Solutions Manual

Second Edition

Field and Wave Electromagnetics

David K. Cheng
Life Fellow, I.E.E.E.;
Fellow, I.E.E.; C. Eng.

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Chapter 2

Vector Analysis

P.2-1 a) $\bar{a}_A = \frac{\bar{A}}{A} = \frac{\bar{a}_x + \bar{a}_y 2 - \bar{a}_z 3}{\sqrt{1^2+2^2+(-3)^2}} = \frac{1}{\sqrt{14}}(\bar{a}_x + \bar{a}_y 2 - \bar{a}_z 3).$

b) $|\bar{A}-\bar{B}| = |\bar{a}_x + \bar{a}_y 6 - \bar{a}_z 4| = \sqrt{1^2+6^2+(-4)^2} = \sqrt{53}.$

c) $\bar{A}\cdot\bar{B} = 0 + 2(-4) + (-3) = -11.$

d) $\theta_{AB} = \cos^{-1}(\bar{A}\cdot\bar{B}/AB) = \cos^{-1}(-11/\sqrt{14}\sqrt{17}) = 135.5°.$

e) $\bar{A}\cdot\bar{a}_c = \bar{A}\cdot\frac{\bar{C}}{C} = \bar{A}\cdot\frac{1}{\sqrt{29}}(\bar{a}_x 5 - \bar{a}_z 2) = \frac{11}{\sqrt{29}}.$

f) $\bar{A}\times\bar{C} = -\bar{a}_x 4 - \bar{a}_y 13 - \bar{a}_z 10.$

g) $\bar{A}\cdot(\bar{B}\times\bar{C}) = (\bar{A}\times\bar{B})\cdot\bar{C} = -42$

h) $(\bar{A}\times\bar{B})\times\bar{C} = \bar{B}(\bar{A}\cdot\bar{C}) - \bar{A}(\bar{C}\cdot\bar{B}) = \bar{a}_x 2 - \bar{a}_y 40 + \bar{a}_z 5.$

$\bar{A}\times(\bar{B}\times\bar{C}) = \bar{B}(\bar{A}\cdot\bar{C}) - \bar{C}(\bar{A}\cdot\bar{B}) = \bar{a}_x 55 - \bar{a}_y 44 - \bar{a}_z 11.$

P.2-2 Let $\bar{C} = \bar{a}_x C_x + \bar{a}_y C_y + \bar{a}_z C_z$,

where $C_x^2 + C_y^2 + C_z^2 = 1.$ ①

For $\bar{C}\perp\bar{A}$: $\bar{C}\cdot\bar{A} = 0 \longrightarrow C_x - 2C_y + 3C_z = 0.$ ②

For $\bar{C}\perp\bar{B}$: $\bar{C}\cdot\bar{B} = 0 \longrightarrow C_x + C_y - 2C_z = 0.$ ③

Solving ①, ②, and ③ simultaneously, we obtain

$$C_x = \frac{1}{\sqrt{35}}, \quad C_y = \frac{5}{\sqrt{35}}, \quad C_z = \frac{3}{\sqrt{35}},$$

and $$\bar{C} = \frac{1}{\sqrt{35}}(\bar{a}_x + \bar{a}_y 5 + \bar{a}_z 3).$$

P.2-3 For $\bar{A} \parallel \bar{B}$ everywhere, $\bar{A}\times\bar{B} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = 0,$

which requires that $\frac{A_x}{B_x} = \frac{A_y}{B_y} = \frac{A_z}{B_z}.$

P.2-4 From $\bar{A}\cdot\bar{B} = \bar{A}\cdot\bar{C}$ we have $\bar{A}\cdot(\bar{B}-\bar{C}) = 0$. ①

From $\bar{A}\times\bar{B} = \bar{A}\times\bar{C}$ we have $\bar{A}\times(\bar{B}-\bar{C}) = 0$. ②

① implies $\bar{A} \perp (\bar{B}-\bar{C})$, and ② implies $\bar{A} \parallel (\bar{B}-\bar{C})$.

Since A is not a null vector, ① and ② cannot hold at the same time unless $(\bar{B}-\bar{C})$ is a null vector. Thus, $\bar{B}-\bar{C}=0$, or $\bar{B}=\bar{C}$.

P.2-5 Expand $\bar{A}\times(\bar{A}\times\bar{X}) = \bar{A}(\bar{A}\cdot\bar{X}) - \bar{X}(\bar{A}\cdot\bar{A})$,

or $\bar{A}\times\bar{B} = p\bar{A} - A^2\bar{X}$.

$$\therefore \quad \bar{X} = \frac{1}{A^2}(p\bar{A} + \bar{B}\times\bar{A}).$$

P.2-6 Position vectors of the three corners:

$\overrightarrow{OP_1} = \bar{a}_y - \bar{a}_z 2$, $\quad \overrightarrow{OP_2} = \bar{a}_x 4 - \bar{a}_y - \bar{a}_z 3$, $\quad \overrightarrow{OP_3} = \bar{a}_x 6 + \bar{a}_y 2 + \bar{a}_z 5$

Vectors representing the three sides of the triangle:

$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \bar{a}_x 4 - \bar{a}_z$, $\overrightarrow{P_2P_3} = \bar{a}_x 2 + \bar{a}_y + \bar{a}_z 3$, $\overrightarrow{P_3P_1} = -\bar{a}_x 6 - \bar{a}_y - \bar{a}_z 7$.

a) $\overrightarrow{P_1P_2}\cdot\overrightarrow{P_2P_3} = 0$. $\quad \therefore \triangle P_1P_2P_3$ is a right triangle.

b) Area of triangle $= \frac{1}{2}|\overrightarrow{P_1P_2}\times\overrightarrow{P_2P_3}| = 17.1$

P.2-7

$\bar{D}_1 = \bar{B}+\bar{A}$, $\quad \bar{D}_2 = \bar{B}-\bar{A}$.

$\bar{D}_1\cdot\bar{D}_2 = (\bar{B}+\bar{A})\cdot(\bar{B}-\bar{A})$

$= \bar{B}\cdot\bar{B} - \bar{A}\cdot\bar{A} = 0$ for a rhombus.

$\therefore \quad \bar{D}_1 \perp \bar{D}_2$.

P.2-8

Let A, B, and C denote the vertices of a triangle, and B' and C' be the midpoints of sides AB and AC, respectively. The following vector relations hold:

$$\overrightarrow{AB'} = \frac{1}{2}\overrightarrow{AB}, \quad \overrightarrow{AC'} = \frac{1}{2}\overrightarrow{AC}.$$

$$\overrightarrow{B'C'} = \overrightarrow{AC'} - \overrightarrow{AB'} = \frac{1}{2}(\overrightarrow{AC} - \overrightarrow{AB}) = \frac{1}{2}\overrightarrow{BC}.$$

Q.E.D.

P.2-9 $\bar{a}_A = \bar{a}_x \cos\alpha + \bar{a}_y \sin\alpha,$
$\bar{a}_B = \bar{a}_x \cos\beta + \bar{a}_y \sin\beta.$

a) $\bar{a}_A \cdot \bar{a}_B = \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$

b)
$$\bar{a}_B \times \bar{a}_A = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix} = \bar{a}_z(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$
$$= \bar{a}_z \sin(\alpha - \beta).$$

$\therefore \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$

P.2-10

$$\bar{A} + \bar{B} + \bar{C} = 0.$$

$\bar{A}\times:$ $\bar{A}\times\bar{B} = \bar{C}\times\bar{A}.$

$\bar{C}\times:$ $\bar{C}\times\bar{A} = \bar{B}\times\bar{C}.$

$\bar{B}\times:$ $\bar{B}\times\bar{C} = \bar{A}\times\bar{B}.$

Magnitude relations:

$AB\sin\theta_{AB} = CA\sin\theta_{CA} = BC\sin\theta_{BC}.$

Hence,
$$\frac{A}{\sin\theta_{BC}} = \frac{B}{\sin\theta_{CA}} = \frac{C}{\sin\theta_{AB}}. \quad \text{(Law of sines.)}$$

P.2-11

$\bar{r}' = -\bar{r}, \quad r' = r = c.$

$$(\bar{c}-\bar{r}')\cdot(\bar{c}-\bar{r}) = (\bar{c}+\bar{r})\cdot(\bar{c}-\bar{r}) = 0.$$

$$\therefore\ (\bar{c}-\bar{r}') \perp (\bar{c}-\bar{r}).$$

P.2-13 Consider line L_1: $\quad b_1x + b_2y = c$,
which has a slope equal to $-b_1/b_2$. Denote the shifted line passing through the origin and parallel to L_1 as L_1^o: $\quad b_1x + b_2y = 0$.
The position vector of a point (x, y) on L_1^o is

$$\bar{r}_1 = \bar{a}_x x + \bar{a}_y y.$$

If we introduce the vector $\bar{n} = \bar{a}_x b_1 + \bar{a}_y b_2$, we can write the equation of L_1^o as

$$\bar{n}\cdot\bar{r}_1 = 0.$$

Thus the vector $\bar{n}$ is $\perp$ to $\bar{r}_1$, and is normal to both L_1 and L_1^o. It follows that the two lines L_1 and L_2 are perpendicular to each other if and only if their normal vectors $\bar{n}$ and $\bar{n}' = \bar{a}_x b_1' + \bar{a}_y b_2'$ are orthogonal: $\bar{n}\cdot\bar{n}' = 0$, which implies

$$b_1b_1' + b_2b_2' = 0, \quad \text{or} \quad \frac{b_2'}{b_1'} = -\frac{b_1}{b_2};$$

that is, the slopes of lines L_2 and L_1 are the negative reciprocals of each other.

P.2-14 a) Letting the position vector of a point in the plane be

$$\bar{R} = \bar{a}_x x + \bar{a}_y y + \bar{a}_z z$$

and introducing the vector $\bar{N} = \bar{a}_x b_1 + \bar{a}_y b_2 + \bar{a}_z b_3$, we can write the given equation as

$$\bar{R}\cdot\bar{N} = c \ \text{(a constant)}.$$

This shows that the projection of the position vector to any point in the plane on $\bar{N}$ is a constant, and that $\bar{N}$ is a normal vector.

b) $$\bar{a}_N = \frac{\bar{N}}{|\bar{N}|} = \frac{\bar{a}_x b_1 + \bar{a}_y b_2 + \bar{a}_z b_3}{\sqrt{b_1^2 + b_2^2 + b_3^2}}.$$

c) The perpendicular distance from the origin to the plane is
$$\bar{a}_N \cdot \bar{R} = \frac{c}{|\bar{N}|}.$$
For our case, $c = 5$, $|\bar{N}| = \sqrt{3^2 + (-2)^2 + 6^2} = 7$, and $\bar{a}_N \cdot \bar{R} = 5/7$.

P.2-15 $\bar{A}_{P_1} = -\bar{a}_y 3 - \bar{a}_z 2$, $\overrightarrow{OP_1} = -\bar{a}_y 2 + \bar{a}_z 3$.

$\overrightarrow{OP_2} = \bar{a}_x (r\cos\phi) + \bar{a}_y (r\sin\phi) + \bar{a}_z = \bar{a}_x \frac{\sqrt{3}}{2} - \bar{a}_y \frac{3}{2} + \bar{a}_z$.

$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \bar{a}_x \frac{\sqrt{3}}{2} + \bar{a}_y \frac{1}{2} - \bar{a}_z 2$, $|\overrightarrow{P_1P_2}| = \sqrt{5}$.

$\bar{A}_{P_1} \cdot \bar{a}_{P_1P_2} = \bar{A}_{P_1} \cdot \frac{\overrightarrow{P_1P_2}}{|\overrightarrow{P_1P_2}|} = \frac{\sqrt{5}}{2} = 1.12$.

P.2-16 a) $x = r\cos\phi = 4\cos(2\pi/3) = -2$.
$y = r\sin\phi = 4\sin(2\pi/3) = 2\sqrt{3}$.
$z = 3$.

b) $R = (r^2 + z^2)^{1/2} = (4^2 + 3^2)^{1/2} = 5$.
$\theta = \tan^{-1}(r/z) = \tan^{-1}(4/3) = 53.1°$.
$\phi = 2\pi/3 = 120°$.

P.2-17 a) $\bar{E}_P = \bar{a}_R \frac{25}{(-3)^2 + 4^2 + (-5)^2} = \bar{a}_R \frac{1}{2}$.

$(E_P)_x = \frac{1}{2}\left(\frac{-3}{\sqrt{(-3)^2 + 4^2 + (-5)^2}}\right) = -0.212$.

b) $\bar{a}_E = \frac{1}{\sqrt{50}}(-\bar{a}_x 3 + \bar{a}_y 4 - \bar{a}_z 5)$, $\bar{a}_B = \frac{\bar{B}}{B} = \frac{1}{3}(\bar{a}_x 2 - \bar{a}_y 2 + \bar{a}_z)$.

$\theta = \cos^{-1}(\bar{a}_E \cdot \bar{a}_B) = \cos^{-1}\left(-\frac{19}{3\sqrt{50}}\right) = 154°$.

P.2-18 $\bar{a}_R = \bar{a}_x \sin\theta\cos\phi + \bar{a}_y \sin\theta\sin\phi + \bar{a}_z\cos\theta = \dfrac{\bar{a}_x x + \bar{a}_y y + \bar{a}_z z}{\sqrt{x^2+y^2+z^2}}$,

$\bar{a}_\theta = \bar{a}_x\cos\theta\cos\phi + \bar{a}_y\cos\theta\sin\phi - \bar{a}_z\sin\theta = \dfrac{\bar{a}_x xz + \bar{a}_y yz - \bar{a}_z(x^2+y^2)}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}}$,

$\bar{a}_\phi = -\bar{a}_x\sin\phi + \bar{a}_y\cos\phi = \dfrac{-\bar{a}_x y + \bar{a}_y x}{\sqrt{x^2+y^2}}$.

P.2-19

a) $\bar{a}_x\cdot\bar{a}_\phi = -\sin\phi$, b) $\bar{a}_\theta\cdot\bar{a}_y = \cos\theta\sin\phi$, c) $\bar{a}_r\times\bar{a}_x = -\bar{a}_z\sin\phi$.

d) $\bar{a}_R\cdot\bar{a}_r = \sin\theta$. e) $\bar{a}_y\cdot\bar{a}_R = \sin\theta\sin\phi$, f) $\bar{a}_R\cdot\bar{a}_z = \cos\theta$.

g) $\bar{a}_R\times\bar{a}_z = -\bar{a}_\phi\sin\theta$, h) $\bar{a}_\theta\cdot\bar{a}_z = -\sin\theta$. i) $\bar{a}_z\times\bar{a}_\theta = \bar{a}_\phi\cos\theta$.

P.2-20 $\bar{F}\cdot d\bar{\ell} = [\bar{a}_x xy + \bar{a}_y(3x-y^2)]\cdot(\bar{a}_x dx + \bar{a}_y dy)$
$= xy\,dx + (3x-y^2)\,dy$.

a) Along direct path ①. The equation of P_1P_2 is

$$y = \tfrac{3}{2}(x-1).$$

$$\underset{(\text{Path ①})}{\int_{P_1}^{P_2}} \bar{F}\cdot d\bar{\ell} = \underset{(\text{Path ①})}{\int_{P_1}^{P_2}} [xy\,dx + (3x-y^2)dy]$$
$$= \int_5^2 \frac{3}{2}x(x-1)dx + \int_6^3 (2y+3-y^2)\,dy$$
$$= -37 + 27 = -10.$$

b) Along path ②. This path has two straight-line segments. From P_1 to A: $x=5$, $dx=0$, $\bar{F}\cdot d\bar{\ell} = (15-y^2)dy$. From A to P_2: $y=3$, $dy=0$, $\bar{F}\cdot d\bar{\ell} = 3x\,dx$. Hence,

$$\underset{(\text{Path ②})}{\int_{P_1}^{P_2}} \bar{F}\cdot d\bar{\ell} = \int_6^3 (15-y^2)\,dy + \int_5^2 3x\,dx = 18 - 24 = -6.$$
$$\neq \underset{(\text{Path ①})}{\int_{P_1}^{P_2}} \bar{F}\cdot d\bar{\ell} \longrightarrow \text{Vector field } \bar{F} \text{ is not conservative}$$

P.2-21 $\int_{P_1}^{P_2} \bar{E}\cdot d\bar{\ell} = \int_{P_1}^{P_2} (y\,dx + x\,dy)$.

a) $x = 2y^2$, $dx = 4y\,dy$: $\int_{P_1}^{P_2} \bar{E}\cdot d\bar{\ell} = \int_1^2 (4y^2dy + 2y^2dy) = 14$

b) $x = 6y - 4,\ dx = 6\,dy$; $\int_{P_1}^{P_2} \bar{E}\cdot d\bar{\ell} = \int_1^2 [6y\,dy + (6y-4)]dy = 14.$

Equal line integrals along two specific paths do not necessarily imply a conservative field. $\bar{E}$ is a conservative field in this case because $\bar{E} = \nabla(xy + c)$.

P.2-22

$$\begin{bmatrix} E_r \\ E_\phi \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} r\sin\phi \\ r\cos\phi \end{bmatrix}.$$

$$\bar{E} = \bar{a}_r\, r\sin 2\phi + \bar{a}_\phi\, r\cos 2\phi.$$

$$\bar{E}\cdot d\bar{\ell} = r\sin 2\phi\, dr + r^2\cos 2\phi\, d\phi.$$

$P_3(3,4,-1) = P_3(5, 53.1°, -1)$; $\quad P_4(4,-3,-1) = P_4(5, -36.9°, -1)$.

There is no change in $r\,(=5)$ from P_3 to P_4.

$$\therefore \int_{P_3}^{P_4} \bar{E}\cdot d\bar{\ell} = 5^2 \int_{53.1°}^{-36.9°} \cos 2\phi\, d\phi = -24.$$

P.2-23 a) $\nabla V = \left[\bar{a}_x\left(\frac{\pi}{2}\cos\frac{\pi}{2}x\right)\left(\sin\frac{\pi}{3}y\right) + \bar{a}_y\left(\sin\frac{\pi}{2}x\right)\left(\frac{\pi}{3}\cos\frac{\pi}{3}y\right) - \bar{a}_z\left(\sin\frac{\pi}{2}x\right)\left(\sin\frac{\pi}{3}y\right)\right]e^{-z}.$

$$(\nabla V)_P = -\left(\bar{a}_y\frac{\pi}{6} + \bar{a}_z\frac{\sqrt{3}}{2}\right)e^{-3} = -(\bar{a}_y 0.026 + \bar{a}_z 0.043).$$

b) $\overline{PO} = -\bar{a}_x - \bar{a}_y 2 - \bar{a}_z 3$; $\quad \bar{a}_{PO} = -\frac{1}{\sqrt{14}}(\bar{a}_x + \bar{a}_y 2 + \bar{a}_z 3).$

$$\therefore (\nabla V)_P\cdot\bar{a}_{PO} = \frac{1}{\sqrt{14}}\left(\frac{\pi}{3} + \frac{3\sqrt{3}}{2}\right)e^{-3} = 0.0485.$$

P.2-24 On the surface of the sphere, $R = 5$.

$$\oint_S (\bar{a}_R 3\sin\theta)\cdot d\bar{s} = \int_0^{2\pi}\int_0^{\pi} (\bar{a}_R 3\sin\theta)\cdot(\bar{a}_R 5^2\sin\theta)\,d\theta\, d\phi$$
$$= \int_0^{2\pi}\int_0^{\pi} 75\sin^2\theta\, d\theta\, d\phi = 75\pi^2.$$

P.2-25 The first step is to find the expression for the unit normal $\bar{a}_n = \bar{a}_x \ell + \bar{a}_y m + \bar{a}_z p$ to the given surface. The given four corner points of the surface lead to the following four equations:

Corner (0,0,2):
$\ell x + my + p(z-2) = 0$ ①

Corner (2,0,2):
$\ell(x-2) + my + p(z-2) = 0.$ ②

Corner (2,2,0):
$\ell(x-2) + m(y-2) + pz = 0.$ ③

Corner (0,2,0):
$\ell x + m(y-2) + pz = 0.$ ④

The direction cosines satisfy the condition:
$\ell^2 + m^2 + p^2 = 1.$ ⑤

From ①-⑤ we obtain $\ell = 0$, and $m = p = 1/\sqrt{2}$.

