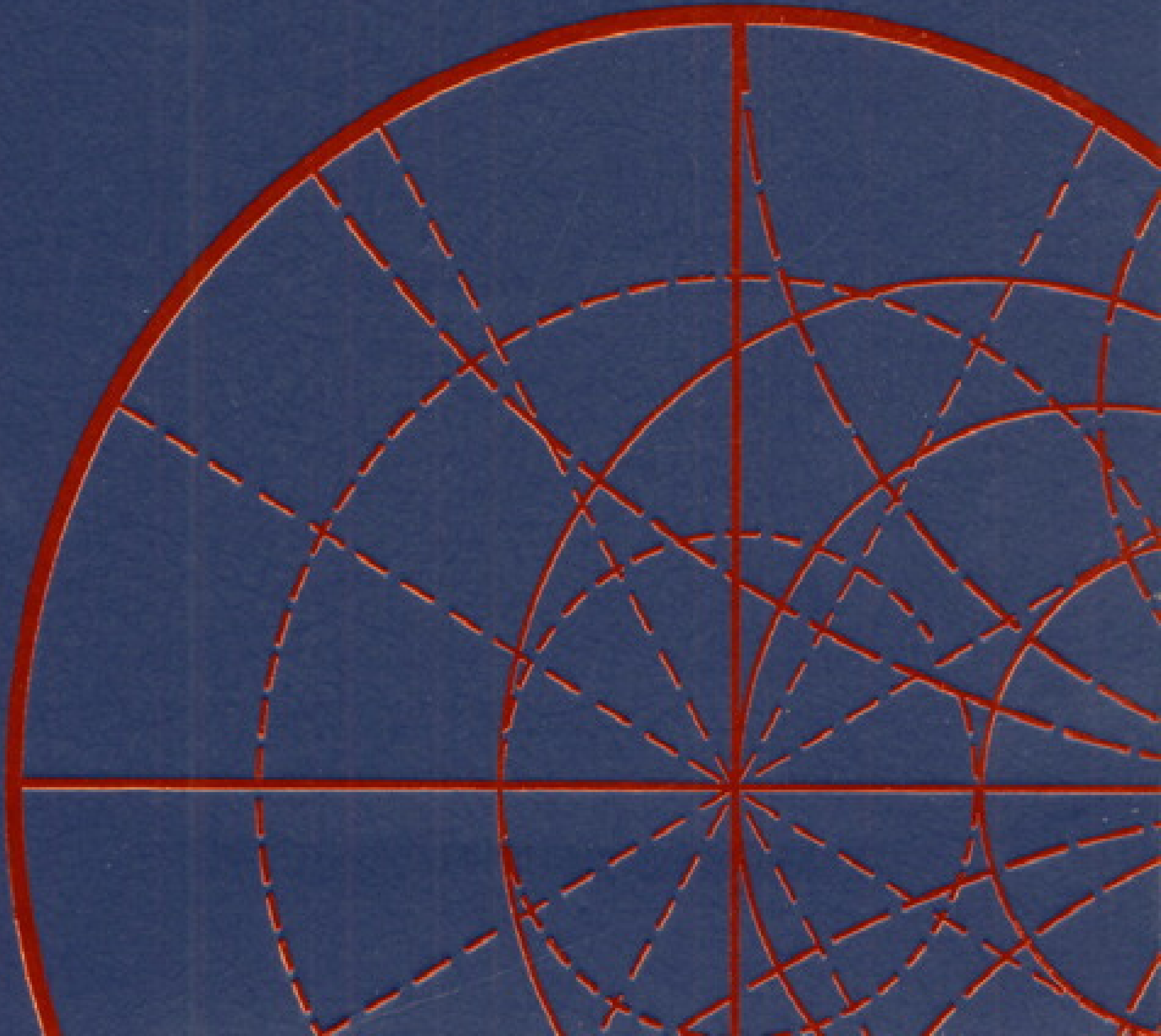
Second Edition

Field and Wave Electromagnetics

David K. Cheng



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Preface

The many books on introductory electromagnetics can be roughly divided into two main groups. The first group takes the traditional development: starting with the experimental laws, generalizing them in steps, and finally synthesizing them in the form of Maxwell's equations. This is an inductive approach. The second group takes the axiomatic development: starting with Maxwell's equations, identifying each with the appropriate experimental law, and specializing the general equations to static and time-varying situations for analysis. This is a deductive approach. A few books begin with a treatment of the special theory of relativity and develop all of electromagnetic theory from Coulomb's law of force; but this approach requires the discussion and understanding of the special theory of relativity first and is perhaps best suited for a course at an advanced level.

Proponents of the traditional development argue that it is the way electromagnetic theory was unraveled historically (from special experimental laws to Maxwell's equations), and that it is easier for the students to follow than the other methods. I feel, however, that the way a body of knowledge was unraveled is not necessarily the best way to teach the subject to students. The topics tend to be fragmented and cannot take full advantage of the conciseness of vector calculus. Students are puzzled at, and often form a mental block to, the subsequent introduction of gradient, divergence, and curl operations. As a process for formulating an electromagnetic model, this approach lacks cohesiveness and elegance.

The axiomatic development usually begins with the set of four Maxwell's equations, either in differential or in integral form, as fundamental postulates. These are equations of considerable complexity and are difficult to master. They are likely to cause consternation and resistance in students who are hit with all of them at the beginning of a book. Alert students will wonder about the meaning of the field vectors and about the necessity and sufficiency of these general equations. At the initial stage students tend to be confused about the concepts of the electromagnetic model, and they are not yet comfortable with the associated mathematical manipulations. In any case, the general Maxwell's equations are soon simplified to apply to static fields, which allow the consideration of electrostatic fields and magnetostatic fields separately. Why then should the entire set of four Maxwell's equations be introduced at the outset?

It may be argued that Coulomb's law, though based on experimental evidence, is in fact also a postulate. Consider the two stipulations of Coulomb's law: that the charged bodies are very small compared with their distance of separation, and that the force between the charged bodies is inversely proportional to the square of their distance. The question arises regarding the first stipulation: How small must the charged bodies be in order to be considered "very small" compared with their distance? In practice the charged bodies cannot be of vanishing sizes (ideal point charges), and there is difficulty in determining the "true" distance between two bodies of finite dimensions. For given body sizes the relative accuracy in distance measurements is better when the separation is larger. However, practical considerations (weakness of force, existence of extraneous charged bodies, etc.) restrict the usable distance of separation in the laboratory, and experimental inaccuracies cannot be entirely avoided. This leads to a more important question concerning the inverse-square relation of the second stipulation. Even if the charged bodies were of vanishing sizes, experimental measurements could not be of an infinite accuracy no matter how skillful and careful an experimentor was. How then was it possible for Coulomb to know that the force was *exactly* inversely proportional to the *square* (not the 2.000001th or the 1.999999th power) of the distance of separation? This question cannot be answered from an experimental viewpoint because it is not likely that during Coulomb's time experiments could have been accurate to the seventh place. We must therefore conclude that Coulomb's law is itself a postulate and that it is a law of nature discovered and assumed on the basis of his experiments of a limited accuracy (see Section 3–2).

This book builds the electromagnetic model using an *axiomatic approach in steps*: first for static electric fields (Chapter 3), then for static magnetic fields (Chapter 6), and finally for time-varying fields leading to Maxwell's equations (Chapter 7). The mathematical basis for each step is Helmholtz's theorem, which states that a vector field is determined to within an additive constant if both its divergence and its curl are specified everywhere. Thus, for the development of the electrostatic model in free space, it is only necessary to define a single vector (namely, the electric field intensity **E**) by specifying its divergence and its curl as postulates. All other relations in electrostatics for free space, including Coulomb's law and Gauss's law, can be derived from the two rather simple postulates. Relations in material media can be developed through the concept of equivalent charge distributions of polarized dielectrics.

Similarly, for the magnetostatic model in free space it is necessary to define only a single magnetic flux density vector **B** by specifying its divergence and its curl as postulates; all other formulas can be derived from these two postulates. Relations in material media can be developed through the concept of equivalent current densities. Of course, the validity of the postulates lies in their ability to yield results that conform with experimental evidence.

For time-varying fields, the electric and magnetic field intensities are coupled. The curl **E** postulate for the electrostatic model must be modified to conform with

Faraday's law. In addition, the curl **B** postulate for the magnetostatic model must also be modified in order to be consistent with the equation of continuity. We have, then, the four Maxwell's equations that constitute the electromagnetic model. I believe that this gradual development of the electromagnetic model based on Helmholtz's theorem is novel, systematic, pedagogically sound, and more easily accepted by students.

