1}{r}\frac{\partial}{\partial r}\left(54r^2\cos\theta\right)\,\mathbf{a}_\theta + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{3r^3}{\sin\theta}\right)\,\mathbf{a}_\phi = \mathbf{J}$$

Thus

$$\mathbf{J} = 54\cot\theta\,\mathbf{a}_r - 108\cos\theta\,\mathbf{a}_\theta + \frac{9r}{\sin\theta}\,\mathbf{a}_\phi$$

The calculation of the other side of Stokes' theorem now involves integrating $\mathbf{J}$ over the surface of the cone, where the outward normal is positive $\mathbf{a}_\theta$, as defined in part a:

$$\int_S (\nabla\times\mathbf{H})\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^5 \left[54\cot\theta\,\mathbf{a}_r - 108\cos\theta\,\mathbf{a}_\theta + \frac{9r}{\sin\theta}\,\mathbf{a}_\phi\right]_{20^\circ} \cdot\mathbf{a}_\theta\, r\sin(20^\circ)\,dr\,d\phi$$
$$= -\int_0^{2\pi}\int_0^5 108\cos(20^\circ)\sin(20^\circ) r\,dr\,d\phi = -2\pi(54)(25)\cos(20^\circ)\sin(20^\circ)$$
$$= \underline{-2.73\times 10^3\text{ A}}$$

7.29. A long straight non-magnetic conductor of 0.2 mm radius carries a uniformly-distributed current of 2 A dc.

a) Find **J** within the conductor: Assuming the current is $+z$ directed,

$$\mathbf{J} = \frac{2}{\pi(0.2\times10^{-3})^2}\mathbf{a}_z = \underline{1.59\times10^7\,\mathbf{a}_z\ \text{A/m}^2}$$

b) Use Ampere's circuital law to find **H** and **B** within the conductor: Inside, at radius ρ, we have

$$2\pi\rho H_\phi = \pi\rho^2 J \quad\Rightarrow\quad \mathbf{H} = \frac{\rho J}{2}\mathbf{a}_\phi = \underline{7.96\times10^6\rho\,\mathbf{a}_\phi\ \text{A/m}}$$

Then $\mathbf{B} = \mu_0\mathbf{H} = (4\pi\times10^{-7})(7.96\times10^6)\rho\mathbf{a}_\phi = \underline{10\rho\,\mathbf{a}_\phi\ \text{Wb/m}^2}$.

c) Show that $\nabla\times\mathbf{H} = \mathbf{J}$ within the conductor: Using the result of part b, we find,

$$\nabla\times\mathbf{H} = \frac{1}{\rho}\frac{d}{d\rho}(\rho H_\phi)\,\mathbf{a}_z = \frac{1}{\rho}\frac{d}{d\rho}\left(\frac{1.59\times10^7\rho^2}{2}\right)\mathbf{a}_z = \underline{1.59\times10^7\,\mathbf{a}_z\ \text{A/m}^2} = \mathbf{J}$$

d) Find **H** and **B** *outside* the conductor (note typo in book): Outside, the entire current is enclosed by a closed path at radius ρ, and so

$$\mathbf{H} = \frac{I}{2\pi\rho}\mathbf{a}_\phi = \underline{\frac{1}{\pi\rho}\mathbf{a}_\phi\ \text{A/m}}$$

Now $\mathbf{B} = \mu_0\mathbf{H} = \underline{\mu_0/(\pi\rho)\,\mathbf{a}_\phi\ \text{Wb/m}^2}$.

e) Show that $\nabla\times\mathbf{H} = \mathbf{J}$ outside the conductor: Here we use **H** outside the conductor and write:

$$\nabla\times\mathbf{H} = \frac{1}{\rho}\frac{d}{d\rho}(\rho H_\phi)\,\mathbf{a}_z = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{1}{\pi\rho}\right)\mathbf{a}_z = \underline{0}\ \ \text{(as expected)}$$

7.30. (an inversion of Problem 8.20). A solid nonmagnetic conductor of circular cross-section has a radius of 2mm. The conductor is inhomogeneous, with $\sigma = 10^6(1+10^6\rho^2)$ S/m. If the conductor is 1m in length and has a voltage of 1mV between its ends, find:

a) **H** inside: With current along the cylinder length (along $\mathbf{a}_z$, and with ϕ symmetry, **H** will be ϕ-directed only. We find $\mathbf{E} = (V_0/d)\mathbf{a}_z = 10^{-3}\mathbf{a}_z$ V/m. Then $\mathbf{J} = \sigma\mathbf{E} = 10^3(1+10^6\rho^2)\mathbf{a}_z$ A/m^2. Next we apply Ampere's circuital law to a circular path of radius ρ, centered on the z axis and normal to the axis:

$$\oint \mathbf{H}\cdot d\mathbf{L} = 2\pi\rho H_\phi = \int\int_S \mathbf{J}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^\rho 10^3(1+10^6(\rho')^2)\mathbf{a}_z\cdot\mathbf{a}_z\rho' d\rho' d\phi$$