Thus, $\bar{a}_n = \frac{1}{\sqrt{2}}(\bar{a}_y + \bar{a}_z)$, $\bar{F}\cdot\bar{a}_n = \frac{5}{\sqrt{2}}$ (a constant),

and $\int_S \bar{F}\cdot d\bar{s} = \frac{5}{\sqrt{2}} S = \frac{5}{\sqrt{2}}(2\times 2\sqrt{2}) = 20.$

P.2-26 In spherical coordinates, $\bar{\nabla}\cdot\bar{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R)$, if $\bar{A} = \bar{a}_R A_R$.

a) $\bar{A} = f_1(\bar{R}) = \bar{a}_R R^n$, $A_R = R^n$.

$$\bar{\nabla}\cdot\bar{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^{n+2}) = (n+2)R^{n-1}.$$

b) $\bar{A} = f_2(\bar{R}) = \bar{a}_R \frac{k}{R^2}$, $A_R = kR^{-2}$.

$$\bar{\nabla}\cdot\bar{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(k) = 0.$$

P.2-27 For radial vector $\bar{R} = \bar{a}_R R$, $\bar{\nabla}\cdot\bar{R} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 R) = 3.$
Using divergence theorem, we have

$$\frac{1}{3}\oint_S \bar{R}\cdot d\bar{s} = \frac{1}{3}\int_V \bar{\nabla}\cdot\bar{R}\, dv = \frac{1}{3}(3V) = V.$$

P.2-29 $\oint_S \bar{A}\cdot d\bar{s} = \left(\int_{\substack{top\\face}} + \int_{\substack{bottom\\face}} + \int_{walls}\right)\bar{A}\cdot d\bar{s}.$

Top face ($z=4$): $\bar{A} = \bar{a}_r r^2 + \bar{a}_z 8$, $\quad d\bar{s} = \bar{a}_z ds$.

$$\int_{\substack{top\\face}} \bar{A}\cdot d\bar{s} = \int_{\substack{top\\face}} 8\, ds = 8(\pi 5^2) = 200\pi.$$

Bottom face ($z=0$): $\bar{A} = \bar{a}_r r^2$; $\quad d\bar{s} = -\bar{a}_z ds$.

$$\int_{\substack{bottom\\face}} \bar{A}\cdot d\bar{s} = 0.$$

Walls ($r=5$): $\bar{A} = \bar{a}_r 25 + \bar{a}_z 2z$, $\quad d\bar{s} = \bar{a}_r ds$.

$$\int_{walls} \bar{A}\cdot d\bar{s} = 25\int_{walls} ds = 25(2\pi 5\times 4) = 1000\pi.$$

$$\therefore \oint_S \bar{A}\cdot d\bar{s} = 200\pi + 0 + 1000\pi = 1{,}200\pi.$$

$\bar{\nabla}\cdot\bar{A} = 3r+2$, $\displaystyle\int_V \bar{\nabla}\cdot\bar{A}\, dv = \int_0^4\int_0^{2\pi}\int_0^5 \bar{\nabla}\cdot\bar{A}\; r\,dr\,d\phi\,dz = 1{,}200\pi = \oint_S \bar{A}\cdot d\bar{s}.$

<u>P.2-30</u> $\quad \bar{\nabla}\cdot\bar{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{\partial}{\partial z}F_z = k_2.$

$$\int_V \bar{\nabla}\cdot\bar{F}\, dv = k_2 V = k_2(\pi 2^2\times 6) = 24\pi k_2 \neq \oint_S \bar{A}\cdot d\bar{s}.$$

Divergence theorem fails here because $\bar{F}$ has a singularity inside the volume at $r=0$.

<u>P.2-31</u> $\quad$ Eq. (2-98): $\quad \bar{\nabla}\cdot\bar{A} = \lim_{\Delta v\to 0}\dfrac{\oint_S \bar{A}\cdot d\bar{s}}{\Delta v}$. $\quad$ ①

Referring to Fig. 2-15, we note that the areas on the opposite sides of a differential volume in cylindrical coordinates are the same in ϕ- and z-directions, but are different in the r-direction. Let us first evaluate the contributions to $\oint_S \bar{A}\cdot d\bar{s}$ of the inside and outside faces. On the inside face:

$$\int_{\substack{inside\\face}} \bar{A}\cdot d\bar{s} = \bar{A}_{\substack{inside\\face}}\cdot \Delta\bar{s}_{\substack{inside\\face}} = -A_r(r_0 - \tfrac{\Delta r}{2}, \phi_0, z_0)(r_0 - \tfrac{\Delta r}{2})\Delta\phi\,\Delta z$$

$$= -\left[A_r(r_0,\phi_0,z_0) - \frac{\Delta r}{2}\frac{\partial A_r}{\partial r}\bigg|_{(r_0,\phi_0,z_0)} + H.O.T.\right](r_0 - \tfrac{\Delta r}{2})\Delta\phi\,\Delta z. \quad ②$$

On the outside face:

$$\int_{\substack{\text{outside}\\\text{face}}} \bar{A}\cdot d\bar{s} = A_r\left(r_0+\frac{\Delta r}{2}, \phi_0, z_0\right)\left(r_0+\frac{\Delta r}{2}\right)\Delta\phi\,\Delta z$$

$$= \left[A_r(r_0,\phi_0,z_0) + \frac{\Delta r}{2}\frac{\partial A_r}{\partial r}\bigg|_{(r_0,\phi_0,z_0)} + H.O.T.\right]\left(r_0+\frac{\Delta r}{2}\right)\Delta\phi\,\Delta z. \quad ②$$

Adding ③ and ②, we have

$$\left[\int_{\substack{\text{inside}\\\text{face}}} + \int_{\substack{\text{outside}\\\text{face}}}\right]\bar{A}\cdot d\bar{s} = \left(A_r + r_0\frac{\partial A_r}{\partial r}\right)\bigg|_{(r_0,\phi_0,z_0)} \Delta r\,\Delta\phi\,\Delta z + H.O.T.$$

$$= \frac{\partial}{\partial r}(rA_r)\bigg|_{(r_0,\phi_0,z_0)} \Delta r\,\Delta\phi\,\Delta z + H.O.T., \quad ④$$

where H.O.T. contain second and higher powers of Δr. The sum of the contributions of the front and back faces (differential area $= \Delta r\,\Delta z$) is

$$\left[\int_{\substack{\text{front}\\\text{face}}} + \int_{\substack{\text{back}\\\text{face}}}\right]\bar{A}\cdot d\bar{s} = \frac{\partial A_\phi}{\partial \phi}\bigg|_{(r_0,\phi_0,z_0)} \Delta r\,\Delta\phi\,\Delta z + H.O.T., \quad ⑤$$

where H.O.T. contain second and higher powers of $\Delta\phi$. Similarly, the sum of the contributions of the top and bottom faces (differential area $= r_0\,\Delta r\,\Delta\phi$) is

$$\left[\int_{\substack{\text{top}\\\text{face}}} + \int_{\substack{\text{bottom}\\\text{face}}}\right]\bar{A}\cdot d\bar{s} = \left(r\frac{\partial A_z}{\partial z}\right)\bigg|_{(r_0,\phi_0,z_0)} \Delta r\,\Delta\phi\,\Delta z + H.O.T., \quad ⑥$$

where H.O.T. contain second and higher powers of Δz.

Combining ④, ⑤, and ⑥ in ①, dividing by $\Delta v = r_0\,\Delta r\,\Delta\phi\,\Delta z$, and letting $\Delta r\,\Delta\phi\,\Delta z \to 0$, we get

$$\bar{\nabla}\cdot\bar{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z},$$

where the subscript 0 has been dropped for simplicity.

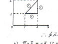
P.2-33 a) $\bar{D} = \bar{a}_R \frac{\cos^2\phi}{R^3}$, $ds = R^2 \sin\theta\, d\theta\, d\phi$.

$$\oint \bar{D}\cdot d\bar{s} = \int_0^{2\pi}\int_0^{\pi}\left(\frac{1}{2}-1\right)\sin\theta\, d\theta\, \cos^2\phi\, d\phi = -\pi.$$

b) $\bar{\nabla}\cdot\bar{D} = -\frac{\cos^2\phi}{R^4}$, $dv = R^2 \sin\theta\, dR\, d\theta\, d\phi$.

$$\int_V \bar{\nabla}\cdot\bar{D}\, dv = \int_0^{2\pi}\int_0^{\pi}\int_1^2\left(-\frac{\cos^2\phi}{R^2}\right)\sin\theta\, dR\, d\theta\, d\phi = -\pi.$$

P.2-34 $\bar{A} = \bar{a}_x 3x^2y^3 - \bar{a}_y x^3y^2$; $d\bar{\ell} = \bar{a}_x dx + \bar{a}_y dy$.



a) $\bar{A}\cdot d\bar{\ell} = 3x^2y^3 dx - x^3y^2 dy$.

Path ①: $x = y$, $\int_{①} \bar{A}\cdot d\bar{\ell} = \int_1^2 2x^5 dx = 21$

Path ②: $x = 2, dx = 0$, $\int_{②} \bar{A}\cdot d\bar{\ell} = \int_2^1 (-2^3y^2)dy = \frac{56}{3}$

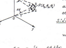
Path ③: $y = 1, dy = 0$, $\int_{③} \bar{A}\cdot d\bar{\ell} = \int_2^1 3x^2 dx = -7$

$$\therefore \oint \bar{A}\cdot d\bar{\ell} = 21 + \frac{56}{3} - 7 = \frac{98}{3} = 32\tfrac{2}{3}.$$

b) $\bar{\nabla}\times\bar{A} = -\bar{a}_z 12x^2y^2$, $d\bar{s} = -\bar{a}_z dx\, dy$.

$$\int_S (\bar{\nabla}\times\bar{A})\cdot d\bar{s} = 12\int_1^2 x^2 dx \int_1^x y^2 dy = 32\tfrac{2}{3}.$$

P.2-35 Eq. (2-126): $(\bar{\nabla}\times\bar{A})_R = \lim_{\Delta s_R \to 0} \frac{1}{\Delta s_R}\left(\oint_{C_R} \bar{A}\cdot d\bar{\ell}\right)$. ①



where $\Delta s_R \doteq R_0^2 \sin\theta_0 \Delta\theta\Delta\phi$ ②
and the contour consists of the four sides 1, 2, 3, & 4.

Side 1: $d\bar{\ell} = \bar{a}_\theta (R_0\Delta\theta)$,

$\bar{A}\cdot d\bar{\ell} = A_\theta(R_0,\theta_0,\phi_0 - \frac{\Delta\phi}{2})R_0\Delta\theta$,

where $A_\theta(R_0,\theta_0,\phi_0 - \frac{\Delta\phi}{2}) = A_\theta(R_0,\theta_0,\phi_0) - \frac{\Delta\phi}{2}\frac{\partial A_\theta}{\partial\phi}\Big|_{(R_0,\theta_0,\phi_0)} + H.O.T.$

$$\int_{\text{side 1}} \bar{A}\cdot d\bar{\ell} = \left\{\left[A_\theta - \frac{\Delta\phi}{2}\frac{\partial A_\theta}{\partial\phi}\right]_{(R_0,\theta_0,\phi_0)} + H.O.T.\right\} R_0\Delta\theta.$$ ③

Side 3: $d\bar{\ell} = -\bar{a}_\theta(R_0\Delta\theta)$, $\bar{A}\cdot d\bar{\ell} = A_\theta(R_0,\theta_0,\phi_0+\frac{\Delta\phi}{2})R_0\Delta\theta$,

$$\int_{side\,3}\bar{A}\cdot d\bar{\ell} = -\left\{\left[A_\theta + \frac{\Delta\phi}{2}\frac{\partial A_\theta}{\partial\phi}\right]_{(R_0,\theta_0,\phi_0)} + H.O.T.\right\}R_0\Delta\theta. \quad ④$$

Combining ① and ④:

$$\int_{sides\,1\,\&\,3}\bar{A}\cdot d\bar{\ell} = \left(-\frac{\partial A_\theta}{\partial\phi} + H.O.T.\right)\bigg|_{(R_0,\theta_0,\phi_0)} R_0\Delta\theta\Delta\phi. \quad ⑤$$

Side 2: $d\bar{\ell} = \bar{a}_\phi R_0\sin(\theta_0+\frac{\Delta\theta}{2})\Delta\phi$, $\bar{A}\cdot d\bar{\ell} = A_\phi(R_0,\theta_0+\frac{\Delta\theta}{2},\phi_0)R_0\sin(\theta_0+\frac{\Delta\theta}{2})\Delta\phi$,

$$\int_{side\,2}\bar{A}\cdot d\bar{\ell} \cong \left\{\left[A_\phi + \frac{\Delta\theta}{2}\frac{\partial A_\phi}{\partial\theta}\right]_{(R_0,\theta_0,\phi_0)} + H.O.T.\right\}R_0\left(\sin\theta_0 + \frac{\Delta\theta}{2}\cos\theta_0\right)\Delta\phi. \quad ⑥$$

Side 4:

$$\int_{side\,4}\bar{A}\cdot d\bar{\ell} = -\left\{\left[A_\phi - \frac{\Delta\theta}{2}\frac{\partial A_\phi}{\partial\theta}\right]_{(R_0,\theta_0,\phi_0)} + H.O.T.\right\}R_0\left(\sin\theta_0 - \frac{\Delta\theta}{2}\cos\theta_0\right)\Delta\phi. \quad ⑦$$

Combining ⑥ and ⑦:

$$\int_{sides\,2\,\&\,4}\bar{A}\cdot d\bar{\ell} = \frac{\partial A_\phi}{\partial\theta}\bigg|_{(R_0,\theta_0,\phi_0)} R_0\sin\theta_0\Delta\theta\Delta\phi + A_\phi R_0\cos\theta_0\Delta\theta\Delta\phi + H.O.T.$$

$$= \frac{\partial}{\partial\theta}(A_\phi\sin\theta)\bigg|_{(R_0,\theta_0,\phi_0)} R_0\Delta\theta\Delta\phi + H.O.T. \quad ⑧$$

Substituting ⑤, ⑥, and ⑧ in ①, we obtain

$$(\bar{\nabla}\times\bar{A})_R = \frac{1}{R\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right],$$

where the subscript 0 has been dropped for simplicity.

P.2-36 $\bar{\nabla}\times\bar{A} = \frac{1}{R\sin\theta}\left(\bar{a}_R\cos\theta\sin\frac{\phi}{2} - \bar{a}_\theta\sin\theta\sin\frac{\phi}{2}\right)$.

$$\int_S(\bar{\nabla}\times\bar{A})\cdot d\bar{s} = \int_0^{2\pi}\int_0^{\pi/2}(\bar{\nabla}\times\bar{A})_{R=b}\cdot(\bar{a}_R b^2\sin\theta\,d\theta\,d\phi) = 4b.$$

$$\oint_C \bar{A}\cdot d\bar{\ell} = \int_0^{2\pi}(\bar{A})_{\substack{R=b\\ \theta=\pi/2}}\cdot(\bar{a}_\phi b\,d\phi) = \int_0^{2\pi} b\sin\frac{\phi}{2}\,d\phi = 4b.$$

P.2-39. $\bar{F} = \bar{a}_x(x+c_1z) + \bar{a}_y(c_2x-3z) + \bar{a}_z(x+c_4y+c_5z)$.

a) $\bar{F}$ irrotational $\longrightarrow \bar{\nabla}\times\bar{F} = 0$,

or

$$\bar{a}_x\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \bar{a}_y\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \bar{a}_z\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial F_y}\right) = 0.$$

which gives three equations:

$$\frac{\partial}{\partial y}(x+c_3y+c_4z)-\frac{\partial}{\partial z}(c_2x-3z)=0 \longrightarrow c_3+3=0 \longrightarrow c_3=-3$$

$$\frac{\partial}{\partial z}(x+c_1z)-\frac{\partial}{\partial x}(x+c_3y+c_4z)=0 \longrightarrow c_1-1=0 \longrightarrow c_1=1.$$

$$\frac{\partial}{\partial x}(c_2x-3z)-\frac{\partial}{\partial y}(x+c_1z)=0 \longrightarrow c_2=0.$$

b) $\bar{F}$ also solenoidal $\longrightarrow \bar{\nabla}\cdot F=0,$

or

$$\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z}=0.$$

or

$$\frac{\partial}{\partial x}(x+c_1z)+\frac{\partial}{\partial y}(c_2x-3z)+\frac{\partial}{\partial z}(x+c_3y+c_4z)=0,$$

or

$$1+c_4=0 \longrightarrow c_4=-1.$$

c) $\bar{F}=-\bar{\nabla}V \longrightarrow \bar{a}_x(x+z)-\bar{a}_y 3z+\bar{a}_z(x-3y-z)$

$$=-a_x\frac{\partial V}{\partial x}-a_y\frac{\partial V}{\partial y}-a_z\frac{\partial V}{\partial z}$$

$$\frac{\partial V}{\partial x}=-(x+z) \longrightarrow V=-\frac{x^2}{2}-xz+f_1(y,z)$$

$$\frac{\partial V}{\partial y}=3z \longrightarrow V=3yz+f_2(x,z).$$

$$\frac{\partial V}{\partial z}=-x+3y+z \longrightarrow V=-xz+3yz+\frac{z^2}{2}+f_3(x,y).$$

$$\therefore\ V=-\frac{x^2}{2}-xz+3yz+\frac{z^2}{2}.$$

Chapter 4

Solution of Electrostatic Problems

P.4-1 Use subscripts d and a to denote dielectric and air regions respectively. $\nabla^2 V = 0$ in both regions.

$$V_d = c_1 y + c_2,\quad \bar{E}_d = -\bar{a}_y c_1,\quad \bar{D}_d = -\bar{a}_y \epsilon_0\epsilon_r c_1.$$

$$V_a = c_3 y + c_4,\quad \bar{E}_a = -\bar{a}_y c_3,\quad \bar{D}_a = -\bar{a}_y \epsilon_0 c_3.$$

B.C.: At $y=0$, $V_d = 0$; at $y=d$, $V_a = V_0$;
at $y = 0.8d$: $V_d = V_a$, $\bar{D}_d = \bar{D}_a$.

Solving: $c_1 = \frac{V_0}{(0.8+0.2\epsilon_r)d}$, $c_2 = 0$, $c_3 = \frac{\epsilon_r V_0}{(0.8+0.2\epsilon_r)d}$, $c_4 = \frac{(1-\epsilon_r)V_0}{1+0.25\epsilon_r}$.

a) $V_d = \frac{5yV_0}{(4+\epsilon_r)d}$, $\bar{E}_d = -\bar{a}_y \frac{5V_0}{(4+\epsilon_r)d}$.

b) $V_a = \frac{5\epsilon_r y - 4(\epsilon_r - 1)d}{(4+\epsilon_r)d}V_0$, $\bar{E}_a = -\bar{a}_y \frac{5\epsilon_r V_0}{(4+\epsilon_r)d}$.

c) $(\rho_s)_{y=d} = -(D_a)_{y=d} = \frac{5\epsilon_0\epsilon_r V_0}{(4+\epsilon_r)d}$.

$(\rho_s)_{y=0} = (D_d)_{y=0} = -\frac{5\epsilon_0\epsilon_r V_0}{(4+\epsilon_r)d}$.

P.4-3 At a point where V is a maximum (minimum) the second derivatives of V with respect to x, y, and z would all be negative (positive); their sum could not vanish, as required by Laplace's equation.

P.4-6 Poisson's eq. $\nabla^2 V = -\frac{A}{\epsilon r} \longrightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = -\frac{A}{\epsilon r}$.

Solution: $V = -\frac{A}{\epsilon}r + c_1 \ln r + c_2$.

B.C.: $\begin{cases} \text{At } r=a, & V_0 = -\frac{A}{\epsilon}a + c_1\ln a + c_2. \\ \text{At } r=b, & 0 = -\frac{A}{\epsilon}b + c_1 \ln b + c_2. \end{cases}$ $c_1 = \frac{\frac{A}{\epsilon}(b-a) - V_0}{\ln(b/a)}$, $c_2 = \frac{V_0 \ln b + \frac{A}{\epsilon}(a\ln b - b\ln a)}{\ln(b/a)}$.

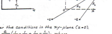
P.4-7

$$\bar{E}\Big|_{y=0} = -\bar{a}_y \frac{Q}{4\pi\epsilon R^2}(2\sin\theta) = -\bar{a}_y \frac{Qd}{2\pi\epsilon(d^2+r^2)^{3/2}}.$$

a) $\rho_s = \bar{a}_y \cdot \epsilon\bar{E}\Big|_{y=0} = -\frac{Qd}{2\pi(d^2+r^2)^{3/2}}$

b) $\int_0^\infty \rho_s\, 2\pi r\, dr = -Q$.

P.4-8



Consider the conditions in the xy-plane ($z=0$).

a) $V_P = \frac{Q}{4\pi\epsilon}\left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{1}{R_3} - \frac{1}{R_4}\right)$, where

$R_1 = [(x-d_1)^2 + (y-d_2)^2]^{1/2}$, $\quad R_2 = [(x-d_1)^2 + (y+d_2)^2]^{1/2}$,

$R_3 = [(x+d_1)^2 + (y+d_2)^2]^{1/2}$, $\quad R_4 = [(x+d_1)^2 + (y-d_2)^2]^{1/2}$.

$$\bar{E}_P = -\bar{\nabla} V_P = -\bar{a}_x \frac{\partial V_P}{\partial x} - \bar{a}_y \frac{\partial V_P}{\partial y}$$

$$= \bar{a}_x \frac{Q}{4\pi\epsilon}\left[-\frac{x-d_1}{R_1^3} + \frac{x-d_1}{R_2^3} - \frac{x+d_1}{R_3^3} + \frac{x+d_1}{R_4^3}\right]$$

$$+ \bar{a}_y \frac{Q}{4\pi\epsilon}\left[-\frac{y-d_2}{R_1^3} + \frac{y+d_2}{R_2^3} - \frac{y+d_2}{R_3^3} + \frac{y-d_2}{R_4^3}\right].$$

E_P will have a z-component if the point P does not lie in the xy-plane.

b) On the conducting half-planes, $\rho_s = D_n = \epsilon E_n$.
Along the x-axis, $y=0$: $R_1 = [(x-d_1)^2 + d_2^2]^{1/2} = R_2$,
and $R_3 = [(x+d_1)^2 + d_2^2]^{1/2} = R_4$.

$E_x = 0$, $E_y = \frac{Q}{2\pi\epsilon}\left[\frac{d_2}{R_1^3} - \frac{d_2}{R_3^3}\right]$.

$$\therefore\ \rho_s(y=0) = \frac{Qd_2}{2\pi}\left\{\frac{1}{[(x-d_1)^2+d_2^2]^{3/2}} - \frac{1}{[(x+d_1)^2+d_2^2]^{3/2}}\right\}$$

$$= \begin{cases} 0, & \text{at } x=0. \\ \text{max.}, & \text{at } x=d_1. \end{cases}$$

Similarly for $\rho_s(x=0)$ on the vertical conducting conducting plane by changing x to y and $d_1 \leftrightarrow d_2$.

P.4-10 Refer to Example 4-4.

$$C' = \frac{2\pi\epsilon_0}{\ln\left[(h/a) + \sqrt{(h/a)^2 - 1}\right]} = \frac{2\pi\epsilon_0}{\cosh^{-1}(h/a)} \quad (\text{F/m}).$$

P.4-11 Same as C_{12} in problem P.3-38.