In the presentation of the material, I strive for lucidity and unity, and for smooth and logical flow of ideas. Many worked-out examples are included to emphasize fundamental concepts and to illustrate methods for solving typical problems. Applications of derived relations to useful technologies (such as ink-jet printers, lightning arresters, electret microphones, cable design, multiconductor systems, electrostatic shielding, Doppler radar, radome design, Polaroid filters, satellite communication systems, optical fibers, and microstrip lines) are discussed. Review questions appear at the end of each chapter to test the students' retention and understanding of the essential material in the chapter. The problems in each chapter are designed to reinforce students' comprehension of the interrelationships between the different quantities in the formulas, and to extend their ability of applying the formulas to solve practical problems. In teaching, I have found the review questions a particularly useful device to stimulate students' interest and to keep them alert in class.

Besides the fundamentals of electromagnetic fields, this book also covers the theory and applications of transmission lines, waveguides and cavity resonators, and antennas and radiating systems. The fundamental concepts and the governing theory of electromagnetism do not change with the introduction of new electromagnetic devices. Ample reasons and incentives for learning the fundamental principles of electromagnetics are given in Section 1–1. I hope that the contents of this book, strengthened by the novel approach, will provide students with a secure and sufficient background for understanding and analyzing basic electromagnetic phenomena as well as prepare them for more advanced subjects in electromagnetic theory.

There is enough material in this book for a two-semester sequence of courses. Chapters 1 through 7 contain the material on fields, and Chapters 8 through 11 on waves and applications. In schools where there is only a one-semester course on electromagnetics, Chapters 1 through 7, plus the first four sections of Chapter 8 would provide a good foundation on fields and an introduction of waves in unbounded media. The remaining material could serve as a useful reference book on applications or as a textbook for a follow-up elective course. Schools on a quarter system could adjust the material to be covered in accordance with the total number of hours assigned to the subject of electromagnetics. Of course, individual instructors have the prerogative to emphasize and expand certain topics, and to deemphasize or delete certain others.

I have given considerable thought to the advisability of including computer programs for the solution of some problems, but have finally decided against it. Diverting students' attention and effort to numerical methods and computer software would distract them from concentrating on learning the fundamentals of electromagnetism. Where appropriate, the dependence of important results on the value of a parameter is stressed by curves; field distributions and antenna patterns are illustrated by graphs; and typical mode patterns in waveguides are plotted. The computer programs for obtaining these curves, graphs, and mode patterns are not always simple. Students in science and engineering are required to acquire a facility for using computers; but the inclusion of some cookbook-style computer programs in a book on the fundamental principles of electromagnetic fields and waves would appear to contribute little to the understanding of the subject matter.

This book was first published in 1983. Favorable reactions and friendly encouragements from professors and students have provided me with the impetus to come out with a new edition. In this second edition I have added many new topics. These include Hall effect, d-c motors, transformers, eddy current, energy-transport velocity for wide-band signals in waveguides, radar equation and scattering cross section, transients in transmission lines, Bessel functions, circular waveguides and circular cavity resonators, waveguide discontinuities, wave propagation in ionosphere and near earth's surface, helical antennas, log-periodic dipole arrays, and antenna effective length and effective area. The total number of problems has been expanded by about 35 percent.

The Addison-Wesley Publishing Company has decided to make this second edition a two-color book. I think the readers will agree that the book is handsomely produced. I would like to take this opportunity to express my appreciation to all the people on the editorial, production, and marketing staff who provided help in bringing out this new edition. In particular, I wish to thank Thomas Robbins, Barbara Rifkind, Karen Myer, Joseph K. Vetere, and Katherine Harutunian.

Chevy Chase, Maryland D. K. C.

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1

The Electromagnetic Model

1–1 Introduction

Stated in a simple fashion, ***electromagnetics*** is the study of the effects of electric charges at rest and in motion. From elementary physics we know that there are two kinds of charges: positive and negative. Both positive and negative charges are sources of an electric field. Moving charges produce a current, which gives rise to a magnetic field. Here we tentatively speak of electric field and magnetic field in a general way; more definitive meanings will be attached to these terms later. A ***field*** is a spatial distribution of a quantity, which may or may not be a function of time. A time-varying electric field is accompanied by a magnetic field, and vice versa. In other words, time-varying electric and magnetic fields are coupled, resulting in an electromagnetic field. Under certain conditions, time-dependent electromagnetic fields produce waves that radiate from the source.

The concept of fields and waves is essential in the explanation of action at a distance. For instance, we learned from elementary mechanics that masses attract each other. This is why objects fall toward the earth's surface. But since there are no elastic strings connecting a free-falling object and the earth, how do we explain this phenomenon? We explain this action-at-a-distance phenomenon by postulating the existence of a gravitational field. The possibilities of satellite communication and of receiving signals from space probes millions of miles away can be explained only by postulating the existence of electric and magnetic fields and electromagnetic waves. In this book, *Field and Wave Electromagnetics*, we study the principles and applications of the laws of electromagnetism that govern electromagnetic phenomena.

Electromagnetics is of fundamental importance to physicists and to electrical and computer engineers. Electromagnetic theory is indispensable in understanding the principle of atom smashers, cathode-ray oscilloscopes, radar, satellite communication, television reception, remote sensing, radio astronomy, microwave devices, optical fiber communication, transients in transmission lines, electromagnetic compatibility

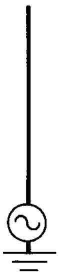

FIGURE 1–1
A monopole antenna.

problems, instrument-landing systems, electromechanical energy conversion, and so on. Circuit concepts represent a restricted version, a special case, of electromagnetic concepts. As we shall see in Chapter 7, when the source frequency is very low so that the dimensions of a conducting network are much smaller than the wavelength, we have a quasi-static situation, which simplifies an electromagnetic problem to a circuit problem. However, we hasten to add that circuit theory is itself a highly developed, sophisticated discipline. It applies to a different class of electrical engineering problems, and it is important in its own right.

Two situations illustrate the inadequacy of circuit-theory concepts and the need for electromagnetic-field concepts. Figure 1–1 depicts a monopole antenna of the type we see on a walkie-talkie. On transmit, the source at the base feeds the antenna with a message-carrying current at an appropriate carrier frequency. From a circuit-theory point of view, the source feeds into an open circuit because the upper tip of the antenna is not connected to anything physically; hence no current would flow, and nothing would happen. This viewpoint, of course, cannot explain why communication can be established between walkie-talkies at a distance. Electromagnetic concepts must be used. We shall see in Chapter 11 that when the length of the antenna is an appreciable part of the carrier wavelength,[†] a nonuniform current will flow along the open-ended antenna. This current radiates a time-varying electromagnetic field in space, which propagates as an electromagnetic wave and induces currents in other antennas at a distance.

In Fig. 1–2 we show a situation in which an electromagnetic wave is incident from the left on a large conducting wall containing a small hole (aperture). Electromagnetic fields will exist on the right side of the wall at points, such as P in the figure, that are not necessarily directly behind the aperture. Circuit theory is obviously inadequate here for the determination (or even the explanation of the existence) of the field at P. The situation in Fig. 1–2, however, represents a problem of practical importance as its solution is relevant in evaluating the shielding effectiveness of the conducting wall.

[†] The product of the wavelength and the frequency of an a-c source is the velocity of wave propagation.

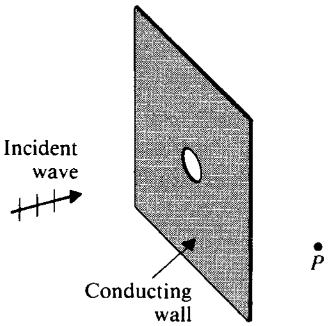

FIGURE 1–2
An electromagnetic problem.

Generally speaking, circuit theory deals with lumped-parameter systems—circuits consisting of components characterized by lumped parameters such as resistances, inductances, and capacitances. Voltages and currents are the main system variables. For d-c circuits the system variables are constants, and the governing equations are algebraic equations. The system variables in a-c circuits are time-dependent; they are scalar quantities and are independent of space coordinates. The governing equations are ordinary differential equations. On the other hand, most electromagnetic variables are functions of time as well as of space coordinates. Many are vectors with both a magnitude and a direction, and their representation and manipulation require a knowledge of vector algebra and vector calculus. Even in static cases the governing equations are, in general, partial differential equations. It is essential that we be equipped to handle vector quantities and variables that are both time- and space-dependent. The fundamentals of vector algebra and vector calculus will be developed in Chapter 2. Techniques for solving partial differential equations are needed in dealing with certain types of electromagnetic problems. These techniques will be discussed in Chapter 4. The importance of acquiring a facility in the use of these mathematical tools in the study of electromagnetics cannot be overemphasized.