Thus

$$H_\phi = \frac{10^3}{\rho}\int_0^\rho \rho' + 10^6(\rho')^3 d\rho' = \frac{10^3}{\rho}\left[\frac{\rho^2}{2} + \frac{10^6}{4}\rho^4\right]$$

Finally, $\mathbf{H} = \underline{500\rho(1+5\times10^5\rho^3)\mathbf{a}_\phi\ \text{A/m}\ (0<\rho<2\text{mm})}$.

b) the total magnetic flux inside the conductor: With field in the ϕ direction, a plane normal to **B** will be that in the region $0<\rho<2$ mm, $0<z<1$ m. The flux will be

$$\Phi = \int\int_S \mathbf{B}\cdot d\mathbf{S} = \mu_0\int_0^1\int_0^{2\times10^{-3}}\left(500\rho + 2.5\times10^8\rho^3\right)d\rho dz = 8\pi\times10^{-10}\ \text{Wb} = \underline{2.5\,\text{nWb}}$$

7.31. The cylindrical shell defined by 1 cm $< \rho <$ 1.4 cm consists of a non-magnetic conducting material and carries a total current of 50 A in the $\mathbf{a}_z$ direction. Find the total magnetic flux crossing the plane $\phi = 0$, $0 < z < 1$:

a) $0 < \rho < 1.2\,$cm: We first need to find **J**, **H**, and **B**: The current density will be:

$$\mathbf{J} = \frac{50}{\pi[(1.4 \times 10^{-2})^2 - (1.0 \times 10^{-2})^2]}\,\mathbf{a}_z = 1.66 \times 10^5\,\mathbf{a}_z\ \text{A/m}^2$$

Next we find H_ϕ at radius ρ between 1.0 and 1.4 cm, by applying Ampere's circuital law, and noting that the current density is zero at radii less than 1 cm:

$$2\pi\rho H_\phi = I_{encl} = \int_0^{2\pi}\int_{10^{-2}}^{\rho} 1.66 \times 10^5 \rho'\,d\rho'\,d\phi$$

$$\Rightarrow \quad H_\phi = 8.30 \times 10^4 \frac{(\rho^2 - 10^{-4})}{\rho}\ \text{A/m} \quad (10^{-2}\,\text{m} < \rho < 1.4 \times 10^{-2}\,\text{m})$$

Then $\mathbf{B} = \mu_0 \mathbf{H}$, or

$$\mathbf{B} = 0.104 \frac{(\rho^2 - 10^{-4})}{\rho}\,\mathbf{a}_\phi\ \text{Wb/m}^2$$

Now,

$$\Phi_a = \int\int \mathbf{B}\cdot d\mathbf{S} = \int_0^1 \int_{10^{-2}}^{1.2\times10^{-2}} 0.104\left[\rho - \frac{10^{-4}}{\rho}\right] d\rho\,dz$$

$$= 0.104\left[\frac{(1.2\times10^{-2})^2 - 10^{-4}}{2} - 10^{-4}\ln\left(\frac{1.2}{1.0}\right)\right] = 3.92 \times 10^{-7}\,\text{Wb} = \underline{0.392\,\mu\text{Wb}}$$

b) $1.0\,\text{cm} < \rho < 1.4\,\text{cm}$ (note typo in book): This is part a over again, except we change the upper limit of the radial integration:

$$\Phi_b = \int\int \mathbf{B}\cdot d\mathbf{S} = \int_0^1 \int_{10^{-2}}^{1.4\times10^{-2}} 0.104\left[\rho - \frac{10^{-4}}{\rho}\right] d\rho\,dz$$

$$= 0.104\left[\frac{(1.4\times10^{-2})^2 - 10^{-4}}{2} - 10^{-4}\ln\left(\frac{1.4}{1.0}\right)\right] = 1.49 \times 10^{-6}\,\text{Wb} = \underline{1.49\,\mu\text{Wb}}$$

c) $1.4\,\text{cm} < \rho < 20\,\text{cm}$: This is entirely outside the current distribution, so we need **B** there: We modify the Ampere's circuital law result of part a to find:

$$\mathbf{B}_{out} = 0.104 \frac{[(1.4\times10^{-2})^2 - 10^{-4}]}{\rho}\,\mathbf{a}_\phi = \frac{10^{-5}}{\rho}\,\mathbf{a}_\phi\ \text{Wb/m}^2$$