P.4-12 a) From Eqs. (4-48) and (4-49):

$$V_P = \frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{\sqrt{(x+b)^2+y^2}}{\sqrt{(x-b)^2+y^2}}.$$

$$\bar{E}_P = -\bar{a}_x\frac{\partial V_P}{\partial x} - \bar{a}_y\frac{\partial V_P}{\partial y} = \frac{\rho_\ell}{2\pi\epsilon_0}\left\{\frac{\bar{a}_x 2b(y^2+b^2-x^2) - \bar{a}_y 4bxy}{[(x+b)^2+y^2][y^2+(x-b)^2]}\right\}.$$

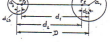
b) Equation for lines everywhere tangent to the electric field lines is obtained by requiring

$$\frac{dy}{dx} = \frac{E_y}{E_x} = -\frac{2xy}{y^2+b^2-x^2}.$$

Which reduces to $\frac{d(x^2+y^2)}{(x^2+y^2)-b^2} = \frac{dy}{y}$.

Integrating, we obtain $x^2+y^2-2Ky = b^2$,
or $x^2+(y-K)^2 = b^2+K^2$,
where K is a constant. —Circles of radii $\sqrt{b^2+K^2}$ having centers at $(0, K)$.

P.4-13



$$V_1 = -\frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{a_1}{d_1}, \quad V_2 = \frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{a_2}{d_2}.$$

Capacitance per unit length

$$C' = \frac{\rho_\ell}{V_1-V_2} = \frac{2\pi\epsilon_0}{\ln\frac{d_1d_2}{a_1a_2}}.$$

Four equations:

$a_1^2 = d_{i1}d_1$, $a_2^2 = d_{i2}d_2$,
$d_1 + d_{i2} = D$, $d_2 + d_{i1} = D$.

We obtain

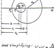
$$\frac{d_1d_2}{a_1a_2} = \frac{a_1a_2}{d_{i1}d_{i2}} \quad \text{and} \quad a_1^2+a_2^2+d_1d_2+d_{i1}d_{i2} = D^2.$$

$$\frac{d_1d_2}{a_1a_2} = \frac{D^2}{2a_1a_2} - \frac{a_1}{2a_2} - \frac{a_2}{2a_1} + \sqrt{\left(\frac{D^2}{2a_1a_2} - \frac{a_1}{2a_2} - \frac{a_2}{2a_1}\right)^2 - 1}.$$

$$\therefore \quad C' = \frac{2\pi\epsilon_0}{\ln\left[\frac{1}{2}\left(\frac{D^2}{a_1a_2} - \frac{a_1}{a_2} - \frac{a_2}{a_1}\right) + \sqrt{\frac{1}{4}\left(\frac{D^2}{a_1a_2} - \frac{a_1}{a_2} - \frac{a_2}{a_1}\right)^2 - 1}\right]}$$

$$= \frac{2\pi\epsilon_0}{\cosh^{-1}\left[\frac{1}{2}\left(\frac{D^2}{a_1a_2} - \frac{a_1}{a_2} - \frac{a_2}{a_1}\right)\right]} \quad (F/m).$$

P. 4-14 Eq. (4-61): $c_1 = \frac{1}{2D}(a_2^2 - a_1^2 - D^2)$; Eq. (4-62): $c_2 = \frac{1}{2D}(a_2^2 - a_1^2 + D^2)$.



Eq. (4-55): $b^2 = c_1^2 - a_1^2$;

Eq. (4-56): $b^2 = c_2^2 - a_2^2$.

a) $V = \frac{\rho_\ell}{2\pi\epsilon_0} \ln \frac{r_2}{r_1}$ (r_2 — Distance to $-\rho_\ell$; r_1 — Distance to $+\rho_\ell$.)

At P_1: $r_2 = b + (c_1 - a_1)$, $r_1 = b - (c_1 - a_1)$.

At P_2: $r_2 = b + (c_2 - a_2)$, $r_1 = b - (c_2 - a_2)$.

$$V_1 - V_2 = \frac{\rho_\ell}{2\pi\epsilon_0} \ln \left[\frac{b + (c_1 - a_1)}{b - (c_1 - a_1)} \cdot \frac{b - (c_2 - a_2)}{b + (c_2 - a_2)}\right].$$

Expressing b, c_1 & c_2 in terms of D, a_1 & a_2 and simplifying:

$$V_1 - V_2 = \frac{\rho_\ell}{2\pi\epsilon_0} \ln\left\{\left(\frac{a_1^2 + a_2^2 - D^2}{2a_1a_2}\right) + \left[\left(\frac{a_1^2 + a_2^2 - D^2}{2a_1a_2}\right)^2 - 1\right]^{1/2}\right\}$$

$$C' = \frac{\rho_\ell}{V_1 - V_2} = \frac{2\pi\epsilon_0}{\ln\left\{\left(\frac{a_1^2 + a_2^2 - D^2}{2a_1a_2}\right) + \left[\left(\frac{a_1^2 + a_2^2 - D^2}{2a_1a_2}\right)^2 - 1\right]^{1/2}\right\}} = \frac{2\pi\epsilon_0}{\cosh^{-1}\left(\frac{a_1^2 + a_2^2 - D^2}{2a_1a_2}\right)} \quad \text{(F/m)}.$$

b) Force per unit length $F' = \frac{\rho_\ell^2}{2\pi\epsilon_0 (4b^2)} = \frac{D^2 \rho_\ell^2}{2\pi\epsilon_0 [(a_1^2 + a_2^2 - D^2)^2 - 4a_1^2 D^2]}$ (N/m)

P. 4-15

$Q_i = -\frac{b}{d}Q$, $\quad d_i = \frac{b^2}{d}$.

a) $V_P = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_Q} - \frac{b}{dR_i}\right)$;

$R_Q = (R^2 + d^2 - 2Rd\cos\theta)^{1/2}$,

$R_i = (R^2 + d_i^2 - 2Rd_i\cos\theta)^{1/2}$.

b) $\rho_s = -\epsilon_0 \frac{\partial V}{\partial R}\Big|_{R=b} = -\frac{Q(b^2 - d^2)}{4\pi b(b^2 + d^2 - 2bd\cos\theta)^{3/2}}$.

P. 4-16 (See next page.)

P. 4-17 Required boundary conditions at $x=0$: $V_1 = V_2$, and $\epsilon_1 \frac{\partial V_1}{\partial x} = \epsilon_2 \frac{\partial V_2}{\partial x}$.

From Fig. 4-23 and the hypotheses in parts a) and b):

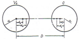
$$V_1 = \frac{Q}{4\pi\epsilon_1\sqrt{(x-d)^2 + y^2 + z^2}} + \frac{Q_1}{4\pi\epsilon_1\sqrt{(x+d)^2 + y^2 + z^2}},$$

$$V_2 = \frac{Q + Q_2}{4\pi\epsilon_2\sqrt{(d-x)^2 + y^2 + z^2}}.$$

In order to satisfy the b.c.'s at $x=0$, we require

$$\frac{Q - Q_1}{\epsilon_1} = \frac{Q + Q_2}{\epsilon_2} \text{ and } Q + Q_1 = Q + Q_2 \longrightarrow Q_1 = Q_2 = \frac{\epsilon_1 - \epsilon_2}{\epsilon_2 + \epsilon_1}Q.$$

P.4-16



a)
Q_0 and system of image charges:

In left sphere	In right sphere
Q_0 at $d_0 = 0$.	$-Q_1 = -\frac{a}{D}Q_0$ at d_1.
$Q_2 = \frac{a^2}{D(D-d_1)}Q_0$ at d_2.	$-Q_3 = -\frac{a}{D-d_2}Q_2 = -\frac{a^3}{D(D-d_1)(D-d_2)}Q_0$ at d_3.
$Q_4 = \frac{a^4}{D(D-d_1)(D-d_2)(D-d_3)}Q_0$ at d_4.	⋮
⋮	⋮
$Q_{2n} = Q_0 \prod_{m=1}^{n} \frac{a}{D-d_{m-1}}$ at d_{2n}. $(n = 2, 4, 6, \cdots)$	$-Q_{2n} = -Q_0 \prod_{m=1}^{n} \frac{a}{D-d_{m-1}}$ at d_n. $(n = 1, 3, 5, \cdots)$

$d_m = \frac{a^2}{D-d_{m-1}}$, $m = 1, 2, 3, \cdots$; $d_0 = 0$.

b) $C = \frac{Q_0 + \sum Q_{2n}}{V_0} = 4\pi\epsilon_0 a \left[1 + \sum_{n=2,4,\cdots} \left(\prod_{m=1}^{n} \frac{a}{D-d_{m-1}}\right)\right]$.

P.4-18 a) $V(x,y) = c_1 xy$. Required boundary conditions:

① $V(0,y) = 0$ → Grounded conducting plane at $x=0$.

② $V(x,0) = 0$ → Grounded conducting plane at $y=0$.

If $V(a,b) = c_1 ab = V_0$ → $c_1 = \frac{V_0}{ab}$ → $V(x,y) = \frac{V_0}{ab}xy$

③ Shape of curved boundary defined by:
$V(x,y) = V_0 = \left(\frac{V_0}{ab}\right)xy$, or $xy = ab$ (a hyperbola).

b) $V(x,y) = c_2 \sin kx \sinh ky$. Required boundary conditions:

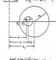
① $V(0,y) = 0$ → Grounded plane at $x=0$.

② $V(\pi/k, y) = 0$ → Grounded plane at $x = \pi/k = b$.

③ $V(x,0) = 0$ → Grounded plane at $y=0$.

④ If $V(b/2, a) = c_2 \sin\left(\frac{\pi}{b}\right)\left(\frac{b}{2}\right)\sinh\left(\frac{\pi}{b}\right)a = c_2 \sinh(a\pi/b) = V_0$,
→ $V(x,y) = \frac{V_0}{\sinh(a\pi/b)} \sin(\pi x/b)\sinh(\pi y/b)$. Curved boundary: $\sinh\left(\frac{\pi}{b}y\right) = \frac{\sinh(a\pi/b)}{\sin(\pi x/b)}$

P.4-14 Eq. (4-61): $c_1 = \frac{1}{2D}(a_2^2 - a_1^2 - D^2)$; Eq. (4-62): $c_2 = \frac{1}{2D}(a_2^2 - a_1^2 + D^2)$.



Eq. (4-55): $b^2 = c_1^2 - a_1^2$;

Eq. (4-56): $b^2 = c_2^2 - a_2^2$.

a) $V = \frac{\rho_\ell}{2\pi\epsilon_0} \ln \frac{r_2}{r_1}$ (r_2 — Distance to $-\rho_\ell$; r_1 — Distance to ρ_ℓ.)

At P_1: $r_2 = b + (c_1 - a_1)$, $r_1 = b - (c_1 - a_1)$.

At P_2: $r_2 = b + (c_2 - a_2)$, $r_1 = b - (c_2 - a_2)$.

$$V_1 - V_2 = \frac{\rho_\ell}{2\pi\epsilon_0} \ln\left[\frac{b + (c_1 - a_1)}{b - (c_1 - a_1)} \, \frac{b - (c_2 - a_2)}{b + (c_2 - a_2)}\right].$$

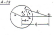
Expressing b, c_1 & c_2 in terms of D, a_1 & a_2 and simplifying:

$$V_1 - V_2 = \frac{\rho_\ell}{2\pi\epsilon_0} \ln\left\{\left(\frac{a_1^2 + a_2^2 - D^2}{2a_1a_2}\right) + \left[\left(\frac{a_1^2 + a_2^2 - D^2}{2a_1a_2}\right)^2 - 1\right]^{1/2}\right\}$$

$$C' = \frac{\rho_\ell}{V_1 - V_2} = \frac{2\pi\epsilon_0}{\ln\left\{\left(\frac{a_1^2 + a_2^2 - D^2}{2a_1a_2}\right) + \left[\left(\frac{a_1^2 + a_2^2 - D^2}{2a_1a_2}\right)^2 - 1\right]^{1/2}\right\}} = \frac{2\pi\epsilon_0}{\cosh^{-1}\left(\frac{a_1^2 + a_2^2 - D^2}{2a_1a_2}\right)} \quad (\text{F/m}).$$

b) Force per unit length $F' = \frac{\rho_\ell^2}{2\pi\epsilon_0(4b^2)} = \frac{D^2\rho_\ell^2}{2\pi\epsilon_0[(a_1^2 + a_2^2 - D^2)^2 - 4a_1^2 D^2]}$ (N/m).

P.4-15



$Q_i = -\frac{b}{d}Q$, $\quad d_i = \frac{b^2}{d}$.

a) $V_P = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_Q} - \frac{b}{dR_i}\right)$;

$R_Q = (R^2 + d^2 - 2Rd\cos\theta)^{1/2}$,

$R_i = (R^2 + d_i^2 - 2Rd_i\cos\theta)^{1/2}$.

b) $\rho_s = -\epsilon_0 \frac{\partial V}{\partial R}\bigg|_{R=b} = -\frac{Q(b^2 - d^2)}{4\pi b(b^2 + d^2 - 2bd\cos\theta)^{3/2}}$.

P.4-16 (See next page.)

P.4-17 Required boundary conditions at $x=0$: $V_1 = V_2$, and $\epsilon_1 \frac{\partial V_1}{\partial x} = \epsilon_2 \frac{\partial V_2}{\partial x}$.

From Fig. 4-23 and the hypotheses in parts a) and b:

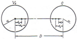
$$V_1 = \frac{Q}{4\pi\epsilon_1\sqrt{(x-d)^2 + y^2 + z^2}} - \frac{Q_1}{4\pi\epsilon_1\sqrt{(x+d)^2 + y^2 + z^2}},$$

$$V_2 = \frac{Q + Q_2}{4\pi\epsilon_2\sqrt{(d-x)^2 + y^2 + z^2}}.$$

In order to satisfy the b.c.'s at $x=0$, we require

$$\frac{Q - Q_1}{\epsilon_1} = \frac{Q + Q_2}{\epsilon_2} \text{ and } Q + Q_1 = Q + Q_2 \longrightarrow Q_1 = Q_2 = \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1}Q.$$

P.4-16



a)

Q_0 and system of image charges:

In left sphere	In right sphere
Q_0 at $d_0 = 0$.	$-Q_1 = -\frac{a}{D}Q_0$ at d_1.
$Q_2 = \frac{a^2}{D(D-d_1)}Q_0$ at d_2.	$-Q_3 = -\frac{a}{D-d_2}Q_2 = -\frac{a^3}{D(D-d_1)(D-d_2)}Q_0$ at d_3.
$Q_4 = \frac{a^4}{D(D-d_1)(D-d_2)(D-d_3)}Q_0$ at d_4.	⋮
⋮	⋮
$Q_{2n} = Q_0 \prod_{m=1}^{n} \frac{a}{D-d_{m-1}}$ at d_n. $(n = 2, 4, 6, \cdots)$	$-Q_{2n} = -Q_0 \prod_{m=1}^{n} \frac{a}{D-d_{m-1}}$ at d_n. $(n = 1, 3, 5, \cdots)$

$$d_m = \frac{a^2}{D-d_{m-1}}, \quad m = 1, 2, 3, \cdots ; \; d_0 = 0.$$

b) $$C = \frac{Q_0 + \sum Q_{2n}}{V_0} = 4\pi\epsilon_0 a\left[1 + \sum_{n=2,4,\cdots}\left(\prod_{m=1}^{n}\frac{a}{D-d_{m-1}}\right)\right].$$

P.4-18 a) $V(x,y) = c_1 xy$. Required boundary conditions:

① $V(0,y) = 0$ → Grounded conducting plane at $x = 0$.

② $V(x,0) = 0$ → Grounded conducting plane at $y = 0$.

If $V(a,b) = c_1 ab = V_0$ → $c_1 = \frac{V_0}{ab}$ → $V(x,y) = \frac{V_0}{ab}xy$

③ Shape of curved boundary defined by:

$V(x,y) = V_0 = (\frac{V_0}{ab})xy$, or $xy = ab$ (a hyperbola).

b) $V(x,y) = c_2 \sin kx \sinh ky$. Required boundary conditions:

① $V(0,y) = 0$ → Grounded plane at $x = 0$.

② $V(\pi/k, y) = 0$ → Grounded plane at $x = \pi/k = b$.

③ $V(x,0) = 0$ → Grounded plane at $y = 0$.

④ If $V(b/2, a) = c_2 \sin(\frac{\pi}{b})(\frac{b}{2})\sinh(\frac{\pi}{b})a = c_2 \sinh(a\pi/b) = V_0$,

→ $V(x,y) = \frac{V_0}{\sinh(a\pi/b)}\sin(\pi x/b)\sinh(\pi y/b)$. Curved boundary: $\sinh(\frac{\pi}{b}y) = \frac{\sinh(a\pi/b)}{\sin(\pi x/b)}$

P.4-19 $V_n(x,y) = C_n' \cosh\frac{n\pi}{b}(x-a)\cos\frac{n\pi}{b}y.$

P.4-20 $V_n(x,y) = C_n \sinh\frac{n\pi}{b}x \sin\frac{n\pi}{b}y$

B.C. at $x=a$: $V_0 = \sum_{n=1}^{\infty} V_n(a,y) = \sum_{n=1}^{\infty} C_n \sinh\frac{n\pi}{b}a \sin\frac{n\pi}{b}y$

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=\text{odd}}^{\infty} \frac{\sinh(n\pi x/b)}{n\sinh(n\pi a/b)}\sin\frac{n\pi}{b}y.$$

P.4-21 $V(x,y) = \sum_n \sin\frac{n\pi}{a}x\left[A_n\sinh\frac{n\pi}{a}y + B_n\cosh\frac{n\pi}{a}y\right].$

At $y=0$, $V(x,0) = V_2 = \sum_n B_n \sin\frac{n\pi}{a}x \longrightarrow B_n = \begin{cases}\frac{4V_2}{n\pi}, & n=\text{odd}.\\ 0, & n=\text{even}.\end{cases}$

At $y=b$, $V(x,b) = V_1 = \sum_n \sin\frac{n\pi}{a}x\left[A_n\sinh\frac{n\pi}{a}b + B_n\cosh\frac{n\pi}{a}b\right]$

$\longrightarrow A_n\sinh\frac{n\pi}{a}b + B_n\cosh\frac{n\pi}{a}b = \begin{cases}\frac{4V_1}{n\pi}, & n=\text{odd}.\\ 0, & n=\text{even}.\end{cases}$

$$\therefore A_n = \begin{cases}\frac{4}{n\pi\sinh(n\pi b/a)}\left(V_1 - V_2\cosh\frac{n\pi}{a}b\right), & n=\text{odd}.\\ 0, & n=\text{even}.\end{cases}$$

P.4-22 $V(x,y,z) = \sum_n\sum_n C_{mn}\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\sinh k_{mn}z,$

where $k_{mn} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}.$

At $z=c$, $V(x,y,c) = V_0 = \sum_m\sum_n C_{mn}\sin\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\sinh k_{mn}c$

$$\longrightarrow C_{mn} = \begin{cases}\frac{16V_0}{mn\pi^2\sinh k_{mn}c}; & m,n=\text{odd}.\\ 0, & m,n=\text{even}.\end{cases}$$

P.4-23 Solution: $V(\phi) = A_0\phi + B_0.$

a) B.C. ①: $V(0) = 0 \longrightarrow B_0 = 0.$
B.C. ②: $V(\alpha) = V_0 = A_0\alpha \longrightarrow A_0 = \frac{V_0}{\alpha}.$
$\therefore V(\phi) = \frac{V_0}{\alpha}\phi,\quad 0\le\phi\le\alpha.$

b) B.C. ①: $V(\alpha) = V_0 = A_1\alpha + B_1$
B.C. ②: $V(2\pi) = 0 = 2\pi A_1 + B$
$\longrightarrow A_1 = -\frac{V_0}{2\pi-\alpha},\ B_1 = \frac{2\pi V_0}{2\pi-\alpha}.$

$$\therefore V(\phi) = \frac{V_0}{2\pi-\alpha}(2\pi-\phi),\quad \alpha\le\phi\le2\pi.$$

Chapter 5

Steady Electric Currents

P.5-1 a) Integrating Eq. (5-16): $V(y)=\left[\frac{9J}{4\epsilon_0}\sqrt{\frac{m}{2e}}\right]^{2/3} y^{4/3}=V_0\left(\frac{y}{d}\right)^{4/3}$. ①

$$E(y)=-\bar{a}_y\frac{dV(y)}{dy}=-\bar{a}_y\frac{4V_0}{3d}\left(\frac{y}{d}\right)^{1/3}.$$

b) $\rho(y)=\epsilon_0\frac{dE(y)}{dy}=-\frac{4\epsilon_0 V_0}{9d^2}\left(\frac{y}{d}\right)^{-2/3}$.

$$Q=\int_0^d \rho(y)S\,dy=-\frac{4\epsilon_0 V_0 S}{9d^{4/3}}\int_0^d y^{-2/3}dy=-\frac{4\epsilon_0 V_0}{3d}S.$$

c) On the anode, $y=d$, $\rho_s=-\epsilon_0 E(d)=\frac{4\epsilon_0 V_0}{3d}$.

Total surface charge on anode $=\rho_s S=\frac{4\epsilon_0 V_0}{3d}S$.

Total charge on cathode $=0$.

d) Substituting ① in Eq. (5-12):

$$u=\frac{dy}{dt}=\left(\frac{2e}{m}V_0\right)^{1/2}\left(\frac{y}{d}\right)^{2/3},\qquad y^{-2/3}dy=\left(\frac{2eV_0}{md^{4/3}}\right)^{1/2}dt.$$

Integrating: $y=\left(\frac{2eV_0}{9md^{4/3}}\right)^{3/2}t^3$.

$\therefore$ Transit time $T_t=3d\left(\frac{m}{2eV_0}\right)^{1/2}$.
$(y=d)$

For $V_0=200\,(V)$, $d=10^{-3}\,(m)$, $m=9.1\times10^{-31}\,(kg)$, and $e=1.6\times10^{-19}\,(C)$, $T_t=3.58\times10^{-9}\,(s)=3.58\,(ns)$.