Students who have mastered circuit theory may initially have the impression that electromagnetic theory is abstract. In fact, electromagnetic theory is no more abstract than circuit theory in the sense that the validity of both can be verified by experimentally measured results. In electromagnetics there is a need to define more quantities and to use more mathematical manipulations in order to develop a logical and complete theory that can explain a much wider variety of phenomena. The challenge of field and wave electromagnetics is not in the abstractness of the subject matter but rather in the process of mastering the electromagnetic model and the associated rules of operation. Dedication to acquiring this mastery will help us to meet the challenge and reap immeasurable satisfaction.

1–2 The Electromagnetic Model

There are two approaches in the development of a scientific subject: the inductive approach and the deductive approach. Using the inductive approach, one follows the historical development of the subject, starting with the observations of some simple experiments and inferring from them laws and theorems. It is a process of reasoning from particular phenomena to general principles. The deductive approach, on the other hand, postulates a few fundamental relations for an idealized model. The postulated relations are axioms, from which particular laws and theorems can be derived. The validity of the model and the axioms is verified by their ability to predict consequences that check with experimental observations. In this book we prefer to use the deductive or axiomatic approach because it is more elegant and enables the development of the subject of electromagnetics in an orderly way.

The idealized model we adopt for studying a scientific subject must relate to real-world situations and be able to explain physical phenomena; otherwise, we would be engaged in mental exercises for no purpose. For example, a theoretical model could be built, from which one might obtain many mathematical relations; but, if these relations disagreed with observed results, the model would be of no use. The mathematics might be correct, but the underlying assumptions of the model could be wrong, or the implied approximations might not be justified.

Three essential steps are involved in building a theory on an idealized model. *First*, some basic quantities germane to the subject of study are defined. *Second*, the rules of operation (the mathematics) of these quantities are specified. *Third*, some fundamental relations are postulated. These postulates or laws are invariably based on numerous experimental observations acquired under controlled conditions and synthesized by brilliant minds. A familiar example is the circuit theory built on a circuit model of ideal sources and pure resistances, inductances, and capacitances. In this case the basic quantities are voltages (V), currents (I), resistances (R), inductances (L), and capacitances (C); the rules of operations are those of algebra, ordinary differential equations, and Laplace transformation; and the fundamental postulates are Kirchhoff's voltage and current laws. Many relations and formulas can be derived from this basically rather simple model, and the responses of very elaborate networks can be determined. The validity and value of the model have been amply demonstrated.

In a like manner, an electromagnetic theory can be built on a suitably chosen electromagnetic model. In this section we shall take the first step of defining the basic quantities of electromagnetics. The second step, the rules of operation, encompasses vector algebra, vector calculus, and partial differential equations. The fundamentals of vector algebra and vector calculus will be discussed in Chapter 2 (Vector Analysis), and the techniques for solving partial differential equations will be introduced when these equations arise later in the book. The third step, the fundamental postulates, will be presented in three substeps in Chapters 3, 6, and 7 as we deal with static electric fields, steady magnetic fields, and electromagnetic fields, respectively.

The quantities in our electromagnetic model can be divided roughly into two categories: source quantities and field quantities. The source of an electromagnetic field is invariably electric charges at rest or in motion. However, an electromagnetic field may cause a redistribution of charges, which will, in turn, change the field; hence the separation between the cause and the effect is not always so distinct.

We use the symbol q (sometimes Q) to denote ***electric charge***. Electric charge is a fundamental property of matter and exists only in positive or negative integral multiples of the charge on an electron, $-e$.[†]

$$\boxed{e = 1.60 \times 10^{-19} \qquad \text{(C)},} \tag{1–1}$$

where C is the abbreviation of the unit of charge, coulomb.[‡] It is named after the French physicist Charles A. de Coulomb, who formulated Coulomb's law in 1785. (Coulomb's law will be discussed in Chapter 3.) A coulomb is a very large unit for electric charge; it takes $1/(1.60 \times 10^{-19})$ or 6.25 million trillion electrons to make up -1 C. In fact, two 1 C charges 1 m apart will exert a force of approximately 1 million tons on each other. Some other physical constants for the electron are listed in Appendix B–2.

The principle of ***conservation of electric charge***, like the principle of conservation of momentum, is a fundamental postulate or law of physics. It states that electric charge is conserved; that is, it can neither be created nor be destroyed. This is a law of nature and cannot be derived from other principles or relations. Its truth has never been questioned or doubted in practice.

Electric charges can move from one place to another and can be redistributed under the influence of an electromagnetic field; but the algebraic sum of the positive and negative charges in a closed (isolated) system remains unchanged. ***The principle of conservation of electric charge must be satisfied at all times and under any circumstances.*** It is represented mathematically by the ***equation of continuity***, which we will discuss in Section 5–4. Any formulation or solution of an electromagnetic problem that violates the principle of conservation of electric charge *must be* incorrect. We recall that the Kirchhoff's current law in circuit theory, which maintains that the sum of all the currents leaving a junction must equal the sum of all the currents entering the junction, is an assertion of the conservation property of electric charge. (Implicit in the current law is the assumption that there is no cumulation of charge at the junction.)

Although, in a microscopic sense, electric charge either does or does not exist at a point in a discrete manner, these abrupt variations on an atomic scale are unimportant when we consider the electromagnetic effects of large aggregates of charges. In constructing a macroscopic or large-scale theory of electromagnetism we find that the use of smoothed-out average density functions yields very good results. (The same approach is used in mechanics where a smoothed-out mass density function is defined, in spite of the fact that mass is associated only with elementary particles in a discrete

---

[†] In 1962, Murray Gell-Mann hypothesized ***quarks*** as the basic building blocks of matter. Quarks were predicted to carry a fraction of the charge of an electron, and their existence has since been verified experimentally.

[‡] The system of units will be discussed in Section 1–3.

manner on an atomic scale.) We define a ***volume charge density***, ρ, as a source quantity as follows:

$$\rho = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta v} \qquad (\text{C/m}^3), \tag{1-2}$$

where Δq is the amount of charge in a very small volume Δv. How small should Δv be? It should be small enough to represent an accurate variation of ρ but large enough to contain a very large number of discrete charges. For example, an elemental cube with sides as small as 1 micron (10^{-6} m or 1 μm) has a volume of 10^{-18} m^3, which will still contain about 10^{11} (100 billion) atoms. A smoothed-out function of space coordinates, ρ, defined with such a small Δv is expected to yield accurate macroscopic results for nearly all practical purposes.

In some physical situations an amount of charge Δq may be identified with an element of surface Δs or an element of line $\Delta \ell$. In such cases it will be more appropriate to define a ***surface charge density***, ρ_s, or a ***line charge density***, ρ_ℓ:

$$\rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} \qquad (\text{C/m}^2), \tag{1-3}$$

$$\rho_\ell = \lim_{\Delta \ell \to 0} \frac{\Delta q}{\Delta \ell} \qquad (\text{C/m}). \tag{1-4}$$

Except for certain special situations, charge densities vary from point to point; hence ρ, ρ_s, and ρ_ℓ are, in general, point functions of space coordinates.

Current is the rate of change of charge with respect to time; that is,

$$I = \frac{dq}{dt} \qquad (\text{C/s or A}), \tag{1-5}$$

where I itself may be time-dependent. The unit of current is coulomb per second (C/s), which is the same as ampere (A). A current must flow through a finite area (a conducting wire of a finite cross section, for instance); hence it is not a point function. In electromagnetics we define a vector point function ***volume current density*** (or simply ***current density***) **J**, which measures the amount of current flowing through a unit area normal to the direction of current flow. The boldfaced **J** is a vector whose magnitude is the current per unit area (A/m^2) and whose direction is the direction of current flow. We shall elaborate on the relation between I and **J** in Chapter 5. For very good conductors, high-frequency alternating currents are confined in the surface layer as a current sheet, instead of flowing throughout the interior of the conductor. In such cases there is a need to define a ***surface current density*** $\mathbf{J}_s$, which is the current per unit width on the conductor surface normal to the direction of current flow and has the unit of ampere per meter (A/m).