We now find

$$\Phi_c = \int_0^1 \int_{1.4\times10^{-2}}^{20\times10^{-2}} \frac{10^{-5}}{\rho}\,d\rho\,dz = 10^{-5}\ln\left(\frac{20}{1.4}\right) = 2.7 \times 10^{-5}\,\text{Wb} = \underline{27\,\mu\text{Wb}}$$

7.32. The free space region defined by $1 < z < 4$ cm and $2 < \rho < 3$ cm is a toroid of rectangular cross-section. Let the surface at $\rho = 3$ cm carry a surface current $\mathbf{K} = 2\mathbf{a}_z$ kA/m.

a) Specify the current densities on the surfaces at $\rho = 2$ cm, $z = 1$cm, and $z = 4$cm. All surfaces must carry equal currents. With this requirement, we find: $\mathbf{K}(\rho = 2) = \underline{-3\,\mathbf{a}_z \text{ kA/m}}$. Next, the current densities on the $z = 1$ and $z = 4$ surfaces must transistion between the current density values at $\rho = 2$ and $\rho = 3$. Knowing the the radial current density will vary as $1/\rho$, we find $\mathbf{K}(z = 1) = \underline{(60/\rho)\mathbf{a}_\rho \text{ A/m}}$ with ρ in meters. Similarly, $\mathbf{K}(z = 4) = \underline{-(60/\rho)\mathbf{a}_\rho \text{ A/m}}$.

b) Find $\mathbf{H}$ everywhere: Outside the toroid, $\mathbf{H} = 0$. Inside, we apply Ampere's circuital law in the manner of Problem 8.14:

$$\oint \mathbf{H} \cdot d\mathbf{L} = 2\pi\rho H_\phi = \int_0^{2\pi} \mathbf{K}(\rho = 2) \cdot \mathbf{a}_z \,(2 \times 10^{-2})\, d\phi$$

$$\Rightarrow \quad \mathbf{H} = -\frac{2\pi(3000)(.02)}{\rho}\mathbf{a}_\phi = \underline{-60/\rho\,\mathbf{a}_\phi \text{ A/m (inside)}}$$

c) Calculate the total flux within the toriod: We have $\mathbf{B} = -(60\mu_0/\rho)\mathbf{a}_\phi$ Wb/m^2. Then

$$\Phi = \int_{.01}^{.04}\int_{.02}^{.03} \frac{-60\mu_0}{\rho}\,\mathbf{a}_\phi \cdot (-\mathbf{a}_\phi)\, d\rho\, dz = (.03)(60)\mu_0 \ln\left(\frac{3}{2}\right) = \underline{0.92\,\mu\text{Wb}}$$

7.33. Use an expansion in rectangular coordinates to show that the curl of the gradient of any scalar field G is identically equal to zero. We begin with

$$\nabla G = \frac{\partial G}{\partial x}\,\mathbf{a}_x + \frac{\partial G}{\partial y}\,\mathbf{a}_y + \frac{\partial G}{\partial z}\,\mathbf{a}_z$$

and

$$\nabla \times \nabla G = \left[\frac{\partial}{\partial y}\left(\frac{\partial G}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial G}{\partial y}\right)\right]\mathbf{a}_x + \left[\frac{\partial}{\partial z}\left(\frac{\partial G}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial G}{\partial z}\right)\right]\mathbf{a}_y$$
$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial G}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial G}{\partial x}\right)\right]\mathbf{a}_z = \underline{0} \text{ for any } G$$

7.34. A filamentary conductor on the z axis carries a current of 16A in the $\mathbf{a}_z$ direction, a conducting shell at $\rho = 6$ carries a total current of 12A in the $-\mathbf{a}_z$ direction, and another shell at $\rho = 10$ carries a total current of 4A in the $-\mathbf{a}_z$ direction.

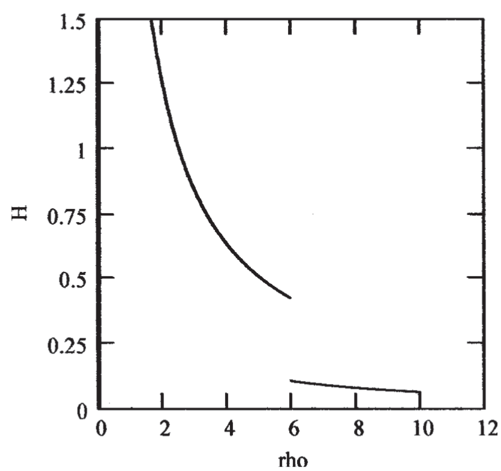
a) Find $\mathbf{H}$ for $0 < \rho < 12$: Ampere's circuital law states that $\oint \mathbf{H} \cdot d\mathbf{L} = I_{encl}$, where the line integral and current direction are related in the usual way through the right hand rule. Therefore, if I is in the positive z direction, $\mathbf{H}$ is in the $\mathbf{a}_\phi$ direction. We proceed as follows:

$$0 < \rho < 6: \quad 2\pi\rho H_\phi = 16 \quad \Rightarrow \quad \mathbf{H} = \underline{16/(2\pi\rho)\mathbf{a}_\phi}$$

$$6 < \rho < 10: \quad 2\pi\rho H_\phi = 16 - 12 \quad \Rightarrow \quad \mathbf{H} = \underline{4/(2\pi\rho)\mathbf{a}_\phi}$$

$$\rho > 10: \quad 2\pi\rho H_\phi = 16 - 12 - 4 = 0 \quad \Rightarrow \quad \mathbf{H} = \underline{0}$$

b) Plot H_ϕ vs. ρ:



c) Find the total flux Φ crossing the surface $1 < \rho < 7$, $0 < z < 1$: This will be

$$\Phi = \int_0^1 \int_1^6 \frac{16\mu_0}{2\pi\rho}\, d\rho\, dz + \int_0^1 \int_6^7 \frac{4\mu_0}{2\pi\rho}\, d\rho\, dz = \frac{2\mu_0}{\pi}\left[4\ln 6 + \ln(7/6)\right] = \underline{5.9\ \mu\text{Wb}}$$

7.35. A current sheet, $\mathbf{K} = 20\,\mathbf{a}_z$ A/m, is located at $\rho = 2$, and a second sheet, $\mathbf{K} = -10\,\mathbf{a}_z$ A/m is located at $\rho = 4$.

a.) Let $V_m = 0$ at $P(\rho = 3, \phi = 0, z = 5)$ and place a barrier at $\phi = \pi$. Find $V_m(\rho, \phi, z)$ for $-\pi < \phi < \pi$: Since the current is cylindrically-symmetric, we know that $\mathbf{H} = I/(2\pi\rho)\,\mathbf{a}_\phi$, where I is the current enclosed, equal in this case to $2\pi(2)K = 80\pi$ A. Thus, using the result of Section 8.6, we find

$$V_m = -\frac{I}{2\pi}\,\phi = -\frac{80\pi}{2\pi}\,\phi = \underline{-40\phi \text{ A}}$$

which is valid over the region $2 < \rho < 4$, $-\pi < \phi < \pi$, and $-\infty < z < \infty$. For $\rho > 4$, the outer current contributes, leading to a total enclosed current of

$$I_{net} = 2\pi(2)(20) - 2\pi(4)(10) = 0$$

With zero enclosed current, $H_\phi = 0$, and the magnetic potential is zero as well.

b) Let $\mathbf{A} = 0$ at P and find $\mathbf{A}(\rho, \phi, z)$ for $2 < \rho < 4$: Again, we know that $\mathbf{H} = H_\phi(\rho)$, since the current is cylindrically symmetric. With the current only in the z direction, and again using symmmetry, we expect only a z component of $\mathbf{A}$ which varies only with ρ. We can then write:

$$\nabla \times \mathbf{A} = -\frac{dA_z}{d\rho}\mathbf{a}_\phi = \mathbf{B} = \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\phi$$

Thus

$$\frac{dA_z}{d\rho} = -\frac{\mu_0 I}{2\pi\rho} \quad \Rightarrow \quad A_z = -\frac{\mu_0 I}{2\pi}\ln(\rho) + C$$

We require that $A_z = 0$ at $\rho = 3$. Therefore $C = [(\mu_0 I)/(2\pi)]\ln(3)$, Then, with $I = 80\pi$, we finally obtain

$$\mathbf{A} = -\frac{\mu_0(80\pi)}{2\pi}\left[\ln(\rho) - \ln(3)\right]\mathbf{a}_z = \underline{40\mu_0 \ln\left(\frac{3}{\rho}\right)\mathbf{a}_z \text{ Wb/m}}$$