P.5-3 R_1 = Resistance per unit length of core $=\frac{1}{\sigma S_1}=\frac{1}{\sigma\pi a^2}$.

R_2 = Resistance per unit length of coating $=\frac{1}{0.1\sigma S_2}$.

Let b = Thickness of coating. $\longrightarrow S_2=\pi(a+b)^2-\pi a^2=\pi(2ab+b^2)$.

a) $R_1=R_2 \longrightarrow b=(\sqrt{11}-1)a=2.32a$.

b) $I_1=I_2=\frac{I}{2}$. $\quad J_1=\frac{I}{2\pi a^2}$, $\quad J_2=\frac{I}{2S_2}=\frac{I}{20S_1}=\frac{I}{20\pi a^2}$.

$$E_1=\frac{J_1}{\sigma}=\frac{I}{2\pi a^2\sigma},\quad E_2=\frac{J_2}{0.1\sigma}=\frac{I}{2\pi a^2\sigma}.$$

Thus, $J_1=10J_2$ and $E_1=E_2$.

P.5-4 $I_1 = 0.1\,(A),\ P_{R1} = 3.33\,(mW);\quad I_2 = 0.02\,(A),\ P_{R2} = 8.00\,(mW);$

$I_3 = 0.0133\,(A),\ P_{R3} = 5.31\,(mW);\quad I_4 = 0.0333\,(A),\ P_{R4} = 8.87\,(mW);$

$I_5 = 0.0667\,(A),\ P_{R5} = 44.5\,(mW).\qquad \sum_n P_{Rn} = V_0 I_1 = 70\,(mW).$

P.5-5 $I_1 = 0.700\,(A),\ P_{R1} = 0.163\,(W);\quad I_2 = 0.140\,(A),\ P_{R2} = 0.392\,(W);$

$I_3 = 0.093\,(A),\ P_{R3} = 0.261\,(W);\quad I_4 = 0.233\,(A),\ P_{R4} = 0.436\,(W);$

$I_5 = 0.467\,(A),\ P_{R5} = 2.178\,(W).$

P.5-6 $\rho_0 = \dfrac{Q_0}{(4\pi/3)b^3} = \dfrac{10^{-3}}{(4\pi/3)(0.1)^3} = 0.239\,(C/m^3),\quad \rho = \rho_0 e^{-(\sigma/\epsilon)t}$

a) $R < b:\ \vec{E}_i = \vec{a}_R \dfrac{(4\pi/3)R^3\rho}{4\pi\epsilon R^2} = \vec{a}_R \dfrac{\rho_0 R}{3\epsilon} e^{-(\sigma/\epsilon)t} = \vec{a}_R\, 7.5\times10^9 R\, e^{-9.42\times10^{12}t}\,(V/m)$

$R > b:\ \vec{E}_o = \vec{a}_R \dfrac{Q_0}{4\pi\epsilon_0 R^2} = \vec{a}_R \dfrac{9}{R^2}\times10^6\ (V/m).$

b) $R < b:\ \vec{J}_i = \sigma\vec{E}_i = \vec{a}_R\, 7.5\times10^{10} R\, e^{-9.42\times10^{12}t}$

$R > b:\ \vec{J}_o = 0.$

P.5-7 a) $e^{-(\sigma/\epsilon)t} = \dfrac{\rho}{\rho_0} = 0.01 \longrightarrow t = \dfrac{\ln 100}{(\sigma/\epsilon)} = 4.88\times10^{-13}(s) = 4.88\,(ps).$

b) $W_i = \dfrac{\epsilon}{2}\int_{V'} E_i^2\,dv' = \dfrac{2\pi\rho_0 b^5}{45\epsilon} e^{-2(\sigma/\epsilon)t} = (W_i)_0 \left[e^{-(\sigma/\epsilon)t}\right]^2.$

$\therefore \dfrac{W_i}{(W_i)_0} = \left[e^{-(\sigma/\epsilon)t}\right]^2 = 0.01^2 = 10^{-4}.$ Energy dissipated as heat loss.

c) Electrostatic energy stored outside the sphere $W_o = \dfrac{\epsilon_0}{2}\int_b^\infty E_o^2\, 4\pi R^2 dR = \dfrac{Q_0^2}{8\pi\epsilon_0 b} = 45\,(kJ)$

— constant.

P.5-8 a) $R = \dfrac{\ell}{\sigma S} = \dfrac{V}{I} \longrightarrow \sigma = \dfrac{\ell I}{SV} = 3.54\times10^7\,(S/m).$

b) $E = \dfrac{V}{\ell} = 6\times10^{-3}\,(V/m).$

c) $P = VI = 1\,(W).$

d) $\rho_e = -\dfrac{\sigma}{\mu_e}$ The given electron mobility $1.4\times10^{-3}\,(m^2/V{\cdot}s)$ is that of a good conductor.

$u = \left|\dfrac{J}{\rho_e}\right| = \left|\dfrac{\mu_e J}{\sigma}\right| = |\mu_e E| = 1.4\times10^{-3}\times(6\times10^{-3})$

$= 8.4\times10^{-6}\ (m/s).$

P.5-9 a) Eq. (3-118): $E_{1t} = E_{2t} \longrightarrow E_2 \sin\alpha_2 = E_1 \sin\alpha_1$.

Eq. (5-58): $J_{1n} = J_{2n} \longrightarrow \sigma_1 E_{1n} = \sigma_2 E_{2n} \longrightarrow \sigma_2 E_2 \cos\alpha_2 = \sigma_1 E_1 \cos\alpha_1$.

$$\therefore \quad E_2 = E_1 \sqrt{\sin^2\alpha_1 + \left(\frac{\sigma_1}{\sigma_2}\cos\alpha_1\right)^2}. \qquad ①$$

$$\tan\alpha_2 = \frac{\sigma_2}{\sigma_1}\tan\alpha_1 \longrightarrow \alpha_2 = \tan^{-1}\left(\frac{\sigma_2}{\sigma_1}\tan\alpha_1\right). \qquad ②$$

b) Eq. (3-121b): $D_{2n} - D_{1n} = \rho_s \longrightarrow \epsilon_2 E_{2n} - \epsilon_1 E_{1n} = \rho_s$.

$$\rho_s = \left(\frac{\sigma_1}{\sigma_2}\epsilon_2 - \epsilon_1\right)E_{1n} = \left(\frac{\sigma_1}{\sigma_2}\epsilon_2 - \epsilon_1\right)E_1 \cos\alpha_1.$$

c) If both media are perfect dielectrics, $\sigma_1 = \sigma_2 = 0$, Eqs. ① and ② revert to Eqs. (3-130) and (3-129) respectively and $\rho_s = 0$.

P.5-10

$$\sigma(y) = \sigma_1 + (\sigma_2 - \sigma_1)\frac{y}{d}$$

a) Neglecting fringing effect and assuming a current density

$$\vec{J} = -\vec{a}_y J_0 \longrightarrow \vec{E} = \frac{\vec{J}}{\sigma} = -\vec{a}_y \frac{J_0}{\sigma(y)}.$$

$$V_0 = -\int_0^d \vec{E}\cdot\vec{a}_y\,dy = \int_0^d \frac{J_0\,dy}{\sigma_1 + (\sigma_2-\sigma_1)\frac{y}{d}} = \frac{J_0 d}{\sigma_2 - \sigma_1}\ln\frac{\sigma_2}{\sigma_1}.$$

$$R = \frac{V_0}{I} = \frac{V_0}{J_0 S} = \frac{d}{(\sigma_2-\sigma_1)S}\ln\frac{\sigma_2}{\sigma_1}.$$

b) $(\rho_s)_u = \epsilon_0 E_y(d) = \frac{\epsilon_0 J_0}{\sigma_2} = \frac{\epsilon_0(\sigma_2-\sigma_1)V_0}{\sigma_2 d \ln(\sigma_2/\sigma_1)}$ on upper plate.

$(\rho_s)_\ell = -\epsilon_0 E_y(0) = -\frac{\epsilon_0 J_0}{\sigma_1} = -\frac{\epsilon_0(\sigma_2-\sigma_1)V_0}{\sigma_1 d \ln(\sigma_2/\sigma_1)}$ on lower plate.

c) $\rho = \vec{\nabla}\cdot\vec{D} = \frac{d}{dy}(\epsilon_0 E) = -\epsilon_0 J_0 \frac{d}{dy}\left[\frac{1}{\sigma_1 + (\sigma_2-\sigma_1)y/d}\right] = \epsilon_0 J_0 \frac{(\sigma_2-\sigma_1)/d}{[\sigma_1 + (\sigma_2-\sigma_1)y/d]^2}$

P.5-11

a) $C_1 = \frac{\epsilon_1 S}{d_1}, \quad G_1 = \frac{\sigma_1 S}{d_1};$

$C_2 = \frac{\epsilon_2 S}{d_2}, \quad G_2 = \frac{\sigma_2 S}{d_2}.$

b) $P = \mathcal{V}^2 G = \mathcal{V}^2 \frac{G_1 G_2}{G_1 + G_2}$

$= \mathcal{V}^2 S \frac{\sigma_1\sigma_2}{\sigma_1 d_2 + \sigma_2 d_1}.$

P.5-12 Refer to Fig. 5-6. In the transient state, the equation of continuity must be satisfied at the interface.

$$-\frac{\partial \rho_{si}}{\partial t} = J_2 - J_1 = \sigma_2 E_2 - \sigma_1 E_1 \qquad ①$$

Now

$$E_1 d_1 + E_2 d_2 = \mathcal{V} \qquad ②$$

$$\epsilon_2 E_2 - \epsilon_1 E_1 = \rho_{si} \qquad ③$$

Solving ③ and ③ for E_1 and E_2 in terms of $\mathcal{V}$ and ρ_{si}:

$$E_1 = \frac{\epsilon_2\mathcal{V} - d_2\rho_{si}}{\epsilon_2 d_1 + \epsilon_1 d_2} \quad ④; \qquad E_2 = \frac{\epsilon_1\mathcal{V} + d_1\rho_{si}}{\epsilon_2 d_1 + \epsilon_1 d_2} \quad ⑤.$$

a) Substituting ④ and ⑤ in ①:

$$-\frac{\partial\rho_{si}}{\partial t} = \frac{\sigma_2 d_1 + \sigma_1 d_2}{\epsilon_2 d_1 + \epsilon_1 d_2}\rho_{si} + \frac{\epsilon_1\sigma_2 - \epsilon_2\sigma_1}{\epsilon_2 d_1 + \epsilon_1 d_2}\mathcal{V}. \qquad ⑥$$

Solution of ⑥:

$$\rho_{si} = \left(\frac{\epsilon_2\sigma_1 - \epsilon_1\sigma_2}{\sigma_2 d_1 + \sigma_1 d_2}\right)\mathcal{V}\left[1 - e^{-t/\tau}\right]. \qquad ⑦$$

where τ = Relaxation time $= \dfrac{\epsilon_2 d_1 + \epsilon_1 d_2}{\sigma_2 d_1 + \sigma_1 d_2}$. ⑧

b) Using ④ and ⑤:

$$E_1 = \frac{\sigma_2\mathcal{V}}{\sigma_2 d_1 + \sigma_1 d_2}\left(1 - e^{-t/\tau}\right) + \frac{\epsilon_2\mathcal{V}}{\epsilon_2 d_1 + \epsilon_1 d_2}e^{-t/\tau};$$

$$E_2 = \frac{\sigma_1\mathcal{V}}{\sigma_2 d_1 + \sigma_1 d_2}\left(1 - e^{-t/\tau}\right) + \frac{\epsilon_1\mathcal{V}}{\epsilon_2 d_1 + \epsilon_1 d_2}e^{-t/\tau}.$$

P.5-13 a) $G_1 = \dfrac{2\pi\sigma_1 L}{\ln(c/a)}, \qquad G_2 = \dfrac{2\pi\sigma_2 L}{\ln(b/c)}$

$$I = V_0 G = V_0\frac{G_1 G_2}{G_1 + G_2} = \frac{2\pi\sigma_1\sigma_2 L V_0}{\sigma_1\ln(b/c) + \sigma_2\ln(c/a)}.$$

$$J_1 = J_2 = \frac{I}{2\pi r L} = \frac{\sigma_1\sigma_2 V_0}{r[\sigma_1\ln(b/c) + \sigma_2\ln(c/a)]}.$$

b) $\rho_{sa} = \epsilon_1 E_1\big|_{r=a} = \dfrac{\epsilon_1\sigma_2 V_0}{a[\sigma_1\ln(b/c) + \sigma_2\ln(c/a)]}.$

$$\rho_{sb} = -\epsilon_2 E_2\big|_{r=b} = -\frac{\epsilon_2\sigma_1 V_0}{b[\sigma_1\ln(b/c) + \sigma_2\ln(c/a)]}.$$

$$\rho_{sc} = -(\epsilon_1 E_1 - \epsilon_2 E_2)\big|_{r=c} = \frac{(\epsilon_2\sigma_1 - \epsilon_1\sigma_2)V_0}{c[\sigma_1\ln(b/c) + \sigma_2\ln(c/a)]}.$$

P.5-14 $\nabla^2 V = 0 \longrightarrow \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial V}{\partial r}\right) = 0.$

Solution: $V(r) = c_1\ln r + c_2$.

Boundary conditions: $V(a) = V_0$; $\quad V(b) = 0$.

$$\longrightarrow V(r) = V_0\frac{\ln(b/r)}{\ln(b/a)}.$$

$$\vec{E}(r) = -\vec{a}_r\frac{\partial V}{\partial r} = \vec{a}_r\frac{V_0}{r\ln(b/a)}.$$

$$\vec{J}(r) = \sigma\vec{E}(r).$$

$$I = \int_S \vec{J}\cdot d\vec{s} = \int_0^{\pi/2}\vec{J}\cdot(\vec{a}_r h r\,d\phi) = \frac{\pi\sigma h V_0}{2\ln(b/a)}.$$

$$R = \frac{V_0}{I} = \frac{2\ln(b/a)}{\pi\sigma h}.$$

P.5-15 Assume a potential difference V_0 between the inner and outer spheres.

$$\nabla^2 V = 0 \longrightarrow \frac{1}{R^2}\frac{d}{dR}(R^2 V)=0 \longrightarrow V = \frac{K}{R} \longrightarrow E_R = \frac{K}{R^2}.$$

$$V_0 = -\int_{R_2}^{R_1} E_R\, dR = -K\int_{R_2}^{R_1}\frac{1}{R^2}dR = K\left(\frac{1}{R_1}-\frac{1}{R_2}\right).$$

$$\longrightarrow K = \frac{V_0}{\frac{1}{R_1}-\frac{1}{R_2}}. \qquad J_R = \sigma E_R = \frac{\sigma V_0}{\frac{1}{R_1}-\frac{1}{R_2}}.$$

$$I = \int_0^{2\pi}\int_0^{\pi} J_R R^2 \sin\theta\, d\theta\, d\phi = \frac{4\pi\sigma V_0}{\frac{1}{R_1}-\frac{1}{R_2}}.$$

$$R = \frac{V_0}{I} = \frac{1}{4\pi\sigma}\left(\frac{1}{R_1}-\frac{1}{R_2}\right),$$ which can be obtained by combining Eqs. (3-140) and (5-81).

P.5-16 Assume a current I between the spherical surfaces.

$$\bar{J} = \bar{a}_R \frac{I}{4\pi R^2} = \sigma \bar{E}.$$

$$V_0 = -\int_{R_2}^{R_1} \bar{E}\cdot d\bar{R} = \int_{R_1}^{R_2} \frac{I\, dR}{4\pi\sigma R^2} = \frac{I}{4\pi\sigma_0}\int_{R_1}^{R_2}\frac{dR}{R^2(1+k/R)}$$

$$= \frac{I}{4\pi\sigma_0}\int_{R_1}^{R_2}\frac{1}{k}\left(\frac{1}{R}-\frac{1}{R+k}\right)dR = \frac{I}{4\pi\sigma_0 k}\ln\frac{R_2(R_1+k)}{R_1(R_2+k)}.$$

$$R = \frac{V_0}{I} = \frac{1}{4\pi\sigma_0 k}\ln\frac{R_2(R_1+k)}{R_1(R_2+k)}.$$

P.5-17 Assume I. $\qquad \bar{J}(R) = \bar{a}_R \dfrac{I}{S(R)}.$

$$S(R) = \int_0^{2\pi}\int_0^{\theta_0} R^2 \sin\theta\, d\theta\, d\phi = 2\pi R^2(1-\cos\theta_0).$$

$$\bar{E}(R) = \frac{1}{\sigma}\bar{J}(R) = \bar{a}_R\frac{I}{2\pi\sigma R^2(1-\cos\theta_0)}.$$

$$V_0 = -\int_{R_2}^{R_1} E(R)\, dR = \frac{I(R_2-R_1)}{2\pi\sigma R_1 R_2(1-\cos\theta_0)}$$

$$R = \frac{V_0}{I} = \frac{1}{2\pi(1-\cos\theta_0)}\left(\frac{1}{R_1}-\frac{1}{R_2}\right).$$

P.5-18 $\bar{\nabla}\cdot\bar{J} = 0 = \bar{\nabla}\cdot(\sigma\bar{E}) = \sigma\,\bar{\nabla}\cdot\bar{E} - (\bar{\nabla}\sigma)\cdot\bar{E} = 0.$

$$\bar{E} = \bar{a}_R E, \quad \bar{\nabla}\cdot\bar{E} = \frac{1}{R^2}\frac{d}{dR}(R^2 E); \quad \bar{\nabla}\sigma = \bar{a}_R\frac{d\sigma}{dR} = -\bar{a}_R\frac{\sigma_0 R_1}{R^2}.$$

Substituting back: $R\frac{dE}{dR} = E \longrightarrow \bar{E} = \bar{a}_R \frac{c}{R}$

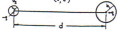
$$V = -\int_{R_2}^{R_1} \bar{E}\cdot d\bar{R} = c\ln\frac{R_2}{R_1} \longrightarrow c = \frac{V_0}{\ln(R_2/R_1)}, \quad \bar{E} = \bar{a}_R\frac{V_0}{R\ln(R_2/R_1)}$$

$$I = \int_S \bar{J}\cdot d\bar{s} = \int_S \sigma\bar{E}\cdot d\bar{s} = \int_0^{2\pi}\int_0^{\theta_0}\left(\frac{\sigma_0 R_1}{R}\right)\left[\frac{V_0}{R\ln(R_2/R_1)}\right]R^2\sin\theta\, d\theta\, d\phi$$

$$= \frac{2\pi\sigma_0 R_1 V_0(1-\cos\theta_0)}{\ln(R_2/R_1)}.$$

$$R = \frac{V_0}{I} = \frac{\ln(R_2/R_1)}{2\pi\sigma_0 R_1(1-\cos\theta_0)}.$$

<u>P.5-19</u> Assume charges $+q$ and $-q$ to concentrate at the centers of spheres 1 and 2 respectively. $d \gg b_1$, $d \gg b_2$.



$$V_1 \cong \frac{q}{4\pi\epsilon}\left(\frac{1}{b_1} - \frac{1}{d-b_1}\right).$$

$$V_2 \cong \frac{q}{4\pi\epsilon}\left(\frac{1}{d-b_2} - \frac{1}{b_2}\right).$$

$$C = \frac{q}{V_1 - V_2} = \frac{4\pi\epsilon}{\frac{1}{b_1} + \frac{1}{b_2} - \frac{1}{d-b_1} - \frac{1}{d-b_2}}$$

$$= G\frac{\epsilon}{\sigma} = \frac{\epsilon}{R\sigma}.$$

$$R = \frac{1}{4\pi\sigma}\left(\frac{1}{b_1} + \frac{1}{b_2} - \frac{1}{d-b_1} - \frac{1}{d-b_2}\right) \cong \frac{1}{4\pi\sigma}\left(\frac{1}{b_1} + \frac{1}{b_2} - \frac{2}{d}\right).$$

<u>P.5-20</u>

The curved flow pattern of the lower half of Fig.(b), if both the conductor and its image are fed with the same current, is exactly the same as that of Fig.(a). All boundary conditions are satisfied.

$$\bar{\nabla}\times\left(\frac{\bar{J}}{\sigma}\right) = 0 \longrightarrow \bar{\nabla}\times\bar{J} = 0.$$

We can write $\bar{J} = -\bar{\nabla}\psi$, where ψ and electrostatic potential V are simply related. The streamlines are similar to the E-lines of a conductor and its image, both carrying a charge $+Q$ in the electrostatic case.

P.5-21 According to problem P.5-20, the current flow pattern would be the same as that of a whole sphere in an unbounded earth medium. Hence the current lines would be radial. Assume a current I.

$$\bar{J} = \bar{a}_R \frac{I}{2\pi R^2}, \qquad \bar{E} = \bar{a}_R \frac{I}{2\pi\sigma R^2}.$$

$$V_0 = -\int_\infty^b E\,dR = -\frac{I}{2\pi\sigma}\int_\infty^b \frac{dR}{R^2} = \frac{I}{2\pi\sigma b}.$$

$$R = \frac{V_0}{I} = \frac{1}{2\pi\sigma b} = \frac{1}{2\pi(10^{-6})(25\times10^{-3})} = 6.36\times10^{6}\ (\Omega).$$

P.5-22 Specified boundary conditions can be satisfied by solutions of Laplace's equation with zero separation constants: $k_x = k_y = 0$. $X(x) = A_0 x + B_0$. $Y(y) = C_0 y + D_0$.

$B_0 = C_0 = 0$. $\quad V(x) = A_0 D_0 x$

a) At $x = a$, $V(a) = V_0 = A_0 D_0 a \longrightarrow A_0 D_0 = \frac{V_0}{a}$

$$\therefore V = \frac{V_0}{a} x$$

b) $\bar{E} = -\bar{\nabla} V = -\bar{a}_x \frac{V_0}{a} \longrightarrow \bar{J} = \sigma\bar{E} = -\bar{a}_x \frac{\sigma V_0}{a}$.