There are four fundamental *vector* field quantities in electromagnetics: ***electric field intensity*** **E**, ***electric flux density*** (or ***electric displacement***) **D**, ***magnetic flux***

TABLE 1–1
Fundamental Electromagnetic Field Quantities

Symbols and Units for Field Quantities	Field Quantity	Symbol	Unit
Electric	Electric field intensity	**E**	V/m
	Electric flux density (Electric displacement)	**D**	C/m^2
Magnetic	Magnetic flux density	**B**	T
	Magnetic field intensity	**H**	A/m

density **B**, and ***magnetic field intensity*** **H**. The definition and physical significance of these quantities will be explained fully when they are introduced later in the book. At this time we want only to establish the following. Electric field intensity **E** is the only vector needed in discussing electrostatics (effects of stationary electric charges) in free space; it is defined as the electric force on a unit test charge. Electric displacement vector **D** is useful in the study of electric field in material media, as we shall see in Chapter 3. Similarly, magnetic flux density **B** is the only vector needed in discussing magnetostatics (effects of steady electric currents) in free space and is related to the magnetic force acting on a charge moving with a given velocity. The magnetic field intensity vector **H** is useful in the study of magnetic field in material media. The definition and significance of **B** and **H** will be discussed in Chapter 6.

The four fundamental electromagnetic field quantities, together with their units, are tabulated in Table 1–1. In Table 1–1, V/m is volt per meter, and T stands for tesla or volt-second per square meter. When there is no time variation (as in static, steady, or stationary cases), the electric field quantities **E** and **D** and the magnetic field quantities **B** and **H** form two separate vector pairs. In time-dependent cases, however, electric and magnetic field quantities are coupled; that is, time-varying **E** and **D** will give rise to **B** and **H**, and vice versa. All four quantities are point functions; they are defined at every point in space and, in general, are functions of space coordinates. Material (or medium) properties determine the relations between **E** and **D** and between **B** and **H**. These relations are called the ***constitutive relations*** of a medium and will be examined later.

The principal objective of studying electromagnetism is to understand the interaction between charges and currents at a distance based on the electromagnetic model. Fields and waves (time- and space-dependent fields) are basic conceptual quantities of this model. Fundamental postulates will relate **E**, **D**, **B**, **H**, and the source quantities; and derived relations will lead to the explanation and prediction of electromagnetic phenomena.

TABLE 1–2
Fundamental SI Units

Quantity	Unit	Abbreviation
Length	meter	m
Mass	kilogram	kg
Time	second	s
Current	ampere	A

1–3 SI Units and Universal Constants

A measurement of any physical quantity must be expressed as a number followed by a unit. Thus we may talk about a length of three meters, a mass of two kilograms, and a time period of ten seconds. To be useful, a unit system should be based on some fundamental units of convenient (practical) sizes. In mechanics, all quantities can be expressed in terms of three basic units (for length, mass, and time). In electromagnetics a fourth basic unit (for current) is needed. The SI (***International System of Units*** or ***Le Système International d'Unités***) is an ***MKSA system*** built from the four fundamental units listed in Table 1–2. All other units used in electromagnetics, including those appearing in Table 1–1, are derived units expressible in terms of meters, kilograms, seconds, and amperes. For example, the unit for charge, coulomb (C), is ampere-second (A·s); the unit for electric field intensity (V/m) is $kg \cdot m/A \cdot s^3$; and the unit for magnetic flux density, tesla (T), is $kg/A \cdot s^2$. More complete tables of the units for various quantities are given in Appendix A.

The official SI definitions, as adopted by the International Committee on Weights and Measures, are as follows:†

Meter. Once the length between two scratches on a platinum-iridium bar (and originally calculated as one ten-millionth of the distance between the North Pole and the equator through Paris, France), is now defined by reference to the *second* (see below) and the speed of light, which in a vacuum is 299,792,458 meters per second.

Kilogram. Mass of a standard bar made of a platinum-iridium alloy and kept inside a set of nested enclosures that protect it from contamination and mishandling. It rests at the International Bureau of Weights and Measures in Sèvres, outside Paris.

Second. 9,192,631,770 periods of the electromagnetic radiation emitted by a particular transition of a cesium atom.

† P. Wallich, "Volts and amps are not what they used to be," *IEEE Spectrum*, vol. 24, pp. 44–49, March 1987.

Ampere. The constant current that, if maintained in two straight parallel conductors of infinite length and negligible circular cross section, and placed one meter apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} newton per meter of length. (A newton is the force that gives a mass of one kilogram an acceleration of one meter per second squared.)

In our electromagnetic model there are three universal constants, in addition to the field quantities listed in Table 1–1. They relate to the properties of the free space (vacuum). They are as follows: ***velocity of electromagnetic wave*** (including light) in free space, c; ***permittivity*** of free space, ϵ_0; and ***permeability*** of free space, μ_0. Many experiments have been performed for precise measurement of the velocity of light, to many decimal places. For our purpose it is sufficient to remember that

$$\boxed{c \cong 3 \times 10^8 \qquad \text{(m/s)}.} \quad \text{(in free space)} \tag{1–6}$$

The other two constants, ϵ_0 and μ_0, pertain to electric and magnetic phenomena, respectively: ϵ_0 is the proportionality constant between the electric flux density **D** and the electric field intensity **E** in free space, such that

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E};} \quad \text{(in free space)} \tag{1–7}$$

μ_0 is the proportionality constant between the magnetic flux density **B** and the magnetic field intensity **H** in free space, such that

$$\boxed{\mathbf{H} = \frac{1}{\mu_0}\mathbf{B}.} \quad \text{(in free space)} \tag{1–8}$$

The values of ϵ_0 and μ_0 are determined by the choice of the unit system, and they are not independent. In the ***SI system*** (rationalized† MKSA system), which is almost universally adopted for electromagnetics work, the permeability of free space is chosen to be

$$\boxed{\mu_0 = 4\pi \times 10^{-7} \qquad \text{(H/m)},} \quad \text{(in free space)} \tag{1–9}$$

where H/m stands for henry per meter. With the values of c and μ_0 fixed in Eqs. (1–6) and (1–9) the value of the permittivity of free space is then derived from the following

† This system of units is said to be ***rationalized*** because the factor 4π does not appear in the Maxwell's equations (the fundamental postulates of electromagnetism). This factor, however, will appear in many derived relations. In the unrationalized MKSA system, μ_0 would be 10^{-7} (H/m), and the factor 4π would appear in the Maxwell's equations.

TABLE 1–3
Universal Constants in SI Units

Universal Constants	Symbol	Value	Unit
Velocity of light in free space	c	3×10^8	m/s
Permeability of free space	μ_0	$4\pi \times 10^{-7}$	H/m
Permittivity of free space	ϵ_0	$\frac{1}{36\pi} \times 10^{-9}$	F/m

relationships:

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \quad \text{(m/s)} \tag{1-10}$$

or

$$\epsilon_0 = \frac{1}{c^2 \mu_0} \cong \frac{1}{36\pi} \times 10^{-9} \cong 8.854 \times 10^{-12} \quad \text{(F/m)}, \tag{1-11}$$

where F/m is the abbreviation for farad per meter. The three universal constants and their values are summarized in Table 1–3.

Now that we have defined the basic quantities and the universal constants of the electromagnetic model, we can develop the various subjects in electromagnetics. But, before we do that, we must be equipped with the appropriate mathematical tools. In the following chapter we discuss the basic rules of operation for vector algebra and vector calculus.

Review Questions

R.1–1 What is electromagnetics?

R.1–2 Describe two phenomena or situations, other than those depicted in Figs. 1–1 and 1–2, that cannot be adequately explained by circuit theory.

R.1–3 What are the three essential steps in building an idealized model for the study of a scientific subject?

R.1–4 What are the four fundamental SI units in electromagnetics?

R.1–5 What are the four fundamental field quantities in the electromagnetic model? What are their units?

R.1–6 What are the three universal constants in the electromagnetic model, and what are their relations?

R.1–7 What are the source quantities in the electromagnetic model?