7.36. Let $\mathbf{A} = (3y - z)\mathbf{a}_x + 2xz\mathbf{a}_y$ Wb/m in a certain region of free space.

a) Show that $\nabla \cdot \mathbf{A} = 0$:

$$\nabla \cdot \mathbf{A} = \frac{\partial}{\partial x}(3y - z) + \frac{\partial}{\partial y}2xz = \underline{0}$$

b) At $P(2, -1, 3)$, find $\mathbf{A}$, $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{J}$: First $\mathbf{A}_P = \underline{-6\mathbf{a}_x + 12\mathbf{a}_y}$. Then, using the curl formula in cartesian coordinates,

$$\mathbf{B} = \nabla \times \mathbf{A} = -2x\mathbf{a}_x - \mathbf{a}_y + (2z - 3)\mathbf{a}_z \quad \Rightarrow \quad \mathbf{B}_P = \underline{-4\mathbf{a}_x - \mathbf{a}_y + 3\mathbf{a}_z \text{ Wb/m}^2}$$

Now

$$\mathbf{H}_P = (1/\mu_0)\mathbf{B}_P = \underline{-3.2 \times 10^6\mathbf{a}_x - 8.0 \times 10^5\mathbf{a}_y + 2.4 \times 10^6\mathbf{a}_z \text{ A/m}}$$

Then $\mathbf{J} = \nabla \times \mathbf{H} = (1/\mu_0)\nabla \times \mathbf{B} = \underline{0}$, as the curl formula in cartesian coordinates shows.

7.37. Let $N = 1000$, $I = 0.8$ A, $\rho_0 = 2$ cm, and $a = 0.8$ cm for the toroid shown in Fig. 8.12b. Find V_m in the interior of the toroid if $V_m = 0$ at $\rho = 2.5$ cm, $\phi = 0.3\pi$. Keep ϕ within the range $0 < \phi < 2\pi$: Well-within the toroid, we have

$$\mathbf{H} = \frac{NI}{2\pi\rho}\mathbf{a}_\phi = -\nabla V_m = -\frac{1}{\rho}\frac{dV_m}{d\phi}\mathbf{a}_\phi$$

Thus

$$V_m = -\frac{NI\phi}{2\pi} + C$$

Then,

$$0 = -\frac{1000(0.8)(0.3\pi)}{2\pi} + C$$

or C = 120. Finally

$$V_m = \underline{\left[120 - \frac{400}{\pi}\phi\right] \text{ A} \quad (0 < \phi < 2\pi)}$$

7.38. A square filamentary differential current loop, dL on a side, is centered at the origin in the $z=0$ plane in free space. The current I flows generally in the $\mathbf{a}_\phi$ direction.

a) Assuming that $r >> dL$, and following a method similar to that in Sec. 4.7, show that

$$d\mathbf{A} = \frac{\mu_0 I (dL)^2 \sin\theta}{4\pi r^2}\,\mathbf{a}_\phi$$

We begin with the expression for the differential vector potential, Eq. (48):

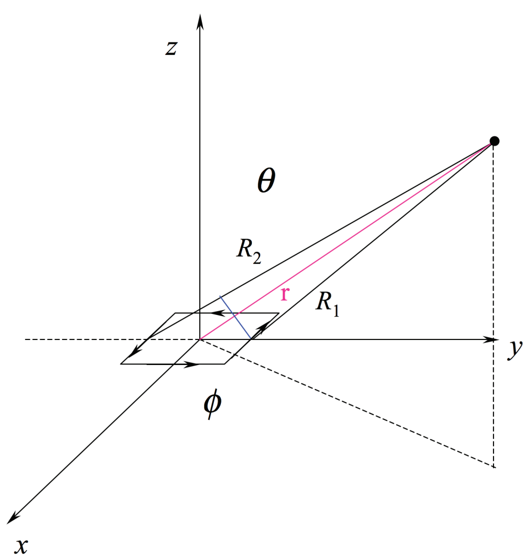
$$d\mathbf{A} = \frac{\mu_0 I d\mathbf{L}}{4\pi R}$$

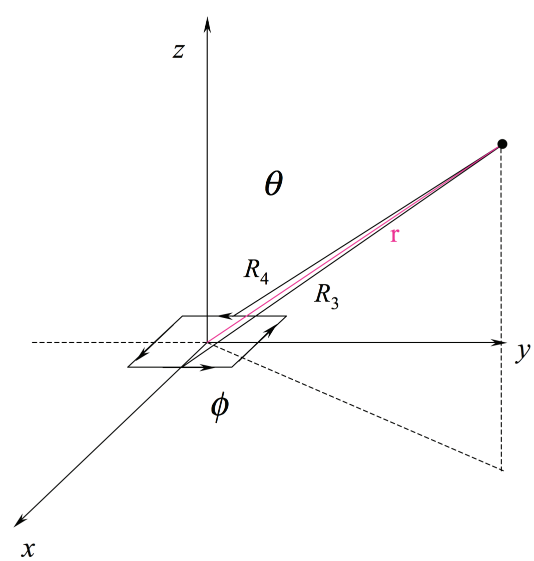
where in our case, we have four differential elements. The net vector potential at some distant point will consist of the vector sum of the four individual potentials. Referring to the figures below, the net potential at the indicated point is initially constructed as:

$$d\mathbf{A} = \frac{\mu_0 I dL}{4\pi}\left[\mathbf{a}_x\left(\frac{1}{R_2} - \frac{1}{R_1}\right) + \mathbf{a}_y\left(\frac{1}{R_3} - \frac{1}{R_4}\right)\right]$$