P.5-23 $V(r,\phi) = \sum_{n=0}^{\infty} (A_n r^n + B_n r^{-n})(C_n \cos n\phi + D_n \sin n\phi)$.

B.C.: $V(r,\phi) = V(r,-\phi) \longrightarrow D_n = 0$.

$r \to \infty$, $V = -\frac{J_0}{\sigma} r\cos\phi \longrightarrow A_n = C_n = 0$ for $n \neq 1$.

Write $V(r,\phi) = \left(K_1 r + \frac{K_2}{r}\right)\cos\phi$. $\quad K_1 = A_1 C_1 = -\frac{J_0}{\sigma}$, $K_2 = B_1 C_1$.

B.C.: $\left.\frac{\partial V}{\partial r}\right|_{r=b} = 0 \longrightarrow K_1 - \frac{K_2}{b^2} = 0$, $\quad K_2 = b^2 K_1 = -\frac{J_0}{\sigma} b^2$.

$$\therefore V(r,\phi) = -\frac{J_0}{\sigma}\left(r + \frac{b^2}{r}\right)\cos\phi.$$

$$\bar{J} = -\sigma\bar{\nabla}V = -\sigma\left(\bar{a}_r \frac{\partial V}{\partial r} + \bar{a}_\phi \frac{\partial V}{r\,\partial\phi}\right)$$

$$= \bar{a}_r J_0\left(1 - \frac{b^2}{r^2}\right)\cos\phi - \bar{a}_\phi J_0\left(1 + \frac{b^2}{r^2}\right)\sin\phi$$

$$= J_0(\bar{a}_r\cos\phi - \bar{a}_\phi\sin\phi) - \frac{J_0 b^2}{r^2}(\bar{a}_r\cos\phi + \bar{a}_\phi\sin\phi)$$

$$= \bar{a}_x J_0 - \frac{J_0 b^2}{r^2}(\bar{a}_r\cos\phi + \bar{a}_\phi\sin\phi), \quad r > b;$$

$J = 0, \quad r < b.$

Static Magnetic Fields

P.6-1

$$\frac{du_y}{dt} = \frac{qB_0}{m}u_z = \omega_0 u_z, \quad ①$$

$$\frac{du_z}{dt} = -\frac{qB_0}{m}u_y = -\omega_0 u_y, \quad ②$$

$$\omega_0 = qB_0/m.$$

Combining ① and ②:

$$\frac{d^2u_z}{dt^2} + \omega_0^2 u_z = 0$$

$$\longrightarrow u_z = A\cos\omega_0 t + B\sin\omega_0 t.$$

At $t=0$, $u_z=0 \longrightarrow A=0$; $u_z = B\sin\omega_0 t$.

Substituting u_z in ②: $u_y = -B\cos\omega_0 t$. At $t=0$, $u_y=u_0 \longrightarrow B=-u_0$.

$$\therefore\ u_y = u_0\cos\omega_0 t \longrightarrow y = \frac{u_0}{\omega_0}\sin\omega_0 t, \quad (t=0,\ y=0); \quad ③$$

$$u_z = -u_0\sin\omega_0 t \longrightarrow z = \frac{u_0}{\omega_0}\cos\omega_0 t + c_1 \quad \left(t=0, z=0 \to c_1 = -\frac{u_0}{\omega_0}\right)$$

$$= -\frac{u_0}{\omega_0}(1-\cos\omega_0 t). \quad ④$$

From ③ and ④: $y^2 + \left(z - \frac{u_0}{\omega_0}\right)^2 = \left(\frac{u_0}{\omega_0}\right)^2$ --- Eq. of a shifted circle.

P.6-2 $\frac{\partial \bar{u}}{\partial t} = -\frac{e}{m}(\bar{E} + \bar{u}\times\bar{B})$.

a) $\bar{E} = \bar{a}_z E_0$, $\bar{B} = \bar{a}_x B_0$.

$$\left.\begin{aligned} &\frac{\partial u_x}{\partial t} = 0, \\ &\frac{\partial u_y}{\partial t} = -\frac{e}{m}B_0 u_z, \\ &\frac{\partial u_z}{\partial t} = -\frac{e}{m}(E_0 - B_0 u_y). \end{aligned}\right. \longrightarrow \left\{\begin{aligned} &u_x = 0, \\ &u_y = \left(u_0 - \frac{E_0}{B_0}\right)\cos\omega_0 t + \frac{E_0}{B_0}, \\ &u_z = \left(\frac{E_0}{B_0} - u_0\right)\sin\omega_0 t; \quad \omega_0 = \frac{e}{m}B_0. \end{aligned}\right.$$

If the electron is injected at the origin ($x=y=z=0$) at $t=0$:

$$x=0, \quad y = \frac{c_1}{\omega_0}\sin\omega_0 t + \frac{E_0}{B_0}t, \quad z = -\frac{c_1}{\omega_0}(1-\cos\omega_0 t); \quad c_1 = u_0 - \frac{E_0}{B_0}.$$

Eq. of motion: $\left(y - \frac{E_0}{B_0}t\right)^2 + \left(z + \frac{c_1}{\omega_0}\right)^2 = \left(\frac{c_1}{\omega_0}\right)^2$.

If $\frac{E_0}{B_0} = u_0$, $u_x = u_z = 0$, $u_y = u_0$;

$x = z = 0$, and $y = u_0 t$

b) $\bar{E} = -\bar{a}_z E_0$, B

$$\frac{\partial u_x}{\partial t} = \frac{e}{m} B_0 u_y = \omega_0 u_y,$$
$$\frac{\partial u_y}{\partial t} = -\omega_0 u_x,$$
$$\frac{\partial u_z}{\partial t} = \frac{e}{m} E_0.$$

(first two equations) Circular motion (See P.6-1) — Helical motion with constant acceleration eE_0/m in z-direction.

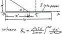
P.6-3 Application of Ampère's circuital law.

$0 \le r \le a$, $\bar{B} = \bar{a}_\phi \frac{\mu r I}{2\pi a^2}$.

$a \le r \le b$, $\bar{B} = \bar{a}_\phi \frac{\mu I}{2\pi r}$.

$b \le r \le c$, $\bar{B} = \bar{a}_\phi \left(\frac{c^2 - r^2}{c^2 - b^2}\right) \frac{\mu I}{2\pi r}$.

P.6-4



a) Using Eq. (6-33a):

$$d\bar{B}_{P_1} = \bar{a}_x dB_x + \bar{a}_y dB_y$$
$$= \bar{a}_x (dB)\sin\alpha + \bar{a}_y (dB)\cos\alpha,$$
$$dB_{P_1} = \frac{\mu_0 (I/w) dx'}{2\pi (x'^2 + d^2)^{1/2}},$$
$$\sin\alpha = \frac{d}{(x'^2+d^2)^{1/2}}, \quad \cos\alpha = \frac{x'}{(x'^2+d^2)^{1/2}}.$$
$$\therefore \bar{B}_{P_1} = \bar{a}_x B_x + \bar{a}_y B_y,$$

where
$$B_x = \frac{\mu_0 I d}{2\pi w} \int_0^w \frac{dx'}{x'^2 + d^2} = \frac{\mu_0 I}{2\pi w} \tan^{-1}\left(\frac{w}{d}\right),$$
$$B_y = \frac{\mu_0 I}{2\pi w} \int_0^w \frac{x' dx'}{x'^2 + d^2} = \frac{\mu_0 I}{4\pi w} \ln\left(1 + \frac{w^2}{d^2}\right).$$

b) To find $\bar{B}$ at $P_2(\frac{2}{3}w, d)$, we add vectorially the contributions of the current strips to the right and to the left of point P_2' using the result in part (a).

$$\bar{B}_{P_2} = \bar{B}_{2R} + \bar{B}_{2L}.$$
$$\bar{B}_{2R} = \frac{\mu_0 I}{2\pi w}\left[\bar{a}_x \tan^{-1}\left(\frac{w}{3d}\right) + \bar{a}_y \frac{1}{2} \ln\left(1 + \frac{w^2}{9d^2}\right)\right].$$
$$\bar{B}_{2L} = \frac{\mu_0 I}{2\pi w}\left[\bar{a}_x \tan^{-1}\left(\frac{2w}{3d}\right) - \bar{a}_y \frac{1}{2} \ln\left(1 + \frac{4w^2}{9d^2}\right)\right].$$
$$\therefore \bar{B}_{P_2} = \frac{\mu_0 I}{2\pi w}\left[\bar{a}_x\left(\tan^{-1}\frac{w}{3d} + \tan^{-1}\frac{2w}{3d}\right) - \bar{a}_y \ln\sqrt{\frac{1+(2w/3d)^2}{1+(w/3d)^2}}\right].$$

P.6-5

We first find $\bar{B}_{P_1}$ at P_1 flush with one end of a wire carrying a current I and making an angle α with the other end as shown.

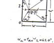
$$d\bar{B}_{P_1} = \frac{\mu_0 I}{4\pi R^2} dz' \times \bar{a}_R \qquad z' = b\cot\theta,\ dz' = -b\csc^2\theta\, d\theta,$$

$$= \frac{\mu_0 I}{4\pi b}(-\bar{a}_\phi \sin\theta\, d\theta). \qquad R = b\csc\theta.$$

$$\bar{a}_z \times \bar{a}_R = \bar{a}_\phi \sin\theta.$$

$$\bar{B}_{P_1} = -\bar{a}_\phi \frac{\mu_0 I}{4\pi b}\int_{\pi/2}^{\alpha} \sin\theta\, d\theta$$

$$= \bar{a}_\phi \frac{\mu_0 I}{4\pi b}\cos\alpha.$$



Applying the above result to the four-sided loop at left, we have

$$\bar{B}_P = \bar{a}_z \frac{\mu_0 I}{4\pi}\Big(\frac{1}{a}\cos\alpha_a + \frac{1}{b}\sin\alpha_a + \frac{1}{b}\cos\alpha_b + \frac{1}{c}\sin\alpha_b + \frac{1}{c}\cos\alpha_c + \frac{1}{d}\sin\alpha_c + \frac{1}{d}\cos\alpha_d + \frac{1}{a}\sin\alpha_d\Big).$$

For this problem, $a = c = \frac{w}{2}$, $b = \frac{w}{4}$, $d = \frac{3}{4}w$.

$\alpha_a = \tan^{-1} 2 = 63.4°$, $\alpha_b = 90° - 63.4° = 26.6°$, $\alpha_c = \tan^{-1}\frac{2}{3} = 33.7°$, $\alpha_d = 56.3°$

$$\therefore \bar{B}_P = \bar{a}_z\, 3.44 \frac{\mu_0 I}{\pi w}.$$

P.6-6 The problem can be decomposed into two subproblems (assuming b = radius of solenoid):

1. A cylindrical tube carrying a uniformly distributed longitudinal surface current $2\pi b n I \sin\alpha$.

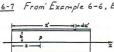
$$\longrightarrow \bar{B}_1 = \begin{cases} 0, & 0 < r < b. \\ \bar{a}_\phi \frac{bnI}{r}\sin\alpha, & r > b. \end{cases}$$

2. A solenoid with n turns per unit length carrying a current $I\cos\alpha$.

$$\longrightarrow \bar{B}_2 = \begin{cases} \bar{a}_z \mu_0 n I \cos\alpha, & 0 < r < b, \\ 0, & r > b. \end{cases}$$

Total $\bar{B} = \bar{B}_1 + \bar{B}_2$.

P.6-7 From Example 6-6, Eq. (6-35):



$$dB = \frac{\mu_0 I b^2}{2[(z'-z)^2+b^2]^{3/2}}\left(\frac{N}{L}\right)dz'.$$

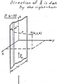
$$B = \frac{\mu_0 N I b^2}{2L}\int_0^L \frac{dz'}{[(z-z')^2+b^2]^{3/2}}$$

$$= \frac{\mu_0 N I}{2L}\left[\frac{L-z}{\sqrt{(L-z)^2+b^2}} + \frac{z}{\sqrt{z^2+b^2}}\right]$$

$$\longrightarrow \mu_0\left(\frac{N}{L}\right)I \quad \text{as } L\to\infty.$$

Direction of $\bar{B}$ is determined by the right-hand rule

P.6-10



$\bar{J}_s = \bar{a}_z J_{s0}$.

At $P(x,y,z)$ the magnetic flux density due to an infinitely long strip of width dx' is

$$d\bar{B} = \frac{\mu_0 J_{s0}\,dx'}{2\pi r}\left(-\bar{a}_x \frac{y}{r} + \bar{a}_y \frac{x-x'}{r}\right),$$

$$r = \sqrt{(x-x')^2+y^2}.$$

$$\therefore\quad \bar{B} = \int d\bar{B} = \bar{a}_x B_x + \bar{a}_y B_y,$$

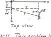
where

$$B_x = -\frac{\mu_0 J_{s0} y}{2\pi}\int_{-w/2}^{w/2}\frac{dx'}{(x-x')^2+y^2}$$

$$= \frac{\mu_0 J_{s0}}{2\pi}\left[\tan^{-1}\left(\frac{x-\frac{w}{2}}{y}\right) - \tan^{-1}\left(\frac{x+\frac{w}{2}}{y}\right)\right],$$

$$B_y = \frac{\mu_0 J_{s0}}{2\pi}\int_{-w/2}^{w/2}\frac{(x-x')\,dx'}{(x-x')^2+y^2}$$

$$= \frac{\mu_0 J_{s0}}{4\pi}\ln\frac{(x+\frac{w}{2})^2+y^2}{(x-\frac{w}{2})^2+y^2}.$$



Top view

P.6-11 This problem is a superposition of two problems:

$$\bar{B} = \bar{B}_1 + \bar{B}_2,$$

where

1. $\bar{B}_1$ is the magnetic flux density at P due to two

Semi-infinite wires carrying equal and opposite currents. Assuming $\bar{a}_z$ points out of paper:

$$\bar{B}_1 = \bar{a}_z \frac{\mu_0 I}{2\pi b}.$$

2. $\bar{B}_2$ is the magnetic flux density at P due to a half-circle. Taking one-half of the result in Eq.(6-38) for $z=0$:

$$\bar{B}_2 = \bar{a}_z \frac{\mu_0 I}{4b}.$$

$$\therefore \quad \bar{B} = \bar{a}_z \frac{\mu_0 I}{2b}\left(\frac{1}{\pi} + \frac{1}{2}\right).$$

P.6-12 Use Eq.(6-38) $B_x = \frac{N\mu_0 I b^2}{2}\left\{\frac{1}{[(\frac{d}{2}+x)^2+b^2]^{3/2}} - \frac{1}{[(\frac{d}{2}-x)^2+b^2]^{3/2}}\right\}.$

a) At $x=0$, $B_x = \frac{N\mu_0 I b^2}{[(\frac{d}{2})^2+b^2]^{3/2}}.$

b) $\frac{dB_x}{dx} = \frac{N\mu_0 I b^2}{2}\left\{-\frac{3(\frac{d}{2}+x)}{[(\frac{d}{2}+x)^2+b^2]^{5/2}} + \frac{3(\frac{d}{2}-x)}{[(\frac{d}{2}-x)^2+b^2]^{5/2}}\right\}.$

At the midpoint, $x=0$, $\frac{dB_x}{dx}=0$.

c) $$\frac{d^2B_x}{dx^2} = -\frac{3N\mu_0 I b^2}{2}\left\{\frac{1}{[(\frac{d}{2}+x)^2+b^2]^{5/2}} - \frac{5(\frac{d}{2}+x)^2}{[(\frac{d}{2}+x)^2+b^2]^{7/2}} + \frac{1}{[(\frac{d}{2}-x)^2+b^2]^{5/2}} - \frac{5(\frac{d}{2}-x)^2}{[(\frac{d}{2}-x)^2+b^2]^{7/2}}\right\}.$$

At $x=0$, $\frac{d^2B_x}{dx^2} = -3N\mu_0 I b^2\left\{\frac{b^2-4(d/2)^2}{[(d/2)^2+b^2]^{7/2}}\right\} \longrightarrow 0$, if $b=d$.

P.6-13 Use Eq.(6-35) for a wire of length $2L$.

$$\bar{B} = \bar{a}_\phi \frac{\mu_0 I L}{2\pi r\sqrt{L^2+r^2}}.$$

In this problem, $\alpha = \frac{\pi}{N}$, $\frac{L}{r} = \tan\alpha = \tan\frac{\pi}{N}$.

$$\bar{B} = \bar{a}_n N\left(\frac{\mu_0 I L}{2\pi r b}\right) = \bar{a}_n \frac{\mu_0 N I}{2\pi b}\tan\frac{\pi}{N}.$$

When N is very large, $\tan\frac{\pi}{N} \cong \frac{\pi}{N}$. $\bar{B} \longrightarrow \bar{a}_n \frac{\mu_0 I}{2b}$, which is the same as Eq.(6-38) with $z=0$.

P.6-14 $B_\phi = \frac{\mu_0 NI}{2\pi r}$, $\quad \Phi = \int_S B_\phi\, ds = \frac{\mu_0 NI}{2\pi}\int_a^b \frac{h}{r}\, dr = \frac{\mu_0 NIh}{2\pi}\ln\frac{b}{a}$.

If B_ϕ at $r = \frac{a+b}{2}$ is used, $\Phi' = \frac{\mu_0 NIh}{\pi}\left(\frac{b-a}{b+a}\right)$.

$\%\ \text{error} = \frac{\Phi' - \Phi}{\Phi}\times 100\% = \left[\frac{2(b-a)}{(b+a)\ln(b/a)} - 1\right]\times 100\%$.

P.6-15 $\bar{J} = \bar{a}_z J$. $\quad \oint \bar{B}\cdot d\bar{\ell} = \mu_0 I$.

If there is no hole,

$2\pi r_1 B_{\phi 1} = \mu_0 \pi r_1^2 J$

$\longrightarrow B_{\phi 1} = \frac{\mu_0 r_1}{2} J \longrightarrow \begin{cases} B_{x1} = -\frac{\mu_0 J}{2} y_1, \\ B_{y1} = +\frac{\mu_0 J}{2} x_1. \end{cases}$

For $-\bar{J}$ in the hole portion:

$B_{\phi 2} = -\frac{\mu_0 r_2}{2} J \longrightarrow \begin{cases} B_{x2} = +\frac{\mu_0 J}{2} y_2, \\ B_{y2} = -\frac{\mu_0 J}{2} x_2. \end{cases}$

Superposing $B_{\phi 1}$ and $B_{\phi 2}$ and noting that $y_1 = y_2$ and $x_1 = x_2 + d$,

we have $B_x = B_{x1} + B_{x2} = 0$, and $B_y = B_{y1} + B_{y2} = \frac{\mu_0 J}{2} d$.

P.6-17 $\bar{B} = \bar{\nabla}\times\bar{A} \longrightarrow \bar{B} = \bar{a}_\phi B = \bar{a}_\phi\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) = -\bar{a}_\phi \frac{\partial A_z}{\partial r}$.

For $0 \le r \le b$, Eq. (6-10) gives $\bar{B}_1 = \bar{a}_\phi \frac{\mu_0 I}{2\pi b^2} r$.

For $r \ge b$, Eq. (6-11) gives $\bar{B}_2 = \bar{a}_\phi \frac{\mu_0 I}{2\pi r}$.

Integrating, $\quad \bar{A}_1 = \bar{a}_z\left[-\frac{\mu_0 I}{4\pi}\left(\frac{r}{b}\right)^2 + c_1\right], \quad 0 \le r \le b.$

$\bar{A}_2 = \bar{a}_z\left[-\frac{\mu_0 I}{2\pi}\ln r + c_2\right], \quad r \ge b.$

At $r = b$, $\bar{A}_1 = \bar{A}_2 \longrightarrow c_2 = \frac{\mu_0 I}{4\pi}(2\ln b - 1) + c_1$

$\therefore\ \bar{A}_2 = \bar{a}_z\left\{-\frac{\mu_0 I}{4\pi}\left[\ln\left(\frac{r}{b}\right)^2 + 1\right] + c_1\right\}, \quad r \ge b.$

P.6-18 Eq. (6-34) for one wire: $\bar{A} = \bar{a}_z \frac{\mu_0 I}{4\pi}\ln\frac{\sqrt{L^2+r^2}+L}{\sqrt{L^2+r^2}-L}$.