2

Vector Analysis

2–1 Introduction

As we noted in Chapter 1, some of the quantities in electromagnetics (such as charge, current, and energy) are scalars; and some others (such as electric and magnetic field intensities) are vectors. Both scalars and vectors can be functions of time and position. At a given time and position, a ***scalar*** is completely specified by its magnitude (positive or negative, together with its unit). Thus we can specify, for instance, a charge of -1 μC at a certain location at $t = 0$. The specification of a ***vector*** at a given location and time, on the other hand, requires both a magnitude and a direction. How do we specify the direction of a vector? In a three-dimensional space, three numbers are needed, and these numbers depend on the choice of a coordinate system. Conversion of a given vector from one coordinate system to another will change these numbers. However, physical laws and theorems relating various scalar and vector quantities certainly must hold irrespective of the coordinate system. The general expressions of the laws of electromagnetism, therefore, do not require the specification of a coordinate system. A particular coordinate system is chosen only when a problem of a given geometry is to be analyzed. For example, if we are to determine the magnetic field at the center of a current-carrying wire loop, it is more convenient to use rectangular coordinates if the loop is rectangular, whereas polar coordinates (two-dimensional) will be more appropriate if the loop is circular in shape. The basic electromagnetic relation governing the solution of such a problem is the same for both geometries.

Three main topics will be dealt with in this chapter on vector analysis:

1. Vector algebra—addition, subtraction, and multiplication of vectors.
2. Orthogonal coordinate systems—Cartesian, cylindrical, and spherical coordinates.
3. Vector calculus—differentiation and integration of vectors; line, surface, and volume integrals; "del" operator; gradient, divergence, and curl operations.

Throughout the rest of this book we will decompose, combine, differentiate, integrate, and otherwise manipulate vectors. It is *imperative* to acquire a facility in vector algebra and vector calculus. In a three-dimensional space a vector relation is, in fact, three scalar relations. The use of vector-analysis techniques in electromagnetics leads to concise and elegant formulations. A deficiency in vector analysis in the study of electromagnetics is similar to a deficiency in algebra and calculus in the study of physics; and it is obvious that these deficiencies cannot yield fruitful results.

In solving practical problems we always deal with regions or objects of a given shape, and it is necessary to express general formulas in a coordinate system appropriate for the given geometry. For example, the familiar rectangular (x, y, z) coordinates are, obviously, awkward to use for problems involving a circular cylinder or a sphere because the boundaries of a circular cylinder and a sphere cannot be described by constant values of x, y, and z. In this chapter we discuss the three most commonly used orthogonal (perpendicular) coordinate systems and the representation and operation of vectors in these systems. Familiarity with these coordinate systems is essential in the solution of electromagnetic problems.

Vector calculus pertains to the differentiation and integration of vectors. By defining certain differential operators we can express the basic laws of electromagnetism in a concise way that is invariant with the choice of a coordinate system. In this chapter we introduce the techniques for evaluating different types of integrals involving vectors, and we define and discuss the various kinds of differential operators.

2–2 Vector Addition and Subtraction

We know that a vector has a magnitude and a direction. A vector **A** can be written as

$$\mathbf{A} = \mathbf{a}_A A, \qquad (2\text{–}1)$$

where A is the magnitude (and has the unit and dimension) of **A**,

$$A = |\mathbf{A}|, \qquad (2\text{–}2)$$

and $\mathbf{a}_A$ is a dimensionless unit vector† with a unity magnitude having the direction of **A**. Thus,

$$\mathbf{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{A}}{A}. \qquad (2\text{–}3)$$

The vector **A** can be represented graphically by a directed straight-line segment of a length $|\mathbf{A}| = A$ with its arrowhead pointing in the direction of $\mathbf{a}_A$, as shown in Fig. 2–1. Two vectors are equal if they have the same magnitude and the same direction, even

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† In some books the unit vector in the direction of **A** is variously denoted by $\hat{\mathbf{A}}$, $\mathbf{u}_A$, or $\mathbf{i}_A$. We prefer to write **A** as in Eq. (2–1) instead of as $\mathbf{A} = \hat{\mathbf{A}}A$. A vector going from point P_1 to point P_2 will then be written as $\mathbf{a}_{P_1P_2}(\overline{P_1P_2})$ instead of as $\widehat{\mathbf{P}_1\mathbf{P}_2}(P_1P_2)$, which is somewhat cumbersome. The symbols **u** and **i** are used for velocity and current, respectively.

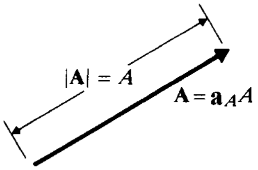


FIGURE 2–1
Graphical representation of vector **A**.

though they may be displaced in space. Since it is difficult to write boldfaced letters by hand, it is a common practice to use an arrow or a bar over a letter ($\vec{A}$ or $\bar{A}$) or a wiggly line under a letter ($\utilde{A}$) to distinguish a vector from a scalar. This distinguishing mark, once chosen, *should never be omitted* whenever and wherever vectors are written.

Two vectors **A** and **B**, which are not in the same direction nor in opposite directions, such as given in Fig. 2–2(a), determine a plane. Their sum is another vector **C** in the same plane. **C** = **A** + **B** can be obtained graphically in two ways.

1. By the parallelogram rule: The resultant **C** is the diagonal vector of the parallelogram formed by **A** and **B** drawn from the same point, as shown in Fig. 2–2(b).
2. By the head-to-tail rule: The head of **A** connects to the tail of **B**. Their sum **C** is the vector drawn from the tail of **A** to the head of **B**; and vectors **A**, **B**, and **C** form a triangle, as shown in Fig. 2–2(c).

It is obvious that vector addition obeys the commutative and associative laws.

$$\text{Commutative law:} \quad \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}. \tag{2–4}$$

$$\text{Associative law:} \quad \mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}. \tag{2–5}$$

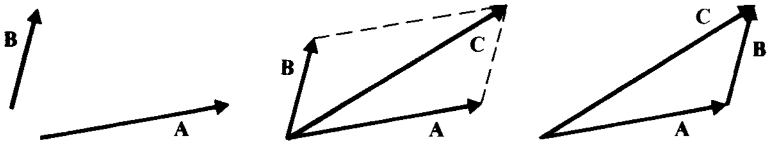
Vector subtraction can be defined in terms of vector addition in the following way:

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}), \tag{2–6}$$

where $-\mathbf{B}$ is the negative of vector **B**; that is, $-\mathbf{B}$ has the same magnitude as **B**, but its direction is opposite to that of **B**. Thus

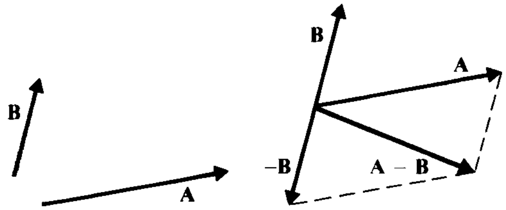
$$-\mathbf{B} = (-\mathbf{a}_B)B. \tag{2–7}$$

The operation represented by Eq. (2–6) is illustrated in Fig. 2–3.



(a) Two vectors, **A** and **B**. (b) Parallelogram rule. (c) Head-to-tail rule.

FIGURE 2–2
Vector addition, **C** = **A** + **B**.



(a) Two vectors, **A** and **B**.

(b) Subtraction of vectors, **A** − **B**.

FIGURE 2–3
Vector subtraction.

2–3 Products of Vectors

Multiplication of a vector **A** by a positive scalar k changes the magnitude of **A** by k times without changing its direction (k can be either greater or less than 1).

$$k\mathbf{A} = \mathbf{a}_A(kA). \tag{2–8}$$

It is not sufficient to say "the multiplication of one vector by another" or "the product of two vectors" because there are two distinct and very different types of products of two vectors. They are (1) scalar or dot products, and (2) vector or cross products. These will be defined in the following subsections.

2–3.1 SCALAR OR DOT PRODUCT

The scalar or dot product of two vectors **A** and **B**, denoted by $\mathbf{A} \cdot \mathbf{B}$, is a scalar, which equals the product of the magnitudes of **A** and **B** and the cosine of the angle between them. Thus,

$$\boxed{\mathbf{A} \cdot \mathbf{B} \triangleq AB \cos \theta_{AB}.} \tag{2–9}$$

In Eq. (2–9) the symbol $\triangleq$ signifies "equal by definition," and θ_{AB} is the *smaller* angle between **A** and **B** and is less than π radians (180°), as indicated in Fig. 2–4. The dot product of two vectors (1) is less than or equal to the product of their magnitudes; (2) can be either a positive or a negative quantity, depending on whether the angle between them is smaller or larger than $\pi/2$ radians (90°); (3) is equal to the product

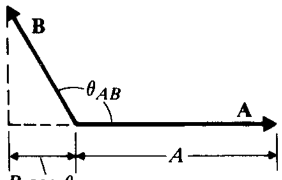


FIGURE 2–4
Illustrating the dot product of **A** and **B**.

of the magnitude of one vector and the projection of the other vector upon the first one; and (4) is zero when the vectors are perpendicular to each other. It is evident that

$$\mathbf{A} \cdot \mathbf{A} = A^2 \tag{2–10}$$

or

$$A = \sqrt[+]{\mathbf{A} \cdot \mathbf{A}}. \tag{2–11}$$

Equation (2–11) enables us to find the magnitude of a vector when the expression of the vector is given in any coordinate system.