The challenge is to determine the four distances, R_1 through R_4, in terms of spherical coordinates, r, θ, and ϕ, thus referencing the four potentials to a common origin. This is where the treatment in Sec. 4.7 is useful, although it is more complicated here because the problem is three-dimensional.

The diagram for the y-displaced elements is shown in the left-hand figure. The three distance lines, r, R_1, and R_2, are approximately parallel because the observation point is in the far zone. Therefore, beyond the blue line segment that crosses the three lines, the lengths are essentially equal. The difference in lengths between r and R_1, for example, is the length of the line segment along r, from the origin to the blue line. This length will be the projection of the distance vector $dL/2\,\mathbf{a}_y$ along $\mathbf{a}_r$, or $dL/2\,\mathbf{a}_y \cdot \mathbf{a}_r$. As a look ahead, this principle is discussed with illustrations in Sec. 14.5, which handles antenna arrays of two elements.





7.38a) (continued). We now may write:

$$R_1 \doteq r - \left[\frac{dL}{2}\,\mathbf{a}_y \cdot \mathbf{a}_r\right] \qquad \text{and} \qquad R_2 \doteq r + \left[\frac{dL}{2}\,\mathbf{a}_y \cdot \mathbf{a}_r\right]$$

Referring to the right-hand figure and applying similar reasoning there leads to

$$R_3 \doteq r - \left[\frac{dL}{2}\,\mathbf{a}_x \cdot \mathbf{a}_r\right] \qquad \text{and} \qquad R_4 \doteq r + \left[\frac{dL}{2}\,\mathbf{a}_x \cdot \mathbf{a}_r\right]$$

where we know that $\mathbf{a}_x \cdot \mathbf{a}_r = \sin\theta\cos\phi$ and $\mathbf{a}_y \cdot \mathbf{a}_r = \sin\theta\sin\phi$ We now substitute all these relations into the original expression for $d\mathbf{A}$:

$$\begin{aligned}
d\mathbf{A} = \frac{\mu_0 I dL}{4\pi}&\left[\left(\left(r + \frac{dL}{2}\sin\theta\sin\phi\right)^{-1} - \left(r - \frac{dL}{2}\sin\theta\sin\phi\right)^{-1}\right)\mathbf{a}_x\right. \\
&+ \left.\left(\left(r - \frac{dL}{2}\sin\theta\cos\phi\right)^{-1} - \left(r + \frac{dL}{2}\sin\theta\cos\phi\right)^{-1}\right)\mathbf{a}_y\right] \\
= \frac{\mu_0 I dL}{4\pi r}&\left[\left(\left(1 + \frac{dL}{2r}\sin\theta\sin\phi\right)^{-1} - \left(1 - \frac{dL}{2r}\sin\theta\sin\phi\right)^{-1}\right)\mathbf{a}_x\right. \\
&+ \left.\left(\left(1 - \frac{dL}{2r}\sin\theta\cos\phi\right)^{-1} - \left(1 + \frac{dL}{2r}\sin\theta\cos\phi\right)^{-1}\right)\mathbf{a}_y\right]
\end{aligned}$$

Now, since $dL/r << 1$, this simplifies to

$$\begin{aligned}
d\mathbf{A} \doteq \frac{\mu_0 I dL}{4\pi r}&\left[\left(\left(1 - \frac{dL}{2r}\sin\theta\sin\phi\right) - \left(1 + \frac{dL}{2r}\sin\theta\sin\phi\right)\right)\mathbf{a}_x\right. \\
&+ \left.\left(\left(1 + \frac{dL}{2r}\sin\theta\cos\phi\right) - \left(1 - \frac{dL}{2r}\sin\theta\cos\phi\right)\right)\mathbf{a}_y\right] \\
= \frac{\mu_0 I (dL)^2 \sin\theta}{4\pi r^2}&\left[-\sin\phi\,\mathbf{a}_x + \cos\phi\,\mathbf{a}_y\right] \\
= \frac{\mu_0 I (dL)^2 \sin\theta}{4\pi r^2}&\,\mathbf{a}_\phi
\end{aligned}$$

b) Show that

$$d\mathbf{H} = \frac{I(dL)^2}{4\pi r^3}\left(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta\right)$$