For two wires carrying equal and opposite currents:

a) $\bar{A} = \bar{a}_z \frac{\mu_0 I}{4\pi}\ln\left[\frac{\sqrt{L^2+r_1^2}+L}{\sqrt{L^2+r_1^2}-L}\cdot\frac{\sqrt{L^2+r_2^2}-L}{\sqrt{L^2+r_2^2}+L}\right] = \bar{a}_z \frac{\mu_0 I}{2\pi}\ln\left[\frac{r_2}{r_1}\frac{\sqrt{L^2+r_1^2}-L}{\sqrt{L^2+r_2^2}+L}\right]$.

b) For a very long two-wire transmission line, $L \to \infty$:

$$\bar{A} = \bar{a}_z \frac{\mu_0 I}{2\pi} \ln\left(\frac{r_1}{r_2}\right) = \bar{a}_z \frac{\mu_0 I}{4\pi} \ln \frac{(\frac{d}{2}+y)^2+x^2}{(\frac{d}{2}-y)^2+x^2}.$$

c) $$\bar{B} = \bar{\nabla}\times\bar{A} = \bar{a}_x \frac{\partial A_z}{\partial y} - \bar{a}_y \frac{\partial A_z}{\partial x}$$

$$= \bar{a}_x \frac{\mu_0 I}{2\pi}\left[\frac{\frac{d}{2}+y}{(\frac{d}{2}+y)^2+x^2} - \frac{\frac{d}{2}-y}{(\frac{d}{2}-y)^2+x^2}\right] - \bar{a}_y \frac{\mu_0 I}{2\pi}\left[\frac{x}{(\frac{d}{2}+y)^2+x^2} - \frac{x}{(\frac{d}{2}-y)^2+x^2}\right]$$

$$= \frac{\mu_0 I}{2\pi}\left[\bar{a}_{\phi 1}\frac{1}{r_1} - \bar{a}_{\phi 2}\frac{1}{r_2}\right].$$

d) To find the equation for magnetic flux lines:

$$\frac{dx}{B_x} = \frac{dy}{B_y} \longrightarrow \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy = 0$$

$$\longrightarrow dA = 0 \longrightarrow A = \text{constant}.$$

Thus, $$\frac{r_1^2}{r_2^2} = \frac{(\frac{d}{2}+y)^2+x^2}{(\frac{d}{2}-y)^2+x^2} = K.$$

P.6-20 Apply divergence theorem to $(\bar{F}\times\bar{C})$, where $\bar{C}$ is a constant vector.

$$\int_V \bar{\nabla}\cdot(\bar{F}\times\bar{C})\,dv = \oint_S (\bar{F}\times\bar{C})\cdot d\bar{s}. \qquad ①$$

Now, from problem P.2-23: $\bar{\nabla}\cdot(\bar{F}\times\bar{C}) = \bar{C}\cdot(\bar{\nabla}\times\bar{F}) - \bar{F}\cdot(\bar{\nabla}\times\bar{C})$
$= \bar{C}\cdot(\bar{\nabla}\times\bar{F})$; ②

from Eq. (2-19): $(\bar{F}\times\bar{C})\cdot d\bar{s} = -\bar{C}\cdot(\bar{F}\times d\bar{s})$ ③

Substituting ② and ③ in ①:

$$\bar{C}\cdot\int(\bar{\nabla}\times\bar{F})\,dv = -\bar{C}\cdot\oint_S(\bar{F}\times d\bar{s}) \longrightarrow \int_V(\bar{\nabla}\times\bar{F})\,dv = -\oint \bar{F}\times d\bar{s}.$$

P.6-21

a) Given $\bar{B}_2 = \mu_2 \bar{H}_2$

$B_{2z} = B_{1z} \longrightarrow \mu H_2 = \mu_0 H_0 \longrightarrow \bar{H}_2 = \bar{a}_z H_2 = \bar{a}_z \frac{\mu_0}{\mu} H_0.$

b) Given $\bar{B}_2 = \mu_0(\bar{H}_2 + \bar{M}_i)$.

$B_{2z} = B_{1z} \longrightarrow \mu_0(H_2 + M_i) = \mu_0 H_0 \longrightarrow \bar{H}_2 = \bar{a}_z(H_0 - M_i).$

P.6-22 a) $r < a$: $\left.\begin{array}{l}\bar{H} = \bar{a}_z nI, \\ \bar{B} = \bar{a}_z \mu nI\end{array}\right\}$ Eq. (6-14). $\quad$ $a < r < b$: $\bar{H} = \bar{a}_z nI$, $\bar{B} = \bar{a}_z \mu_0 nI$, $\bar{M} = 0$.

$\bar{M} = \frac{\bar{B}}{\mu_0} - \bar{H} = \bar{a}_z\left(\frac{\mu}{\mu_0} - 1\right)nI.$

b) $\bar{J}_m = \bar{\nabla}\times\bar{M} = 0$; $\quad \bar{J}_{ms} = \bar{M}\times\bar{a}_n = (\bar{a}_z\times\bar{a}_r)\left(\frac{\mu}{\mu_0} - 1\right)nI = \bar{a}_\phi\left(\frac{\mu}{\mu_0} - 1\right)nI.$

P.6-24

a) $V_m = \frac{I}{4\pi}\int \frac{d\bar{s}\cdot\bar{a}_R}{R^2} = \frac{I}{4\pi}\Omega.$

$d\bar{s}\cdot\bar{a}_R = (\cos\alpha)\rho\, d\rho\, d\phi = \frac{z}{\sqrt{z^2+\rho^2}}\rho\, d\rho\, d\phi,$

$R = \sqrt{z^2+\rho^2}.$

$\therefore V_m = \frac{I}{4\pi}\int_0^{2\pi}\int_0^{b} \frac{z}{(z^2+\rho^2)^{3/2}}\rho\, d\rho\, d\phi$

$= \frac{I}{2}\left(1 - \frac{z}{\sqrt{z^2+b^2}}\right).$

b) $\bar{B} = -\mu_0\bar{\nabla}V_m = -\bar{a}_z\mu_0\frac{\partial V_m}{\partial z} = \bar{a}_z\frac{\mu_0 I b^2}{2(z^2+b^2)^{3/2}}$, which is the same as Eq.(6-38).

P.6-25 A cylindrical bar magnet having a uniform magnetization $\bar{M} = \bar{a}_z M_0$ is equivalent to a $\bar{J}_m = \bar{\nabla}\times\bar{M} = 0$ and a $\bar{J}_{ms} = \bar{M}\times\bar{a}_n = (\bar{a}_z M_0)\times\bar{a}_r = \bar{a}_\phi M_0$ on the cylinder wall. At a distant point, $\bar{B}$ due to this $\bar{J}_{ms}$ flowing on a cylindrical wall of length L and radius b is the same as that due to a circular loop of radius b carrying a current $I = M_0 L$. It is given by Eq.(6-44), which is the same as Eq.(6-73) obtained in Example 6-9 where the total dipole moment of the cylindrical magnet is $M_T = I\pi b^2 = M_0 L\pi b^2$.

P.6-26

$\bar{M} = \bar{a}_z M_0$

a) $\bar{J}_m = \bar{\nabla}\times\bar{M} = 0.$

$\bar{J}_{ms} = (\bar{a}_R\cos\theta - \bar{a}_\theta\sin\theta)M\times\bar{a}_R$

$= \bar{a}_\phi M_0\sin\theta.$

b) Apply Eq.(6-38) to a loop of radius $b\sin\theta$ carrying a current $J_{ms}\, b\, d\theta$:

$d\bar{B} = \bar{a}_z\frac{\mu_0(J_{ms}b\,d\theta)(b\sin\theta)^2}{2(b^2)^{3/2}}$

$= \bar{a}_z\frac{\mu_0 M_0}{2}\sin^3\theta.$

$\bar{B} = \int d\bar{B} = \bar{a}_z\frac{\mu_0 M_0}{2}\int_0^{\pi}\sin^3\theta\, d\theta = \bar{a}_z\frac{2}{3}\mu_0 M_0 = \frac{2}{3}\mu_0\bar{M}.$

P.6-27 a) $\mathcal{R}_g = \frac{\ell_g}{\mu_0 S} = \frac{3\times10^{-3}}{4\pi\times10^{-7}\times(\pi\times0.025)^2} = 1.21\times10^6\ (H^{-1}),$

$\mathcal{R}_c = \frac{2\pi\times0.08 - 0.003}{3000\times(4\pi\times10^{-7})\times(\pi\times0.025)^2} = 6.75\times10^4\ (H^{-1}).$

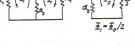
b) $\bar{B}_g = \bar{B}_c = \bar{a}_\phi \frac{10^{-5}}{\pi\times0.025^2} = \bar{a}_\phi\, 5.09\times10^{-3}\ (T),$

$\bar{H}_g = \frac{1}{\mu_0}\bar{B}_g = \bar{a}_\phi \frac{5.09\times10^{-3}}{4\pi\times10^{-7}} = \bar{a}_\phi\, 4.05\times10^3\ (A/m),$

$\bar{H}_c = \frac{1}{\mu_0\mu_r}\bar{B}_c = \bar{a}_\phi \frac{4.05\times10^3}{3000} = \bar{a}_\phi\, 1.35\ \ (A/m).$

c) $NI = \Phi(\mathcal{R}_c + \mathcal{R}_g),\ \ I = \frac{1}{N}\Phi(\mathcal{R}_c + \mathcal{R}_g) = 0.0256\,(A) = 25.6\,(mA)$

P.6-28 Magnetic circuit:



$\Phi_1 = \Phi_0/2$

$\frac{1}{\mu_0 S} = \frac{1}{(4\pi\,10^{-7})\times10^{-3}} = 7.95\times10^{8}$

Neglecting leakage flux and assuming constant flux density over S:

$\mathcal{R}_0 = \frac{0.002}{\mu_0 S} - \frac{0.24 - 0.002}{\mu_0\mu_r S}$

$= 1.60\times10^6\ (H^{-1}),$

$\mathcal{R}_1 = \frac{0.24 + 2\times0.2}{\mu_0\mu_r S} = 0.102\times10^6\ (H^{-1}).$

a) $\Phi_0 = \frac{NI}{\mathcal{R}_0 + \mathcal{R}_1/2} = 3.63\times10^{-4}\ (Wb);\qquad \Phi_1 = \frac{\Phi_0}{2} = 1.82\times10^{-4}\ (Wb).$

b) $H_1 = \frac{\Phi_1}{\mu_0\mu_r S} = 28.9\ (A/m),$

$(H_0)_g = \frac{1}{\mu_0 S}\Phi_0 = 28.9\times10^4\ (A/m)$ in air gap,

$(H_0)_c = (H_0)_g/\mu_r = 57.8\ \ (A/m).$

P.6-29 a) Work required per unit length in time dt:

$$P_1\,dt = nI\,d\Phi.$$

Work per unit volume in dt: $dW = \frac{1}{S}P_1\,dt = nI\,dB = H\,dB.$

Thus. $\quad W_1 = \int_0^{B_f} H\,dB.$

P.6-31 $\bar{H}_1 = -\bar{\nabla}V_{m1},\quad \bar{H}_2 = -\bar{\nabla}V_{m2}.$

Boundary conditions: $\mu_1 H_{1n} = \mu_2 H_{2n} \longrightarrow \mu_1\frac{\partial V_{m1}}{\partial n} = \mu_2\frac{\partial V_{m2}}{\partial n},$

$H_{1t} = H_{2t} \longrightarrow V_{m1} = V_{m2}$

(assuming absence of current)

P.6-32

a) $\bar{B}_1 = \bar{a}_x 0.5 - \bar{a}_y 10$ (mT).

$\bar{B}_2 = \bar{a}_x B_{2x} - \bar{a}_y B_{2y}$.

$H_{2x} = \frac{B_{2x}}{5000\mu_0} = H_{1x} = \frac{0.5}{\mu_0}$

$\longrightarrow B_{2x} = 2{,}500$ (mT),

$B_{2y} = B_{1y} = -10$ (mT).

$\therefore \bar{B}_2 = \bar{a}_x 2500 - \bar{a}_y 10$ (mT).

$\tan\alpha_2 = \frac{\mu_2}{\mu_1}\tan\alpha_1 = 5000\left(\frac{B_{1x}}{B_{1y}}\right) = 250 \longrightarrow \alpha_2 = 89.8°, \alpha_2' = 0.2°.$

b) If $\bar{B}_2 = \bar{a}_x 10 + \bar{a}_y 0.5$ (mT), $\quad \bar{B}_1 = \bar{a}_x B_{1x} + \bar{a}_y B_{1y}$.

$H_{1x} = \frac{B_{1x}}{\mu_1} = H_{2x} = \frac{B_{2x}}{\mu_2} \longrightarrow B_{1x} = \frac{1}{\mu_{r2}} B_{2x} = \frac{10}{5000} = 0.002.$

$B_{1y} = B_{2y} = 0.5. \qquad \therefore \bar{B}_1 = \bar{a}_x 0.002 + \bar{a}_y 0.5$ (mT)

$\alpha_1 = \tan^{-1}\frac{B_{1x}}{B_{1y}} \cong \frac{0.002}{0.5} = 0.004$ (rad) $= 0.23°$

P.6-33 a) Consider two situations: ① I and I_i both in air; and ② I and $-I_i$ both in magnetic medium with relative permeability μ_r.

Find B_{1y} and H_{1x} at $P\,(y=0)$.

$B_{1y} = \frac{\mu_0}{2\pi r}(I+I_i)\cos\alpha = \frac{\mu_0\mu_r}{\pi(\mu_r+1)}\frac{x}{r^2}I,$

$B_{1x} = \frac{\mu_0}{2\pi r}(I-I_i)\sin\alpha = \frac{\mu_0}{\pi(\mu_r+1)}\frac{d}{r^2}I,$

$H_{1x} = \frac{B_{1x}}{\mu_0} = \frac{I}{\pi(\mu_r+1)}\frac{d}{r^2}.$

Find B_{2y} and H_{2x} at $P\,(x=0)$.

$B_{2y} = \frac{\mu_0\mu_r}{2\pi r}(I-I_i)\cos\alpha = \frac{\mu_0\mu_r}{\pi(\mu_r+1)}\frac{x}{r^2}I,$

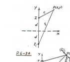
$B_{2x} = \frac{\mu_0\mu_r}{2\pi r}(I-I_i)\sin\alpha = \frac{\mu_0\mu_r}{\pi(\mu_r+1)}\frac{d}{r^2}I,$

$H_{2x} = \frac{B_{2x}}{\mu_0\mu_r} = \frac{I}{\pi(\mu_r+1)}\frac{d}{r^2}.$

$\therefore B_{1y} = B_{2y}$ and $H_{1x} = H_{2x}$ (Boundary conditions satisfied)

b) For $\mu_r >> 1$, $I_i = \frac{\mu_r - 1}{\mu_r + 1} I \cong I$.

Refer to the following figure.



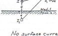
$$\bar{B}_I = \frac{\mu_0 I}{2\pi r_1}\left(-\bar{a}_x \frac{y-d}{r_1} + \bar{a}_y \frac{x}{r_1}\right),$$

$$\bar{B}_{I_i} = \frac{\mu_0 I}{2\pi r_2}\left(-\bar{a}_x \frac{y+d}{r_2} + \bar{a}_y \frac{x}{r_2}\right).$$

$$\therefore\ \bar{B} = \bar{B}_I + \bar{B}_{I_i}$$

$$= -\bar{a}_x \frac{\mu_0 I}{2\pi}\left[\frac{y-d}{(y-d)^2+x^2} + \frac{y+d}{(y+d)^2+x^2}\right] + \bar{a}_y \frac{\mu_0 I}{2\pi}\left[\frac{1}{(y-d)^2+x^2} + \frac{1}{(y+d)^2+x^2}\right]$$

P.6-34



a) (i) If $\sigma_2 \to \infty$, $\bar{B}_2 = \bar{H}_2 = 0$.

B_n continuous $\longrightarrow B_{1n} = H_{2n} = 0$;

$\bar{a}_y \times \bar{H}_1 = \bar{J}_s \longrightarrow \bar{J}_s = -\bar{a}_z H_{1x}$.

Image I_i $(=-I)$ flowing out of the paper.

(ii) If $\mu_2 \to \infty$, $\bar{H}_2 = 0$, but $\bar{B}_2$ is finite.

No surface current. $\longrightarrow H_{1t} = H_{2t} = 0$;

B_n continuous $\longrightarrow B_{1n} = B_{2n}$.

Image I_i $(=I)$ flowing into the paper.

b) (i) $\bar{H}_P = \bar{H}_1 + (\bar{H}_2)_i$, where $\bar{H}_1 = \frac{I}{2\pi}\left[\bar{a}_x \frac{y-d}{x^2+(y-d)^2} - \bar{a}_y \frac{x}{x^2+(y-d)^2}\right]$,

$(\bar{H}_2)_i = \frac{I}{2\pi}\left[-\bar{a}_x \frac{y+d}{x^2+(y+d)^2} + \bar{a}_y \frac{x}{x^2+(y+d)^2}\right]$.

(ii) $\bar{H}_P' = \bar{H}_1 + (\bar{H}_2)_{i2} = \bar{H}_1 - (\bar{H}_2)_i$.

c) (i) $\bar{J}_s = -\bar{a}_z (H_P)_x\big|_{y=0} = \bar{a}_z\left(\frac{Id}{x^2+d^2}\right)$.

(ii) $\bar{J}_s = 0$.

P.6-35

$\bar{B} = \bar{a}_\phi B_\phi = \bar{a}_\phi \frac{\mu_0 N I}{2\pi r}$, $\quad r = r_0 - \rho\cos\alpha$

$\Phi = \frac{\mu_0 N I}{2\pi}\int_0^b\int_0^{2\pi} \frac{\rho\, d\alpha\, d\rho}{r_0 - \rho\cos\alpha} = \mu_0 N I\left(r_0 - \sqrt{r_0^2 - b^2}\right)$.

$\therefore\ L = \frac{N\Phi}{I} = \mu_0 N^2\left(r_0 - \sqrt{r_0^2 - b^2}\right)$.

If $r_0 \gg b$, $B_\phi \cong \frac{\mu_0 N I}{2\pi r_0}$ (constant).

$\Phi \cong B_\phi S = B_\phi(\pi b^2) = \frac{\mu_0 N b^2 I}{2 r_0} \longrightarrow L \cong \frac{\mu_0 N^2 b^2}{2 r_0}$

P.6-36 For $b \le r \le (b+d)$, $\bar{B}_3 = \bar{a}_\phi B_{3\phi} = \bar{a}_\phi \frac{\mu_0 I}{2\pi r}\left[1 - \frac{\pi(r^2-b^2)}{\pi(b+d)^2 - \pi b^2}\right]$

$$= \bar{a}_\phi \frac{\mu_0 I}{2\pi r}\left[\frac{(b+d)^2 - r^2}{(b+d)^2 - b^2}\right].$$

Magnetic energy per unit length stored in the outer conductor:

$$W'_m = \frac{1}{2\mu_0}\int_b^{b+d} B_{3\phi}^2\, 2\pi r\, dr$$

$$= \frac{\mu_0 I^2}{4\pi}\left\{\frac{(b+d)^4}{[(b+d)^2-b^2]^2}\ln\left(1+\frac{d}{b}\right) + \frac{b^2-3(b+d)^2}{4[(b+d)^2-b^2]}\right\}$$

From Eqs. (6-175), (6-176a), and (6-176b) we have

$$L' = \frac{2}{I^2}\left(W'_{m1} + W'_{m2} + W'_{m3}\right)$$

$$= \frac{\mu_0}{2\pi}\left\{\frac{1}{4} + \ln\frac{b}{a} + \frac{(b+d)^4}{[(b+d)^2-b^2]^2}\ln\left(1+\frac{d}{b}\right) - \frac{b^2-3(b+d)^2}{4[(b+d)^2-b^2]}\right\} \text{(H)}$$

P.6-37

$\bar{B}$ at a distance r from an infinitely long line carrying a current I: $\bar{B} = \bar{a}_\phi \frac{\mu_0 I}{2\pi r}$

For a unit length the flux due to I in line A that links with the second line pair B-B' is

$$\Phi'_A = \frac{\mu_0 I}{2\pi}\int_{AB}^{AC}\frac{dr}{r} = \frac{\mu_0 I}{2\pi}\ln\frac{AC}{AB}.$$

That due to I in line A' is

$$\Phi'_{A'} = \frac{\mu_0 I}{2\pi}\ln\frac{A'C'}{A'B'}$$

Total flux linkage per unit length

$$\Lambda'_{12} = \Phi'_A + \Phi'_{A'} = \frac{\mu_0 I}{2\pi}\ln\frac{(AC)(A'C')}{(AB)(A'B')}$$

$$= \frac{\mu_0 I}{2\pi}\ln\frac{(AB')(A'B)}{(AB)(A'B')} = \frac{\mu_0 I}{2\pi}\ln\frac{D^2+d^2}{D^2}$$

$$\therefore M'_{12} = \frac{\Lambda'_{12}}{I} = \frac{\mu_0}{2\pi}\ln\left(1+\frac{d^2}{D^2}\right).$$

P.6-38 For I in the long straight wire, $\bar{B} = \bar{a}_\phi \frac{\mu_0 I}{2\pi r}$.

$$\Lambda_{12} = \int_S \bar{B}\cdot d\bar{s} = \int B_\phi \frac{2}{\sqrt{3}}(r-d)\,dr = \frac{\mu_0 I}{\pi\sqrt{3}}\int_d^{d+\frac{\sqrt{3}}{2}b}\left(\frac{r-d}{r}\right)dr$$

$$= \frac{\mu_0 I}{\pi\sqrt{3}}\left[\frac{\sqrt{3}}{2}b - d\ln\left(1+\frac{\sqrt{3}b}{2d}\right)\right] \longrightarrow L_{12} = \frac{\mu_0}{\pi}\left[\frac{b}{2} - \frac{d}{\sqrt{3}}\ln\left(1+\frac{\sqrt{3}b}{2d}\right)\right].$$

P.6-39

Assume a current I.

B at $P(r,\theta)$ is $\bar{a}_\phi \dfrac{\mu_0 I}{2\pi(d+r\cos\theta)}$.

$$\Lambda_{12} = \frac{\mu_0 I}{2\pi}\int_0^b\int_0^{2\pi} \frac{r\,dr\,d\theta}{d+r\cos\theta}$$

$$= \frac{\mu_0 I}{2\pi}\int_0^b \frac{2\pi r\,dr}{\sqrt{d^2-r^2}} = \mu_0 I\left(d-\sqrt{d^2-b^2}\right).$$

$$L_{12} = \mu_0\left(d-\sqrt{d^2-b^2}\right).$$

P.6-40 Approximate the magnetic flux due to the long loop linking with the small loop by that due to two infinitely long wires carrying equal and opposite current I.

$$\Lambda_{12} = \frac{\mu_0 h_2 I}{2\pi}\int_0^{w_2}\left(\frac{1}{d+x} - \frac{1}{w_1+d+x}\right)dx = \frac{\mu_0 h_2 I}{2\pi}\ln\left(\frac{w_2+d}{d}\cdot\frac{w_1+d}{w_1+w_2+d}\right).$$

$$L_{12} = \frac{\Lambda_{12}}{I} = \frac{\mu_0 h_2}{2\pi}\ln\frac{(w_1+d)(w_2+d)}{d(w_1+w_2+d)}.$$

P.6-41 Eq. (6-161): $W_2 = \frac{1}{2}L_1 I_1^2 + M I_1 I_2 + \frac{1}{2}L_2 I_2^2$.

a) $W_2 = \dfrac{I_2^2}{2}\left[L_1\left(\dfrac{I_1}{I_2}\right)^2 + 2M\left(\dfrac{I_1}{I_2}\right) + L_2\right] = \dfrac{I_2^2}{2}\left(L_1 x^2 + 2Mx + L_2\right)$, $x = \dfrac{I_1}{I_2}$.