The dot product is commutative and distributive.

$$\text{Commutative law:} \quad \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}. \tag{2–12}$$

$$\text{Distributive law:} \quad \mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}. \tag{2–13}$$

The commutative law is obvious from the definition of the dot product in Eq. (2–9), and the proof of Eq. (2–13) is left as an exercise. The associative law does not apply to the dot product, since no more than two vectors can be so multiplied and an expression such as $\mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C}$ is meaningless.

EXAMPLE 2–1 Prove the law of cosines for a triangle.

Solution The law of cosines is a scalar relationship that expresses the length of a side of a triangle in terms of the lengths of the two other sides and the angle between them. Referring to Fig. 2–5, we find the law of cosines states that

$$C = \sqrt{A^2 + B^2 - 2AB\cos\alpha}.$$

We prove this by considering the sides as vectors; that is,

$$\mathbf{C} = \mathbf{A} + \mathbf{B}.$$

Taking the dot product of $\mathbf{C}$ with itself, we have, from Eqs. (2–10) and (2–13),

$$\begin{aligned} C^2 = \mathbf{C} \cdot \mathbf{C} &= (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} + \mathbf{B}) \\ &= \mathbf{A} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{B} + 2\mathbf{A} \cdot \mathbf{B} \\ &= A^2 + B^2 + 2AB\cos\theta_{AB}. \end{aligned}$$

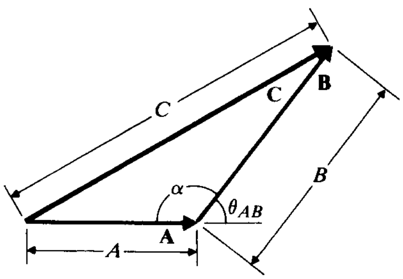


FIGURE 2–5
Illustrating Example 2–1.

Note that θ_{AB} is, by definition, the *smaller* angle between **A** and **B** and is equal to $(180° - \alpha)$; hence $\cos\theta_{AB} = \cos(180° - \alpha) = -\cos\alpha$. Therefore,

$$C^2 = A^2 + B^2 - 2AB\cos\alpha,$$

and the law of cosines follows directly. ■

2–3.2 VECTOR OR CROSS PRODUCT

The vector or cross product of two vectors **A** and **B**, denoted by $\mathbf{A} \times \mathbf{B}$, is a vector perpendicular to the plane containing **A** and **B**; its magnitude is $AB \sin\theta_{AB}$, where θ_{AB} is the *smaller* angle between **A** and **B**, and its direction follows that of the thumb of the right hand when the fingers rotate from **A** to **B** through the angle θ_{AB} (the right-hand rule).

$$\boxed{\mathbf{A} \times \mathbf{B} \triangleq \mathbf{a}_n |AB \sin\theta_{AB}|.} \tag{2–14}$$

This is illustrated in Fig. 2–6. Since $B\sin\theta_{AB}$ is the height of the parallelogram formed by the vectors **A** and **B**, we recognize that the magnitude of $\mathbf{A} \times \mathbf{B}$, $|AB\sin\theta_{AB}|$, which is always positive, is numerically equal to the area of the parallelogram.

Using the definition in Eq. (2–14) and following the right-hand rule, we find that

$$\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}. \tag{2–15}$$

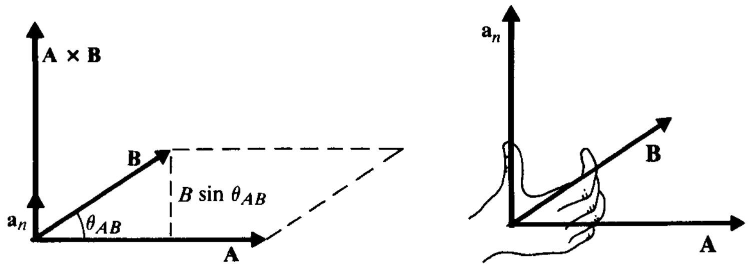
Hence the cross product is *not* commutative. We can see that the cross product obeys the distributive law,

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}. \tag{2–16}$$

Can you show this in general without resolving the vectors into rectangular components?

The vector product is obviously *not* associative; that is,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}. \tag{2–17}$$



(a) $\mathbf{A} \times \mathbf{B} = \mathbf{a}_n|AB\sin\theta_{AB}|$.

(b) The right-hand rule.

FIGURE 2–6
Cross product of **A** and **B**, $\mathbf{A} \times \mathbf{B}$.

The vector representing the triple product on the left side of the expression above is perpendicular to **A** and lies in the plane formed by **B** and **C**, whereas that on the right side is perpendicular to **C** and lies in the plane formed by **A** and **B**. The order in which the two vector products are performed is therefore vital, and *in no case should the parentheses be omitted.*

EXAMPLE 2-2 The motion of a rigid disk rotating about its axis shown in Fig. 2-7(a) can be described by an angular velocity vector $\boldsymbol{\omega}$. The direction of $\boldsymbol{\omega}$ is along the axis and follows the right-hand rule; that is, if the fingers of the right hand bend in the direction of rotation, the thumb points to the direction of $\boldsymbol{\omega}$. Find the vector expression for the lineal velocity of a point on the disk, which is at a distance d from the axis of rotation.

Solution From mechanics we know that the magnitude of the lineal velocity, **v**, of a point P at a distance d from the rotating axis is ωd and the direction is always tangential to the circle of rotation. However, since the point P is moving, the direction of **v** changes with the position of P. How do we write its vector representation?

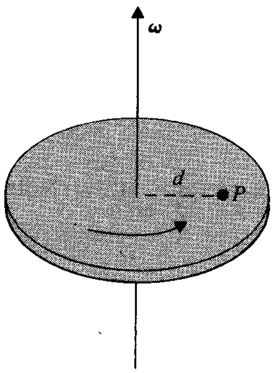
Let O be the origin of the chosen coordinate system. The position vector of the point P can be written as **R**, as shown in Fig. 2-7(b). We have

$$|\mathbf{v}| = \omega d = \omega R \sin\theta.$$

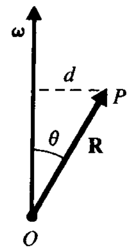
No matter where the point P is, the direction of **v** is always perpendicular to the plane containing the vectors $\boldsymbol{\omega}$ and **R**. Hence we can write, very simply,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{R},$$

which represents correctly both the magnitude and the direction of the lineal velocity of P.



(a) A rotating disk.



(b) Vector representation.

FIGURE 2-7
Illustrating Example 2-2.

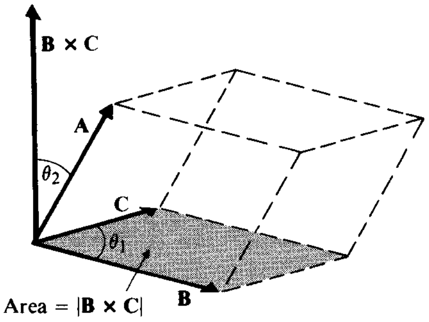


FIGURE 2–8
Illustrating scalar triple product $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$.

2–3.3 PRODUCT OF THREE VECTORS

There are two kinds of products of three vectors; namely, the ***scalar triple product*** and the ***vector triple product***. The scalar triple product is much the simpler of the two and has the following property:

$$\boxed{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}).} \tag{2–18}$$

Note the cyclic permutation of the order of the three vectors **A**, **B**, and **C**. Of course,

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= -\mathbf{A}\cdot(\mathbf{C}\times\mathbf{B})\\ &= -\mathbf{B}\cdot(\mathbf{A}\times\mathbf{C})\\ &= -\mathbf{C}\cdot(\mathbf{B}\times\mathbf{A}).\end{aligned} \tag{2–19}$$

As can be seen from Fig. 2–8, each of the three expressions in Eq. (2–18) has a magnitude equal to the volume of the parallelepiped formed by the three vectors **A**, **B**, and **C**. The parallelepiped has a base with an area equal to $|\mathbf{B}\times\mathbf{C}| = |BC\sin\theta_1|$ and a height equal to $|A\cos\theta_2|$; hence the volume is $|ABC\sin\theta_1\cos\theta_2|$.