Using the part a expression, we construct $d\mathbf{H} = (1/\mu_0)\nabla\times d\mathbf{A}$, where in this case, we have a ϕ-directed $d\mathbf{A}$ that varies with r and θ. The curl expression in spherical coordinates reduces to:

$$\nabla \times d\mathbf{A} = \frac{1}{r\sin\theta}\,\frac{\partial(dA\sin\theta)}{\partial\theta}\,\mathbf{a}_r \;-\; \frac{1}{r}\frac{\partial(r dA)}{\partial r}\,\mathbf{a}_\theta$$

or

$$\begin{aligned}
\nabla \times d\mathbf{A} &= \frac{1}{r\sin\theta}\,\frac{\partial}{\partial\theta}\left(\frac{I(dL)^2\sin^2\theta}{4\pi r^2}\right)\mathbf{a}_r \;-\; \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{I(dL)^2\sin\theta}{4\pi r}\right)\mathbf{a}_\theta \\
&= \frac{I(dL)^2}{4\pi r^3}\left(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta\right)
\end{aligned}$$

7.39. Planar current sheets of $\mathbf{K} = 30\mathbf{a}_z$ A/m and $-30\mathbf{a}_z$ A/m are located in free space at $x = 0.2$ and $x = -0.2$ respectively. For the region $-0.2 < x < 0.2$:

a) Find $\mathbf{H}$: Since we have parallel current sheets carrying equal and opposite currents, we use Eq. (12), $\mathbf{H} = \mathbf{K} \times \mathbf{a}_N$, where $\mathbf{a}_N$ is the unit normal directed into the region between currents, and where either one of the two currents are used. Choosing the sheet at $x = 0.2$, we find

$$\mathbf{H} = 30\mathbf{a}_z \times -\mathbf{a}_x = \underline{-30\mathbf{a}_y \text{ A/m}}$$

b) Obtain and expression for V_m if $V_m = 0$ at $P(0.1, 0.2, 0.3)$: Use

$$\mathbf{H} = -30\mathbf{a}_y = -\nabla V_m = -\frac{dV_m}{dy}\mathbf{a}_y$$

So

$$\frac{dV_m}{dy} = 30 \quad \Rightarrow \quad V_m = 30y + C_1$$

Then

$$0 = 30(0.2) + C_1 \quad \Rightarrow \quad C_1 = -6 \quad \Rightarrow \quad V_m = \underline{30y - 6 \text{ A}}$$

c) Find $\mathbf{B}$: $\mathbf{B} = \mu_0 \mathbf{H} = \underline{-30\mu_0 \mathbf{a}_y \text{ Wb/m}^2}$.

d) Obtain an expression for $\mathbf{A}$ if $\mathbf{A} = 0$ at P: We expect $\mathbf{A}$ to be z-directed (with the current), and so from $\nabla \times \mathbf{A} = \mathbf{B}$, where $\mathbf{B}$ is y-directed, we set up

$$-\frac{dA_z}{dx} = -30\mu_0 \quad \Rightarrow \quad A_z = 30\mu_0 x + C_2$$

Then $0 = 30\mu_0(0.1) + C_2 \quad \Rightarrow \quad C_2 = -3\mu_0$. So finally $\mathbf{A} = \underline{\mu_0(30x - 3)\mathbf{a}_z \text{ Wb/m}}$.

7.40. Show that the line integral of the vector potential $\mathbf{A}$ about any closed path is equal to the magnetic flux enclosed by the path, or $\oint \mathbf{A} \cdot d\mathbf{L} = \int \mathbf{B} \cdot d\mathbf{S}$.

We use the fact that $\mathbf{B} = \nabla \times \mathbf{A}$, and substitute this into the desired relation to find

$$\oint \mathbf{A} \cdot d\mathbf{L} = \int \nabla \times \mathbf{A} \cdot d\mathbf{S}$$

This is just a statement of Stokes' theorem (already proved), so we are done.