$\dfrac{dW_2}{dx} = \dfrac{I_2^2}{2}\left(2L_1 x + 2M\right) = 0$, $\quad \dfrac{d^2 W_2}{dx^2} = I_2^2 L_1 > 0$.

$\therefore\ x = \dfrac{I_1}{I_2} = -\dfrac{M}{L_1}$ for minimum W_2.

b) $(W_2)_{min} = \dfrac{I_2^2}{2}\left(-\dfrac{M^2}{L_1} + L_2\right) \ge 0 \longrightarrow M \le \sqrt{L_1 L_2}$.

P.6-42

$I_1 = I_2 = I_3 = 25$ (A); $\quad d = 0.15$ (m).

$\bar{B}_2 = \bar{a}_x 2B_{12}\cos 30° = \bar{a}_x \dfrac{\sqrt{3}\mu_0 I}{2\pi d}$.

Force per unit length on wire 2:

$\bar{F}_2 = -\bar{a}_y I B_2 = -\bar{a}_y \dfrac{\sqrt{3}\mu_0 I^2}{2\pi d}$

$= -\bar{a}_y 1150\mu_0 = -\bar{a}_y 1.44\times10^{-3}$ (N/m).

Forces on all three wires are of equal magnitude and toward the center of the triangle.

P.6-43 Magnetic field intensity at the wire due to the current $dI = \frac{I}{w}dy$ in an elemental dy is

$$|d\bar{H}| = \frac{dI}{2\pi r} = \frac{I\,dy}{2\pi w\sqrt{D^2+y^2}}.$$

Symmetry ⟶ H at the wire has only a y-component.

$$\bar{H} = \bar{a}_y \int (dH)\cdot\left(\frac{D}{r}\right) = \bar{a}_y 2\int_0^{w/2} \frac{ID\,dy}{2\pi w(D^2+y^2)}$$
$$= \bar{a}_y \frac{I}{\pi w}\tan^{-1}\left(\frac{w}{2D}\right).$$

$$\bar{f}' = \bar{I}\times\bar{B} = (-\bar{a}_z I)\times(\mu_0\bar{H}) = \bar{a}_x \frac{\mu_0 I^2}{\pi w}\tan^{-1}\left(\frac{w}{2D}\right) \quad (N/m).$$

P.6-44 From Problem P.6-4 we have the y-component of the magnetic flux density at an arbitrary point $P(d,y)$ on the right-hand strip due to I_1 in the left-hand strip

$$B_{Py} = -\frac{\mu_0 I_1}{2\pi w}\left[\tan^{-1}\left(\frac{y}{d}\right) + \tan^{-1}\left(\frac{w-y}{d}\right)\right].$$

The x-component of the force on a strip of width dy due to I_2 in the right-hand conductor is

$$dF'_{2x} = \left(\frac{I_2}{w}dy\right)B_{Py} \quad \text{(in the +x direction, a repulsive force).}$$

$$\bar{F}'_2 = \bar{a}_x \int dF_{2x} = \bar{a}_x \frac{\mu_0 I_1 I_2}{2\pi w^2}\int_0^{w}\left[\tan^{-1}\left(\frac{y}{d}\right) + \tan^{-1}\left(\frac{w-y}{d}\right)\right]dy$$
$$= \bar{a}_x \frac{\mu_0 I_1 I_2}{2\pi w^2}\left[2w\tan^{-1}\left(\frac{w}{d}\right) - d\cdot\ln\left(1+\frac{w^2}{d^2}\right)\right] \quad \text{per unit length}$$

There is no net force in the y-direction.

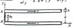
P.6-45 B due to I_1 in the straight wire in the z-direction at an elemental arc $b\,d\theta$ on the circular loop is

$$\bar{B} = \bar{a}_\phi \frac{\mu_0 I_1}{2\pi(d+b\cos\theta)}.$$

$$\bar{F}_{\text{on loop}} = -\bar{a}_x 2\int_0^{\pi}(I_2 b\,d\theta)\cos\theta$$
$$= -\bar{a}_x \frac{\mu_0 I_1 I_2 b}{\pi}\int_0^{\pi}\frac{\cos\theta}{d+b\cos\theta}d\theta$$
$$= \bar{a}_x \mu_0 I_1 I_2\left[\frac{1}{\sqrt{1-(b/d)^2}} - 1\right] \quad \text{(Repulsive force).}$$

$\bar{F}$ has no net y-component.

P.6-46 $\bar{B}_{\text{at } y} = \bar{a}_z \frac{\mu_0 I}{4\pi}\left(\frac{1}{y}+\frac{1}{d-y}\right)$. $d\bar{\ell} = \bar{a}_y\,dy$



$$d\bar{F} = I\,d\bar{\ell}\times\bar{B}$$
$$= -\bar{a}_x \frac{\mu_0 I^2}{4\pi}\left(\frac{1}{y}+\frac{1}{d-y}\right)dy$$
$$\bar{F} = -\bar{a}_x \frac{\mu_0 I^2}{4\pi}\int_b^{d-b}\left(\frac{1}{y}+\frac{1}{d-y}\right)dy$$
$$= -\bar{a}_x \frac{\mu_0 I^2}{2\pi}\ln\left(\frac{d}{b}-1\right).$$

(A rail-gun problem.)

P.6-47

$\bar{B}=\bar{a}_y 0.5\,(T)$; $\bar{B}\times$	$I(\vec{AB})$	$I(\vec{CA})$	$I(\vec{BC})$
	$(\bar{a}_y 0.5)\times 10(-\bar{a}_x 0.1-\bar{a}_y 0.2)$,	$10(-\bar{a}_x 0.1+\bar{a}_y 0.2)$,	$10\,\bar{a}_x 0.2$
Force:	$\bar{a}_z 0.5$	$\bar{a}_z 0.5$	$\bar{a}_z 1.0$ (N)

Torque on loop:

$$\bar{T} = \bar{m}\times\bar{B} = (\bar{a}_z I S)\times\bar{B}$$
$$= (\bar{a}_z 10\times\tfrac{1}{2}\times 0.2\times 0.2)\times(\bar{a}_y 0.5) = -\bar{a}_x 0.1\ (N\cdot m).$$

P.6-48 Let x-axis be the center line of the coaxial cable. The magnetic energy stored in a section of length x is

$$W_m = \tfrac{1}{2}LI^2.$$
$$L = \frac{\Phi}{I} = \frac{x}{I}\int_a^b B_\phi\,dr = \frac{x}{I}\int_a^b \frac{\mu_0 I}{2\pi r}dr = \frac{\mu_0 x}{2\pi}\ln\frac{b}{a}.$$
$$\bar{F}_x = \bar{a}_x\frac{\partial W_m}{\partial x} = \bar{a}_x\left(\frac{I^2}{2}\right)\frac{\partial L}{\partial x} = \bar{a}_x\frac{\mu_0 I^2}{4\pi}\ln\frac{b}{a}.$$

P.6-49 Resolve the circular loop into many small loops, each with a magnetic dipole moment $d\bar{m} = I_2\,d\bar{s}$, $d\bar{T} = d\bar{m}\times\bar{B}$.

$$\bar{T} = \int d\bar{T} = I_2\int d\bar{s}\times\bar{B} = -\bar{a}_x I_2\sin\alpha\int B\,ds = -\bar{a}_x\mu_0 I_1 I_2\left(d-\sqrt{d^2-b^2}\right)\sin\alpha$$

From Problem P.6-39

—— This torque is in the direction of aligning the flux produced by I_2 in the loop with that of $\bar{B}$ due to I_1 in the straight wire.

P.6-50 $\bar{B}_2$ at the of the large circular turn of wire carrying a current I_2 is (by setting $z=0$ in Eq. 6-38):

$$\bar{B}_2 = \bar{a}_{z2}\frac{\mu_0 I_2}{2r_2}.$$

Torque on the small circular wire: $\bar{T} = \bar{m}_1 \times \bar{B}_2 = (\bar{a}_{z1} I_1 \pi r_1^2) \times (\bar{a}_{z2}\frac{\mu_0 I_2}{2r_2}) = (\bar{a}_{z1} \times \bar{a}_{z2})\frac{\mu_0 I_2 I_1 \pi r_1^2}{2r_2}$

$\longrightarrow$ Magnitude $= \frac{\mu_0 I_1 I_2 \pi r_1^2}{2r_2}\sin\theta$ in a direction to align the magnetic fluxes produced by I_1 & I_2.

P.6-51

$\bar{B}_m$ (magnetized compass needle)

$$= \frac{\mu_0 m}{4\pi R^3}(\bar{a}_R 2\cos\theta + \bar{a}_\theta \sin\theta) \quad \text{from Eq. (6-49)}$$

$$= \frac{(4\pi\times10^{-7})\times2}{4\pi(0.15)^3}(\bar{a}_R 2\cos\theta + \bar{a}_\theta \sin\theta)$$

$$= 0.59\times10^{-4}(\bar{a}_R 2\cos\theta + \bar{a}_\theta \sin\theta)\ (T).$$

$\bar{B}_e(\text{earth}) = -\bar{a}_\theta 10^{-4}\ (T)$.

Max. deflection occurs when $\left|\frac{B_R}{B_\theta}\right|$ is max. or when

$$\left|\frac{B_\theta}{B_R}\right| = \left|\frac{(0.59\sin\theta - 1)\times10^{-4}}{1.18\times10^{-4}\cos\theta}\right| \text{ is min.}$$

Set $\frac{d}{d\theta}\left(\frac{1-0.59\sin\theta}{1.18\cos\theta}\right) = 0$ —— $\sin\theta = 0.59$, or $\theta = 36.2°$

At $\theta = 36.2°$, $|B_R/B_\theta| = 1.47$, and $\alpha = \tan^{-1}1.47 = 55.8°$.

P.6-52

$$F = \frac{\Phi^2}{\mu_0 S} = \frac{(NI)^2}{\mu_0 S\left(\frac{2\ell_g}{\mu_0 S} + \frac{\ell_i}{\mu_0\mu_r S}\right)^2} = \frac{(NI)^2\mu_0 S}{\left(2\ell_g - \frac{\ell_i}{\mu_r}\right)^2}$$

$F = 100\times9.8 = 980\ (N)$, $S = 0.01\ (m^2)$, $\ell_g = 2\times10^{-3}\ (m)$, $\ell_i = 3\ (m)$, $\mu_r = 4000$.

Solving: $\quad \text{mmf} = NI = 1.33\times10^3\ (A\text{-}t)$.

P.6-53 $\quad W_m = \frac{1}{2}\int \mu H^2\,dv$

Assume a virtual displacement, Δx, of the iron core.

$$W_m(x+\Delta x) = W_m(x) + \frac{1}{2}\int_{S\Delta x}(\mu - \mu_0)H^2\,dv$$

$$= W_m(x) + \frac{1}{2}\mu_0(\mu_r - 1)n^2 I^2 S\Delta x.$$

$$(F_I)_x = \frac{\partial W_m}{\partial x} = \frac{\mu_0}{2}(\mu_r - 1)n^2 I^2 S, \text{ in the direction of increasing } x.$$

Chapter 7

Time-Varying Field and Maxwell's Equations

P.7-1 $\mathcal{V} = -\int_S \frac{\partial \bar{B}}{\partial t}\cdot d\bar{s} = -\int_S \frac{\partial}{\partial t}(\nabla\times\bar{A})\cdot d\bar{s} = -\oint_C \frac{\partial \bar{A}}{\partial t}\cdot d\bar{\ell}.$

P.7-2. $\bar{B} = \bar{a}_z 3\cos(5\pi 10^7 t - \frac{2}{3}\pi x)\cdot 10^{-6}$ (T).

$$\int_S \bar{B}\cdot d\bar{s} = \int_0^{0.6} \bar{a}_z 3\cos(5\pi 10^7 t - \tfrac{2}{3}\pi x)10^{-6}\cdot(\bar{a}_z 0.2\,dx)$$

$$= -\frac{0.18}{2\pi}\left[\sin(5\pi 10^7 t - 0.4\pi) - \sin 5\pi 10^7 t\right]\cdot 10^{-6}\ \text{(Wb)},$$

$$\mathcal{V} = -\frac{d}{dt}\int \bar{B}\cdot d\bar{s} = 45\left[\cos(5\pi 10^7 t - 0.4\pi) - \cos 5\pi 10^7 t\right]\ \text{(V)},$$

$$i = \frac{\mathcal{V}}{2R} = 1.5\left[\cos(5\pi 10^7 t - 0.4\pi) - \cos 5\pi 10^7 t\right]$$

$$= 1.76\sin(5\pi 10^7 t - 0.2\pi)\ \text{(A)}.$$

P.7-3 In the rectangur loop with the assigned direction for i_2

$$L_{12}\frac{di_1}{dt} = L\frac{di_2}{dt} + Ri_2. \qquad ①$$

where $L_{12} = \frac{\Phi_{12}}{i_1} = \frac{h}{i_1}\int_d^{d+w} B_{12}\,dr = \frac{h}{i_1}\int_d^{d+w}\frac{\mu_0 i_1}{2\pi r}\,dr$

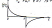
$$= \frac{\mu_0 h}{2\pi}\ln\left(1+\frac{w}{d}\right). \qquad ②$$

a) At $t=0$, $i_1(t) = I_1 U(t)$ is applied and ① becomes

$$L\frac{di_2}{dt} + Ri_2 = L_{12} I_1 \delta(t). \qquad ③$$

Solution of ③: $i_2 = \frac{L_{12}}{L} I_1 e^{-(R/L)t}, \quad 0<t<T \qquad ④$

At $t=T$, $i_2 = \frac{L_{12}}{L} I_1 e^{-RT/L}$, when a negative step function $-I_1 U(t)$ is applied. If $T \gg L/R$, then i_2 for $t>T$ is the reverse of i_2 for $0<t<T$; i.e., $i_2 = -\frac{L_{12}}{L} I_1 e^{-(R/L)(t-T)}, \ t>T.$



b) Energy dissipated in R.

$$W \cong 2\left(\frac{L_{12}}{L}I_1\right)^2 R\int_0^\infty e^{-(2R/L)t}\,dt$$

$$= \frac{1}{L}(L_{12}I_1)^2.$$

P.7-4 $\bar{B} = \bar{a}_\phi \frac{\mu_0 I \sin\omega t}{2\pi r}$, $\Phi = \int_S \bar{B}\cdot d\bar{s}$, $d\bar{s} = \bar{a}_\phi 2z\,dr$, $z = \frac{\sqrt{3}}{3}(r-d)$.

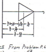
a) $\Phi = \frac{\sqrt{3}}{3}\frac{\mu_0 I \sin\omega t}{\pi}\int_d^{\frac{\sqrt{3}}{2}b+d}\left(1-\frac{d}{r}\right)dr = \frac{\sqrt{3}\mu_0 I \sin\omega t}{3\pi}\left[\frac{\sqrt{3}}{2}b - d\ln\left(\frac{\frac{\sqrt{3}}{2}b+d}{d}\right)\right]$.

$d = \frac{b}{2}$, $\mathcal{V} = -\frac{\partial\Phi}{\partial t} = -\frac{\sqrt{3}\mu_0 I\omega b}{3\pi}\left[\frac{\sqrt{3}}{2} - \frac{1}{2}\ln(\sqrt{3}+1)\right]\cos\omega t$

$= V_m \cos\omega t$.

$V_{rms} = \frac{\sqrt{2}}{2}|V_m| = \frac{\sqrt{6}\mu_0 I\omega b}{12\pi}\left[\sqrt{3} - \ln(\sqrt{3}+1)\right]$

$= 0.0472\,\mu_0 I\omega b$ (V).



b) $z = \frac{1}{\sqrt{3}}\left[\frac{b}{2}\left(1+\frac{4}{\sqrt{3}}\right) - r\right]$.

$\int \bar{B}\cdot d\bar{s} = \frac{\mu_0 I \sin\omega t}{\sqrt{3}\pi}\int_{\frac{b}{2}(1+\frac{1}{\sqrt{3}})}^{\frac{b}{2}(1+\frac{4}{\sqrt{3}})}\left[\frac{b}{2}\left(1+\frac{4}{\sqrt{3}}\right)\frac{1}{r} - 1\right]dr$

$= \frac{\mu_0 I \sin\omega t}{\sqrt{3}\pi}\left[\frac{b}{2}\left(1+\frac{4}{\sqrt{3}}\right)\ln\left(\frac{4+\sqrt{3}}{1+\sqrt{3}}\right) - \frac{\sqrt{3}}{2}b\right]$

$V_{rms} = \frac{1}{\sqrt{2}}\frac{\mu_0 I\omega}{\sqrt{3}\pi}\frac{b}{2}\left|\left(1+\frac{4}{\sqrt{3}}\right)\ln\left(\frac{4+\sqrt{3}}{1+\sqrt{3}}\right) - \sqrt{3}\right|$

$= 0.0469\,\mu_0 I\omega b$ (V).

P.7-5 From Problem P.6-40: $\Phi_{12} = \mu_0 I(\sin\omega t)(d - \sqrt{d^2-b^2})$.

a) $\mathcal{V} = -\frac{d\Phi}{dt} = -\mu_0 I\omega(\cos\omega t)(d-\sqrt{d^2-b^2}) = V_m\cos\omega t$,

$3\times10^{-6} = \frac{|V_m|}{\sqrt{2}R} = \frac{\mu_0 I\omega(d-\sqrt{d^2-b^2})}{\sqrt{2}R}$.

$I = \frac{\sqrt{2}R\times3\times10^{-6}}{\mu_0\omega(d-\sqrt{d^2-b^2})} = \frac{3\sqrt{2}\times10^{-6}}{4\pi10^{-7}(2\pi60)\times0.0382} = 0.234$ (A).

b) $\alpha = \cos^{-1}(0.2/0.3) = 48.2°$.

P.7-6 a) Flux enclosed in the ring in Fig. 7-12(a): $\Phi = \pi r^2 B(t)$. ①

The induced emf in the ring referring to the assigned direction for current: $\mathcal{V} = iR_r = \frac{d\Phi}{dt} = \pi r^2\frac{dB(t)}{dt}$. ②

Resistance of differential circular ring: $R_r = \frac{2\pi r}{\sigma h\,dr}$. ③

Combining ② and ③: $i = \frac{\pi r^2}{R_r}\frac{dB(t)}{dt} = \frac{\sigma h}{2}r\,dr\left(\frac{dB}{dt}\right)$. ④

$dp = i^2R_r = \frac{\pi\sigma h}{2}r^3dr\left(\frac{dB}{dt}\right)^2$. ⑤

$P = \int dp = \frac{\pi\sigma h}{8}a^4\left(\frac{dB}{dt}\right)^2 = \frac{\pi\sigma h}{8}a^4\omega^2B_0^2\cos^2\omega t$. ⑥

$P_{av} = \frac{\pi\sigma h}{16}a^4\omega^2B_0^2$. ⑦

b) For N insulated filamentary parts, each with an area $S = \frac{0.95\pi a^2}{N} = \pi b^2$ —— $b = \sqrt{\frac{S}{\pi}} = a\sqrt{\frac{0.95}{N}}$.

Power loss in N filaments in Fig. 7-12(b) from ⑥: $p' = N\left(\frac{\pi\sigma h}{8}\right)\left(a\sqrt{\frac{0.95}{N}}\right)^4 \omega^2 B_0^2 \cos^2\omega t = \frac{0.95^2}{N} p$.

$P'_{av} = \frac{0.95^2}{N} P_{av}$.

P.7-7 $\Phi(t) = \bar{B}(t)\cdot\bar{S}(t) = -(5\cos\omega t)\times 0.2\,(0.7-x)$
$= -0.35\cos\omega t\,(1+\cos\omega t)$ (mT),
$i = -\frac{1}{R}\frac{d\Phi}{dt} = -\frac{1}{R}\,0.35\,\omega(\sin\omega t + \sin 2\omega t)$
$= -1.75\,\omega\sin\omega t\,(1+2\cos\omega t)$ (mA).

P.7-8

Assuming the loop to have N turns each with an area a×b, the torque on the loop is $\bar{T} = \bar{m}\times\bar{B} = -\bar{a}_x NabIB\sin\alpha$.

Mechanical work done by the motor in rotating through an angle $(-\Delta\alpha)$ is
$W_m = T(-\Delta\alpha) = NabIB(\Delta\alpha)\sin\alpha$.

Flux linking with the loop, $\Phi = NabB\cos\alpha$.
Emf induced in the loop, $\mathcal{V} = -\frac{d\Phi}{dt} = NabB\left(\frac{d\alpha}{dt}\right)\sin\alpha$.

Electric energy required to send current I against this emf in time Δt, $W_e = \mathcal{V}I(\Delta t) = NabB(\Delta\alpha)\sin\alpha = W_m$.

P.7-9 $i = \frac{\mathcal{V}}{R} = -\frac{1}{R}\frac{d}{dt}\int_S \bar{B}\cdot d\bar{s} = -\frac{1}{R}\frac{d}{dt}(B_0 hw\cos\omega t)$
$= \frac{\omega B_0 hw}{R}\sin\omega t$,
$p_d = i^2R = \frac{(\omega B_0 hw)^2}{R}\sin^2\omega t$ (Power dissipated in R).

On the other hand,
for side 1-2: $\bar{F}_{12} = \bar{a}_z ihB_0$, $\bar{u}_{12} = \frac{\omega w}{2}(\bar{a}_y\cos\omega t - \bar{a}_z\sin\omega t)$;
for side 4-3: $\bar{F}_{43} = -\bar{a}_z ihB_0$, $\bar{u}_{43} = \frac{\omega w}{2}(-\bar{a}_y\cos\omega t + \bar{a}_z\sin\omega t)$.