The vector triple product $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ can be expanded as the difference of two simple vectors as follows:

$$\boxed{\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}).} \tag{2–20}$$

Equation (2–20) is known as the "***back-cab*" *rule*** and is a useful vector identity. (Note "BAC-CAB" on the right side of the equation!)

EXAMPLE 2–3† Prove the back-cab rule of vector triple product.

† The back-cab rule can be verified in a straightforward manner by expanding the vectors in the Cartesian coordinate system (Problem P.2–12). Only those interested in a general proof need to study this example.

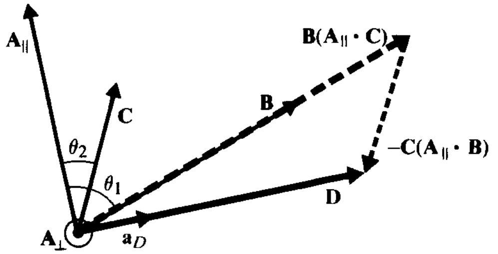


FIGURE 2–9
Illustrating the back-cab rule of vector triple product.

Solution In order to prove Eq. (2–20) it is convenient to expand **A** into two components:

$$\mathbf{A} = \mathbf{A}_{||} + \mathbf{A}_{\perp},$$

where $\mathbf{A}_{||}$ and $\mathbf{A}_{\perp}$ are parallel and perpendicular, respectively, to the plane containing **B** and **C**. Because the vector representing $(\mathbf{B} \times \mathbf{C})$ is also perpendicular to the plane, the cross product of $\mathbf{A}_{\perp}$ and $(\mathbf{B} \times \mathbf{C})$ vanishes. Let $\mathbf{D} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$. Since only $\mathbf{A}_{||}$ is effective here, we have

$$\mathbf{D} = \mathbf{A}_{||} \times (\mathbf{B} \times \mathbf{C}).$$

Referring to Fig. 2–9, which shows the plane containing **B**, **C**, and $\mathbf{A}_{||}$, we note that **D** lies in the same plane and is normal to $\mathbf{A}_{||}$. The magnitude of $(\mathbf{B} \times \mathbf{C})$ is $BC \sin(\theta_1 - \theta_2)$, and that of $\mathbf{A}_{||} \times (\mathbf{B} \times \mathbf{C})$ is $A_{||}BC \sin(\theta_1 - \theta_2)$. Hence,

$$\begin{aligned} D = \mathbf{D} \cdot \mathbf{a}_D &= A_{||}BC \sin(\theta_1 - \theta_2) \\ &= (B \sin\theta_1)(A_{||}C \cos\theta_2) - (C \sin\theta_2)(A_{||}B \cos\theta_1) \\ &= [\mathbf{B}(\mathbf{A}_{||} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A}_{||} \cdot \mathbf{B})] \cdot \mathbf{a}_D. \end{aligned}$$

The expression above does not alone guarantee the quantity inside the brackets to be **D**, since the former may contain a vector that is normal to **D** (parallel to $\mathbf{A}_{||}$); that is, $\mathbf{D} \cdot \mathbf{a}_D = \mathbf{E} \cdot \mathbf{a}_D$ does not guarantee $\mathbf{E} = \mathbf{D}$. In general, we can write

$$\mathbf{B}(\mathbf{A}_{||} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A}_{||} \cdot \mathbf{B}) = \mathbf{D} + k\mathbf{A}_{||},$$

where k is a scalar quantity. To determine k, we scalar-multiply both sides of the above equation by $\mathbf{A}_{||}$ and obtain

$$(\mathbf{A}_{||} \cdot \mathbf{B})(\mathbf{A}_{||} \cdot \mathbf{C}) - (\mathbf{A}_{||} \cdot \mathbf{C})(\mathbf{A}_{||} \cdot \mathbf{B}) = 0 = \mathbf{A}_{||} \cdot \mathbf{D} + kA_{||}^2.$$

Since $\mathbf{A}_{||} \cdot \mathbf{D} = 0$, then $k = 0$ and

$$\mathbf{D} = \mathbf{B}(\mathbf{A}_{||} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A}_{||} \cdot \mathbf{B}),$$

which proves the back-cab rule, inasmuch as $\mathbf{A}_{||} \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C}$ and $\mathbf{A}_{||} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{B}$. ▬

Division by a vector is not defined, and expressions such as $k/\mathbf{A}$ and $\mathbf{B}/\mathbf{A}$ are meaningless.

2–4 Orthogonal Coordinate Systems

We have indicated before that although the laws of electromagnetism are invariant with coordinate system, solution of practical problems requires that the relations derived from these laws be expressed in a coordinate system appropriate to the geometry of the given problems. For example, if we are to determine the electric field at a certain point in space, we at least need to describe the position of the source and the location of this point in a coordinate system. In a three-dimensional space a point can be located as the intersection of three surfaces. Assume that the three families of surfaces are described by u_1 = constant, u_2 = constant, and u_3 = constant, where the u's need not all be lengths. (In the familiar Cartesian or rectangular coordinate system, u_1, u_2, and u_3 correspond to x, y, and z, respectively.) When these three surfaces are mutually perpendicular to one another, we have an ***orthogonal coordinate system.*** Nonorthogonal coordinate systems are not used because they complicate problems.

Some surfaces represented by u_i = constant (i = 1, 2, or 3) in a coordinate system may not be planes; they may be curved surfaces. Let $\mathbf{a}_{u_1}$, $\mathbf{a}_{u_2}$, and $\mathbf{a}_{u_3}$ be the unit vectors in the three coordinate directions. They are called the ***base vectors.*** In a general right-handed, orthogonal, curvilinear coordinate system the base vectors are arranged in such a way that the following relations are satisfied:

$$\mathbf{a}_{u_1} \times \mathbf{a}_{u_2} = \mathbf{a}_{u_3}, \tag{2–21a}$$

$$\mathbf{a}_{u_2} \times \mathbf{a}_{u_3} = \mathbf{a}_{u_1}, \tag{2–21b}$$

$$\mathbf{a}_{u_3} \times \mathbf{a}_{u_1} = \mathbf{a}_{u_2}. \tag{2–21c}$$

These three equations are not all independent, as the specification of one automatically implies the other two. We have, of course,

$$\mathbf{a}_{u_1} \cdot \mathbf{a}_{u_2} = \mathbf{a}_{u_2} \cdot \mathbf{a}_{u_3} = \mathbf{a}_{u_3} \cdot \mathbf{a}_{u_1} = 0 \tag{2–22}$$

and

$$\mathbf{a}_{u_1} \cdot \mathbf{a}_{u_1} = \mathbf{a}_{u_2} \cdot \mathbf{a}_{u_2} = \mathbf{a}_{u_3} \cdot \mathbf{a}_{u_3} = 1. \tag{2–23}$$

Any vector **A** can be written as the sum of its components in the three orthogonal directions, as follows:

$$\boxed{\mathbf{A} = \mathbf{a}_{u_1} A_{u_1} + \mathbf{a}_{u_2} A_{u_2} + \mathbf{a}_{u_3} A_{u_3}.} \tag{2–24}$$

From Eq. (2–24) the magnitude of **A** is

$$A = |\mathbf{A}| = (A_{u_1}^2 + A_{u_2}^2 + A_{u_3}^2)^{1/2}. \tag{2–25}$$

EXAMPLE 2–4 Given three vectors **A**, **B**, and **C**, obtain the expressions of (a) $\mathbf{A} \cdot \mathbf{B}$, (b) $\mathbf{A} \times \mathbf{B}$, and (c) $\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$ in the orthogonal curvilinear coordinate system (u_1, u_2, u_3).