7.41. Assume that $\mathbf{A} = 50\rho^2 \mathbf{a}_z$ Wb/m in a certain region of free space.

a) Find **H** and **B**: Use

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial \rho}\mathbf{a}_\phi = \underline{-100\rho\,\mathbf{a}_\phi \text{ Wb/m}^2}$$

Then $\mathbf{H} = \mathbf{B}/\mu_0 = \underline{-100\rho/\mu_0\,\mathbf{a}_\phi \text{ A/m}}$.

b) Find **J**: Use

$$\mathbf{J} = \nabla \times \mathbf{H} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_\phi)\mathbf{a}_z = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{-100\rho^2}{\mu_0}\right)\mathbf{a}_z = \underline{-\frac{200}{\mu_0}\mathbf{a}_z \text{ A/m}^2}$$

c) Use **J** to find the total current crossing the surface $0 \le \rho \le 1$, $0 \le \phi < 2\pi$, $z = 0$: The current is

$$I = \int\int \mathbf{J} \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^1 \frac{-200}{\mu_0}\mathbf{a}_z \cdot \mathbf{a}_z\,\rho\,d\rho\,d\phi = \frac{-200\pi}{\mu_0} \text{ A} = \underline{-500 \text{ MA}}$$

d) Use the value of H_ϕ at $\rho = 1$ to calculate $\oint \mathbf{H} \cdot d\mathbf{L}$ for $\rho = 1$, $z = 0$: Have

$$\oint \mathbf{H} \cdot d\mathbf{L} = I = \int_0^{2\pi} \frac{-100}{\mu_0}\mathbf{a}_\phi \cdot \mathbf{a}_\phi\,(1)d\phi = \frac{-200\pi}{\mu_0} \text{ A} = \underline{-500 \text{ MA}}$$

7.42. Show that $\nabla_2(1/R_{12}) = -\nabla_1(1/R_{12}) = \mathbf{R}_{21}/R_{12}^3$. First

$$\nabla_2\left(\frac{1}{R_{12}}\right) = \nabla_2\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{-1/2}$$

$$= -\frac{1}{2}\left[\frac{2(x_2 - x_1)\mathbf{a}_x + 2(y_2 - y_1)\mathbf{a}_y + 2(z_2 - z_1)\mathbf{a}_z}{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{3/2}}\right] = \frac{-\mathbf{R}_{12}}{R_{12}^3} = \frac{\mathbf{R}_{21}}{R_{12}^3}$$

Also note that $\nabla_1(1/R_{12})$ would give the same result, but of opposite sign.

7.43. Compute the vector magnetic potential within the outer conductor for the coaxial line whose vector magnetic potential is shown in Fig. 8.20 if the outer radius of the outer conductor is $7a$. Select the proper zero reference and sketch the results on the figure: We do this by first finding $\mathbf{B}$ within the outer conductor and then "uncurling" the result to find $\mathbf{A}$. With $-z$-directed current I in the outer conductor, the current density is

$$\mathbf{J}_{out} = -\frac{I}{\pi(7a)^2 - \pi(5a)^2}\mathbf{a}_z = -\frac{I}{24\pi a^2}\mathbf{a}_z$$

Since current I flows in both conductors, but in opposite directions, Ampere's circuital law inside the outer conductor gives:

$$2\pi\rho H_\phi = I - \int_0^{2\pi}\int_{5a}^{\rho}\frac{I}{24\pi a^2}\rho'\,d\rho'\,d\phi \quad\Rightarrow\quad H_\phi = \frac{I}{2\pi\rho}\left[\frac{49a^2-\rho^2}{24a^2}\right]$$

Now, with $\mathbf{B} = \mu_0\mathbf{H}$, we note that $\nabla\times\mathbf{A}$ will have a ϕ component only, and from the direction and symmetry of the current, we expect $\mathbf{A}$ to be z-directed, and to vary only with ρ. Therefore

$$\nabla\times\mathbf{A} = -\frac{dA_z}{d\rho}\mathbf{a}_\phi = \mu_0\mathbf{H}$$

and so

$$\frac{dA_z}{d\rho} = -\frac{\mu_0 I}{2\pi\rho}\left[\frac{49a^2-\rho^2}{24a^2}\right]$$

Then by direct integration,

$$A_z = \int\frac{-\mu_0 I(49)}{48\pi\rho}\,d\rho + \int\frac{\mu_0 I\rho}{48\pi a^2}\,d\rho + C = \frac{\mu_0 I}{96\pi}\left[\frac{\rho^2}{a^2} - 98\ln\rho\right] + C$$

As per Fig. 8.20, we establish a zero reference at $\rho = 5a$, enabling the evaluation of the integration constant:

$$C = -\frac{\mu_0 I}{96\pi}\left[25 - 98\ln(5a)\right]$$

Finally,

$$\underline{A_z = \frac{\mu_0 I}{96\pi}\left[\left(\frac{\rho^2}{a^2} - 25\right) + 98\ln\left(\frac{5a}{\rho}\right)\right]\ \text{Wb/m}}$$

A plot of this continues the plot of Fig. 8.20, in which the curve goes negative at $\rho = 5a$, and then approaches a minimum of $-.09\mu_0 I/\pi$ at $\rho = 7a$, at which point the slope becomes zero.