Mechanical power required to rotate coil:
$p_m = -(\bar{F}_{12}\cdot\bar{u}_{12} + \bar{F}_{43}\cdot\bar{u}_{43}) = \omega B_0 hwi\sin\omega t = p_d$.
(Alternatively, $p_m = \bar{T}\cdot\bar{\omega}$; where $\bar{T} = -\bar{a}_x(ihB_0)w\sin\omega t$, and $\bar{\omega} = -\bar{a}_x\omega$)

P.7-10 a) $\mu_r = 1 + \chi_m, \quad \chi_m = 5000 - 1 = 4999.$

$$\bar{H} = \frac{\bar{M}}{\chi_m} = \bar{a}_z \frac{M_0}{4999}; \quad \bar{B} = \mu_0 \mu_r \bar{H} = \bar{a}_z \frac{5000}{4999} \mu_0 M_0.$$

b) $$V_0 = \oint \bar{u} \times \bar{B} \cdot d\bar{\ell} = \int_b^a (\bar{a}_\phi \omega r) \times (\bar{a}_z B) \cdot \bar{a}_r \, dr$$

$$= -\frac{\omega B}{2}(b^2 - a^2) = -\frac{2500}{4999} \mu_0 M_0 \omega (b^2 - a^2).$$

c) $$E_r = \bar{E}_r' - \bar{u} \times \bar{B} = \frac{\bar{J}_r}{\sigma} - (\bar{a}_\phi u) \times (\bar{a}_z B) = \bar{a}_r \left(\frac{i}{2\pi r h \sigma} - \omega r B \right)$$

(at r)

Induced voltage $$V = \int_a^b E_r \, dr = \frac{i}{2\pi h \sigma} \ln \frac{b}{a} - \frac{\omega B}{2}(b^2 - a^2)$$

$$= iR + V_0.$$

Short circuit: $V_0 = 0$, $i_{sc} = \frac{\omega B}{2R}(b^2 - a^2)$, where $R = \frac{\ln(b/a)}{2\pi h \sigma}$.

P.7-13 a) $$\bar{E} = -\bar{\nabla} V - \frac{\partial \bar{A}}{\partial t} = -\bar{\nabla}\left(V - \frac{\partial \psi}{\partial t}\right) - \frac{\partial \bar{A}'}{\partial t} \longrightarrow V' = V - \frac{\partial \psi}{\partial t}.$$

b) Eq. (7-62): $\bar{\nabla} \cdot \bar{A}' + \mu \epsilon \frac{\partial V'}{\partial t} = 0.$

$$\longrightarrow \bar{\nabla} \cdot (\bar{A} + \bar{\nabla} \psi) + \mu \epsilon \frac{\partial}{\partial t}\left(V - \frac{\partial \psi}{\partial t}\right) = 0.$$

$$\longrightarrow \nabla^2 \psi - \mu \epsilon \frac{\partial^2 \psi}{\partial t^2} = 0.$$

P.7-14 Eq. (7-53b): $\bar{\nabla} \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \longrightarrow \bar{\nabla} \times \left(\frac{\bar{B}}{\mu}\right) = \bar{J} + \epsilon \frac{\partial \bar{E}}{\partial t}$ ①

Eqs. (7-55) & (7-57): $\bar{B} = \bar{\nabla} \times \bar{A}, \quad \bar{E} = -\bar{\nabla} V - \frac{\partial \bar{A}}{\partial t}$ ②, ③

Substituting ② & ③ in ①: $\mu \bar{\nabla} \times \left(\frac{1}{\mu} \bar{\nabla} \times \bar{A}\right) = \mu \bar{J} - \mu \epsilon \frac{\partial^2 \bar{A}}{\partial t^2} - \mu \epsilon \bar{\nabla}\left(\frac{\partial V}{\partial t}\right).$

Using gauge condition for potentials in an inhomogeneous medium:

$$\bar{\nabla} \cdot (\epsilon \bar{A}) + \mu \epsilon^2 \frac{\partial V}{\partial t} = 0.$$

We obtain the wave equation for vector potential: $-\bar{\nabla} \times \left(\frac{1}{\mu} \bar{\nabla} \times \bar{A}\right) + \epsilon \bar{\nabla}\left[\frac{1}{\mu \epsilon^2} \bar{\nabla} \cdot (\epsilon \bar{A})\right] - \epsilon \frac{\partial^2 \bar{A}}{\partial t^2} = -\bar{J}$

Eq. (7-53c): $\bar{\nabla} \cdot \bar{D} = \rho \longrightarrow \bar{\nabla} \cdot (\epsilon \bar{\nabla} V) + \frac{\partial}{\partial t} \bar{\nabla} \cdot (\epsilon \bar{A}) = -\rho.$

Wave equation for scalar potential: $\frac{1}{\epsilon} \bar{\nabla} \cdot (\epsilon \bar{\nabla} V) - \mu \epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}.$

P.7-18 a) Eq. (3-89): $\bar{\nabla} \cdot \bar{P} = -\rho_p \longrightarrow \bar{a}_{n2} \cdot (\bar{P}_1 - \bar{P}_2) = -\rho_{ps}.$

b) $\epsilon_0 \bar{E} = \bar{D} - \bar{P}$ $\begin{cases} \bar{\nabla} \cdot \bar{D} = \rho_f \longrightarrow \bar{a}_{n2} \cdot (\bar{D}_1 - \bar{D}_2) = \rho_{fs} & ① \\ -\bar{\nabla} \cdot \bar{P} = \rho_p \longrightarrow \bar{a}_{n2} \cdot (\bar{P}_1 - \bar{P}_2) = -\rho_{ps} & ② \end{cases}$

Combining ① & ②: $\bar{a}_{n2} \cdot (\bar{E}_1 - \bar{E}_2) = \frac{1}{\epsilon_0}(\rho_{fs} + \rho_{ps}).$ Subscript f signifies free charge.

P.7-19 Medium 1: Free space.

Medium 2: $\mu_2 \to \infty$. H_2 must be zero so that B_2 is not infinite.

Boundary Conditions: $\bar{a}_{n2} \times \bar{H}_1 = \bar{J}_s$, $B_{1n} = B_{2n}$.

$E_{1t} = E_{2t}$, $\bar{a}_{n2} \cdot (\bar{D}_1 - \bar{D}_2) = \rho_s$.

P.7-21 We wish to prove $\nabla^2 V - \frac{1}{u^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}$, ①

where $V(R,t) = \frac{1}{4\pi\epsilon}\int_{V'} \frac{\rho(t-R/u)}{R}\,dv'$. ②

We need: $\bar{\nabla}^2\left(\frac{\rho}{R}\right) = \frac{1}{R}\bar{\nabla}^2\rho + \rho\bar{\nabla}^2\left(\frac{1}{R}\right) + 2(\bar{\nabla}\rho)\cdot\bar{\nabla}\left(\frac{1}{R}\right)$. ③

{Formula: $\bar{\nabla}^2(fg) = \bar{\nabla}\cdot\bar{\nabla}(fg) = g\bar{\nabla}^2 f + f\bar{\nabla}^2 g + 2(\bar{\nabla}f)\cdot(\bar{\nabla}g)$}

Let $\xi = t - R/u$. $\nabla^2\rho(\xi) = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\rho}{\partial R}\right) = \frac{1}{u^2}\frac{d^2\rho}{d\xi^2} - \frac{2}{uR}\frac{d\rho}{d\xi}$. ④

$\bar{\nabla}^2\left(\frac{1}{R}\right) = -4\pi\delta(R)$ ⑤, $(\bar{\nabla}\rho)\cdot\left(\bar{\nabla}\frac{1}{R}\right) = \frac{\partial\rho}{\partial R}\left(-\frac{1}{R^2}\right) = \frac{1}{uR^2}\frac{d\rho}{d\xi}$. ⑥

Substituting ④, ⑤ & ⑥ in ③: $\bar{\nabla}^2\left(\frac{\rho}{R}\right) = \frac{1}{u^2 R}\frac{d^2\rho}{d\xi^2} - 4\pi\rho\delta(R)$. ⑦

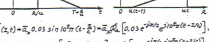
From ②: $\nabla^2 V = \frac{1}{4\pi\epsilon}\bar{\nabla}^2\int_{V'}\frac{\rho}{R}\,dv' = \frac{1}{4\pi\epsilon}\int_{V'}\left[\frac{1}{u^2 R}\frac{d^2\rho}{d\xi^2} - 4\pi\rho\delta(R)\right]dv'$. ⑧

$\frac{\partial^2 V}{\partial t^2} = \frac{1}{4\pi\epsilon}\int_{V'}\frac{1}{R}\frac{d^2\rho}{d\xi^2}\,dv'$

$\therefore\ \bar{\nabla}^2 V - \frac{1}{u^2}\frac{\partial^2 V}{\partial t^2} = \frac{1}{4\pi\epsilon}\int_{V'}\left[\frac{1}{u^2 R}\frac{d^2\rho}{d\xi^2} - 4\pi\rho\delta(R) - \frac{1}{u^2 R}\frac{d^2\rho}{d\xi^2}\right]dv'$

$= -\frac{\rho}{\epsilon}$ Q.E.D.

P.7-22



P.7-23 $\bar{E}_1(z,t) = \bar{a}_x 0.03\sin 10^8\pi\left(t - \frac{z}{c}\right) = \bar{a}_x \mathcal{R}_e\left[0.03e^{-j\pi/2}e^{j10^8\pi(t-z/c)}\right]$.

$\bar{E}_2(z,t) = \bar{a}_x 0.04\cos\left[10^8\pi\left(t-\frac{z}{c}\right) - \frac{\pi}{3}\right] = \bar{a}_x \mathcal{R}_e\left[0.04e^{-j\pi/3}e^{j10^8\pi(t-z/c)}\right]$.

Phasors: $\bar{E} = \bar{E}_1 + \bar{E}_2 = \bar{a}_x\left[0.03e^{-j\pi/2} + 0.04e^{-j\pi/3}\right]$

$= \bar{a}_x\left[-j0.03 + (0.02 - j0.02\sqrt{3})\right] = \bar{a}_x(0.065e^{-j1.27}) = \bar{a}_x E_0 e^{j\theta}$.

$\therefore\ E_0 = 0.065$, $\theta = -1.27$ (rad), or $-72.8°$

P.7-24 $\quad \bar{\nabla}\times\bar{E} = -\mu\frac{\partial\bar{H}}{\partial t}$ ① $\qquad \bar{\nabla}\times\bar{H} = \bar{J} + \epsilon\frac{\partial\bar{E}}{\partial t}$ ②

$\quad \bar{\nabla}\cdot\bar{E} = \frac{\rho}{\epsilon}$ ③ $\qquad \bar{\nabla}\cdot\bar{H} = 0$ ④

$\bar{\nabla}\times$①: $\bar{\nabla}\times\bar{\nabla}\times\bar{E} = -\mu\frac{\partial}{\partial t}(\bar{\nabla}\times\bar{H}) = -\mu\frac{\partial}{\partial t}(\bar{J}+\epsilon\frac{\partial\bar{E}}{\partial t}) = \bar{\nabla}(\bar{\nabla}\cdot\bar{E}) - \bar{\nabla}^2\bar{E}.$

Wave equation for $\bar{E}$: $\quad \bar{\nabla}^2\bar{E} - \mu\epsilon\frac{\partial^2\bar{E}}{\partial t^2} = \mu\frac{\partial\bar{J}}{\partial t} + \frac{1}{\epsilon}\bar{\nabla}\rho.$

$\bar{\nabla}\times$②: $\bar{\nabla}\times\bar{\nabla}\times\bar{H} = \bar{\nabla}\times\bar{J} + \epsilon\frac{\partial}{\partial t}(\bar{\nabla}\times\bar{E}) = \bar{\nabla}\times\bar{J} - \mu\epsilon\frac{\partial^2\bar{H}}{\partial t^2} = \bar{\nabla}(\bar{\nabla}\cdot\bar{H}) - \bar{\nabla}^2\bar{H}.$

Wave equation for $\bar{H}$: $\quad \bar{\nabla}^2\bar{H} - \mu\epsilon\frac{\partial^2\bar{H}}{\partial t^2} = -\bar{\nabla}\times\bar{J}.$

For sinusoidal time dependence: $\frac{\partial}{\partial t}\longrightarrow j\omega,\ \frac{\partial^2}{\partial t^2}\longrightarrow -\omega^2.$

Helmholtz's equations (for phasors): $\quad \bar{\nabla}^2\bar{E} + \omega^2\mu\epsilon\bar{E} = j\omega\mu\bar{J} + \frac{1}{\epsilon}\bar{\nabla}\rho;$

$\qquad \bar{\nabla}^2\bar{H} + \omega^2\mu\epsilon\bar{H} = -\bar{\nabla}\times\bar{J}.$

P.7-25 $\quad \bar{E} = \bar{a}_y\, 0.1\sin(10\pi x)\cos(6\pi 10^9 t - \beta z)\quad$ (V/m).

Use phasors: $\bar{H} = -\frac{1}{j\omega\mu_0}\bar{\nabla}\times\bar{E} = \frac{j}{\omega\mu_0}\left[\bar{a}_x j0.1\beta\sin(10\pi x) + \bar{a}_z 0.1(10\pi)\cos(10\pi x)\right]e^{-j\beta z}.$ ①

$\bar{E} = \frac{1}{j\omega\epsilon_0}\bar{\nabla}\times\bar{H} = \bar{a}_y\frac{0.1}{\omega^2\mu_0\epsilon_0}\left[(10\pi)^2+\beta^2\right]\sin(10\pi x)e^{-j\beta z}.$ ②

Phasor form for given $\bar{E}$: $\bar{E} = \bar{a}_y\, 0.1\sin(10\pi x)e^{-j\beta z}.$ ③

Equating ② and ③: $(10\pi)^2 + \beta^2 = \omega^2\mu_0\epsilon_0 = 400\pi^2$

$\longrightarrow \beta = \sqrt{300}\,\pi = 54.4$ (rad/m).

From ①: $\bar{H}(x,z;t) = \mathcal{R}e(\bar{H}e^{j\omega t})$

$= -\bar{a}_x 2.30\times10^{-4}\sin(10\pi x)\cos(6\pi10^9 t - 54.4z)$

$-\bar{a}_z 1.33\times10^{-4}\cos(10\pi x)\sin(6\pi10^9 t - 54.4z)$ (A/m).

P.7-26 $\quad \bar{H}(x,z;t) = \bar{a}_y 2\cos(15\pi x)\sin(6\pi10^9 t - \beta z)\quad$ (A/m).

Phasor: $\bar{H} = \bar{a}_y 2\cos(15\pi x)e^{-j\beta z}.$

Similar to Problem P.7-25: $(15\pi)^2 + \beta^2 = \omega^2\mu_0\epsilon_0 = \frac{(6\pi10^9)^2}{(3\times10^8)^2} = 400\pi^2$

$\longrightarrow \beta = 13.2\pi = 41.6$ (rad/m).

$\bar{E} = \frac{1}{j\omega\epsilon_0}\bar{\nabla}\times\bar{H} = \left[\bar{a}_x 158\pi\cos(15\pi x) + \bar{a}_z j180\pi\sin(15\pi x)\right]e^{-j\beta z}.$

$\bar{E}(x,z;t) = \mathcal{I}m(\bar{E}e^{j\omega t}) = \bar{a}_x 496\cos(15\pi x)\sin(6\pi10^9 t - 41.6z)$

$+\bar{a}_z 565\sin(15\pi x)\cos(6\pi10^9 t - 41.6z)$ (V/m).

P.7-27 Use phasors: $\bar{E} = \bar{a}_\theta \frac{E_0}{R} \sin\theta \cdot e^{-jkR}$.

$$\bar{\nabla}\times\bar{E} = \bar{a}_\phi \frac{1}{R}\frac{\partial}{\partial R}(RE_\theta) = \bar{a}_\phi(-jk)\frac{E_0}{R}\sin\theta\cdot e^{-jkR}$$

$$= -j\omega\mu_0\bar{H} \longrightarrow \bar{H} = \bar{a}_\phi \frac{kE_0}{\omega\mu_0 R}\sin\theta\cdot e^{-jkR}$$

In free space, $k = \omega\sqrt{\mu_0\epsilon_0} \longrightarrow \bar{H} = \bar{a}_\phi \frac{E_0}{R}\sqrt{\frac{\epsilon_0}{\mu_0}}\sin\theta\cdot e^{-j\omega\sqrt{\mu_0\epsilon_0}R}$,

$$\bar{H}(R,\theta;t) = \bar{a}_\phi \frac{E_0}{R}\sqrt{\frac{\epsilon_0}{\mu_0}}\sin\theta\cos\omega(t-\sqrt{\mu_0\epsilon_0}R).$$

P.7-28 Maxwell's curl eqs: $\bar{\nabla}\times\bar{E} = -j\omega\mu\bar{H}$, ①

$\bar{\nabla}\times\bar{H} = j\omega\epsilon\bar{E}$. ②

From $\bar{\nabla}\cdot\bar{E} = 0$, define $\bar{A}_e$ such that $\bar{E} = \bar{\nabla}\times\bar{A}_e$. ③

From ①, $\bar{H} = \frac{j}{\omega\mu}\bar{\nabla}\times\bar{E} = \frac{j}{\omega\mu}\bar{\nabla}\times\bar{\nabla}\times\bar{A}_e$

$= \frac{j}{\omega\mu}\left[\bar{\nabla}(\bar{\nabla}\cdot\bar{A}_e) - \bar{\nabla}^2\bar{A}_e\right]$. ④

From ②, $\bar{\nabla}\times(\bar{H} - j\omega\epsilon\bar{A}_e) = 0$. Let $\bar{H} - j\omega\epsilon\bar{A}_e = -\bar{\nabla}V_m$. ⑤

Subtracting ⑤ from ④: $\omega\epsilon\bar{A}_e = \frac{1}{\omega\mu}\left[\bar{\nabla}(\bar{\nabla}\cdot\bar{A}_e) - \bar{\nabla}^2\bar{A}_e\right] - j\bar{\nabla}V_m$.

Choose $\bar{\nabla}\cdot\bar{A}_e = j\omega\mu V_m$. ⑥

a) Eq. ⑤ becomes $\bar{H} = j\omega\epsilon\bar{A}_e + \frac{j}{\omega\mu}\bar{\nabla}(\bar{\nabla}\cdot\bar{A}_e)$.

b) Eq. ⑥ becomes $\bar{\nabla}^2\bar{A}_e + \omega^2\mu\epsilon\bar{A}_e = 0$, a homogeneous Helmholtz's eq.

P.7-29 $\bar{H} = j\omega\epsilon_0\bar{\nabla}\times\bar{\pi}_e$. ①

$\bar{\nabla}\times\bar{E} = -j\omega\mu_0\bar{H} = \omega^2\mu_0\epsilon_0\bar{\nabla}\times\bar{\pi}_e$.

$\longrightarrow \bar{\nabla}\times(\bar{E} - k_0^2\bar{\pi}_e) = 0$. Let $\bar{E} - k_0^2\bar{\pi}_e = \bar{\nabla}V_e$. ②

$\bar{\nabla}\times\bar{H} = j\omega\bar{D} = j\omega(\epsilon_0\bar{E} + \bar{P}) = j\omega\epsilon_0(\bar{E} + \frac{\bar{P}}{\epsilon_0})$. ③

Substituting ① and ② in ③:

$j\omega\epsilon_0\bar{\nabla}\times\bar{\nabla}\times\bar{\pi}_e = j\omega\epsilon_0(k_0^2\bar{\pi}_e + \bar{\nabla}V_e + \frac{\bar{P}}{\epsilon_0})$

$= j\omega\epsilon_0(\bar{\nabla}\bar{\nabla}\cdot\bar{\pi}_e - \bar{\nabla}^2\bar{\pi}_e)$. ④

Choose $\bar{\nabla}\cdot\bar{\pi}_e = V_e$. Eq. ④ becomes

b) $\bar{\nabla}^2\bar{\pi}_e + k_0^2\bar{\pi}_e = -\frac{\bar{P}}{\epsilon_0}$. (7-119)

a) Eq. ② becomes

$$\bar{E} = k_0^2 \bar{\pi}_e + \bar{\nabla}\bar{\nabla}\cdot\bar{\pi}_e$$
$$= k_0^2 \bar{\pi}_e + (\bar{\nabla}^2 \bar{\pi}_e + \bar{\nabla}\times\bar{\nabla}\times\bar{\pi}_e). \quad ⑤$$

Combination of Eqs. (7-119) and ⑤ gives

$$\bar{E} = \bar{\nabla}\times\bar{\nabla}\times\bar{\pi}_e - \frac{\bar{P}}{\epsilon_0}.$$

P.7-30

a) $$\left|\frac{\text{Displacement current}}{\text{Conduction current}}\right| = \frac{\omega\epsilon}{\sigma} = \frac{(2\pi\times 100\times 10^6)\times\frac{1}{36\pi}\times 10^{-9}}{5.70\times 10^7}$$
$$= 9.75\times 10^{-11}.$$

b) In a source-free conductor:

$$\bar{\nabla}\times\bar{H} = \sigma\bar{E}, \quad ①$$
$$\bar{\nabla}\times\bar{E} = -j\omega\mu\bar{H}. \quad ②$$
$$\bar{\nabla}\times①:\ \bar{\nabla}\times\bar{\nabla}\times\bar{H} = \bar{\nabla}(\bar{\nabla}\cdot\bar{H}) - \bar{\nabla}^2\bar{H} = \sigma\bar{\nabla}\times\bar{E}. \quad ③$$

But $\bar{\nabla}\cdot\bar{H} = 0$, Eq. ③ becomes

$$\bar{\nabla}^2\bar{H} + \sigma\bar{\nabla}\times\bar{E} = 0. \quad ④$$

Combining ② and ④:

$$\bar{\nabla}^2\bar{H} - j\omega\mu\sigma\bar{H} = 0.$$