Solution First we write **A**, **B**, and **C** in the orthogonal coordinates (u_1, u_2, u_3):

$$\mathbf{A} = \mathbf{a}_{u_1}A_{u_1} + \mathbf{a}_{u_2}A_{u_2} + \mathbf{a}_{u_3}A_{u_3},$$
$$\mathbf{B} = \mathbf{a}_{u_1}B_{u_1} + \mathbf{a}_{u_2}B_{u_2} + \mathbf{a}_{u_3}B_{u_3},$$
$$\mathbf{C} = \mathbf{a}_{u_1}C_{u_1} + \mathbf{a}_{u_2}C_{u_2} + \mathbf{a}_{u_3}C_{u_3}.$$

a) $$\mathbf{A}\cdot\mathbf{B} = (\mathbf{a}_{u_1}A_{u_1} + \mathbf{a}_{u_2}A_{u_2} + \mathbf{a}_{u_3}A_{u_3})\cdot(\mathbf{a}_{u_1}B_{u_1} + \mathbf{a}_{u_2}B_{u_2} + \mathbf{a}_{u_3}B_{u_3}) = A_{u_1}B_{u_1} + A_{u_2}B_{u_2} + A_{u_3}B_{u_3}, \qquad (2\text{–}26)$$

in view of Eqs. (2–22) and (2–23).

b) $$\mathbf{A}\times\mathbf{B} = (\mathbf{a}_{u_1}A_{u_1} + \mathbf{a}_{u_2}A_{u_2} + \mathbf{a}_{u_3}A_{u_3})\times(\mathbf{a}_{u_1}B_{u_1} + \mathbf{a}_{u_2}B_{u_2} + \mathbf{a}_{u_3}B_{u_3})$$
$$= \mathbf{a}_{u_1}(A_{u_2}B_{u_3} - A_{u_3}B_{u_2}) + \mathbf{a}_{u_2}(A_{u_3}B_{u_1} - A_{u_1}B_{u_3}) + \mathbf{a}_{u_3}(A_{u_1}B_{u_2} - A_{u_2}B_{u_1})$$
$$= \begin{vmatrix} \mathbf{a}_{u_1} & \mathbf{a}_{u_2} & \mathbf{a}_{u_3} \\ A_{u_1} & A_{u_2} & A_{u_3} \\ B_{u_1} & B_{u_2} & B_{u_3} \end{vmatrix}. \qquad (2\text{–}27)$$

Equations (2–26) and (2–27) express the dot and cross products, respectively, of two vectors in orthogonal curvilinear coordinates. They are important and should be remembered.

c) The expression for $\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$ can be written down immediately by combining the results in Eqs. (2–26) and (2–27):

$$\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = C_{u_1}(A_{u_2}B_{u_3} - A_{u_3}B_{u_2}) + C_{u_2}(A_{u_3}B_{u_1} - A_{u_1}B_{u_3}) + C_{u_3}(A_{u_1}B_{u_2} - A_{u_2}B_{u_1})$$
$$= \begin{vmatrix} C_{u_1} & C_{u_2} & C_{u_3} \\ A_{u_1} & A_{u_2} & A_{u_3} \\ B_{u_1} & B_{u_2} & B_{u_3} \end{vmatrix}. \qquad (2\text{–}28)$$

Eq. (2–28) can be used to prove Eqs. (2–18) and (2–19) by observing that a permutation of the order of the vectors on the left side leads simply to a rearrangement of the rows in the determinant on the right side. ■

In vector calculus (and in electromagnetics work) we are often required to perform line, surface, and volume integrals. In each case we need to express the differential length-change corresponding to a differential change in one of the coordinates. However, some of the coordinates, say u_i ($i = 1, 2$, or 3), may not be a length; and a conversion factor is needed to convert a differential change du_i into a change in length $d\ell_i$:

$$d\ell_i = h_i\,du_i, \qquad (2\text{–}29)$$

where h_i is called a ***metric coefficient*** and may itself be a function of u_1, u_2, and u_3. For example, in the two-dimensional polar coordinates $(u_1, u_2) = (r, \phi)$, a differential change $d\phi$ $(=du_2)$ in ϕ $(=u_2)$ corresponds to a differential length-change $d\ell_2 = r\,d\phi$ $(h_2 = r = u_1)$ in the $\mathbf{a}_\phi$ $(=\mathbf{a}_{u_2})$-direction. A directed differential length-change in an arbitrary direction can be written as the vector sum of the component length-changes:

$$d\boldsymbol{\ell} = \mathbf{a}_{u_1}\, d\ell_1 + \mathbf{a}_{u_2}\, d\ell_2 + \mathbf{a}_{u_3}\, d\ell_3 \tag{2–30}$$ [†]

or

$$\boxed{d\boldsymbol{\ell} = \mathbf{a}_{u_1}(h_1\, du_1) + \mathbf{a}_{u_2}(h_2\, du_2) + \mathbf{a}_{u_3}(h_3\, du_3).} \tag{2–31}$$

In view of Eq. (2–25) the magnitude of $d\boldsymbol{\ell}$ is

$$\begin{aligned} d\ell &= [(d\ell_1)^2 + (d\ell_2)^2 + (d\ell_3)^2]^{1/2} \\ &= [(h_1\, du_1)^2 + (h_2\, du_2)^2 + (h_3\, du_3)^2]^{1/2}. \end{aligned} \tag{2–32}$$

The differential volume dv formed by differential coordinate changes du_1, du_2, and du_3 in directions $\mathbf{a}_{u1}$, $\mathbf{a}_{u2}$, and $\mathbf{a}_{u3}$, respectively, is $(d\ell_1\, d\ell_2\, d\ell_3)$, or

$$\boxed{dv = h_1 h_2 h_3\, du_1\, du_2\, du_3.} \tag{2–33}$$

Later we will have occasion to express the current or flux flowing through a differential area. In such cases the cross-sectional area perpendicular to the current or flux flow must be used, and it is convenient to consider the differential area a vector with a direction normal to the surface; that is,

$$\boxed{d\mathbf{s} = \mathbf{a}_n\, ds.} \tag{2–34}$$

For instance, if current density $\mathbf{J}$ is not perpendicular to a differential area of a magnitude ds, the current, dI, flowing through ds must be the component of $\mathbf{J}$ normal to the area multiplied by the area. Using the notation in Eq. (2–34), we can write simply

$$\begin{aligned} dI &= \mathbf{J} \cdot d\mathbf{s} \\ &= \mathbf{J} \cdot \mathbf{a}_n\, ds. \end{aligned} \tag{2–35}$$

In general orthogonal curvilinear coordinates the differential area ds_1 normal to the unit vector $\mathbf{a}_{u_1}$ is

$$ds_1 = d\ell_2\, d\ell_3$$

or

$$\boxed{ds_1 = h_2 h_3\, du_2\, du_3.} \tag{2–36}$$

Similarly, the differential areas normal to unit vectors $\mathbf{a}_{u_2}$ and $\mathbf{a}_{u_3}$ are, respectively,

$$\boxed{ds_2 = h_1 h_3\, du_1\, du_3} \tag{2–37}$$

[†] The $\boldsymbol{\ell}$ here is the symbol of a vector of length ℓ.

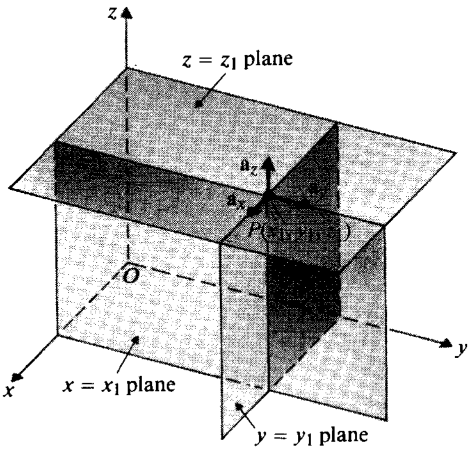


FIGURE 2–10
Cartesian coordinates.

and

$$\boxed{ds_3 = h_1 h_2\, du_1\, du_2.} \tag{2–38}$$

Many orthogonal coordinate systems exist; but we shall be concerned only with the three that are most common and most useful:

1. Cartesian (or rectangular) coordinates.[†]
2. Cylindrical coordinates.
3. Spherical coordinates.

These will be discussed separately in the following subsections.

2–4.1 CARTESIAN COORDINATES

$$(u_1, u_2, u_3) = (x, y, z)$$

A point $P(x_1, y_1, z_1)$ in Cartesian coordinates is the intersection of *three planes* specified by $x = x_1$, $y = y_1$, and $z = z_1$, as shown in Fig. 2–10. It is a right-handed system with base vectors $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ satisfying the following relations:

$$\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z \tag{2–39a}$$

$$\mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x \tag{2–39b}$$

$$\mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y. \tag{2–39c}$$

[†] The term "Cartesian coordinates" is preferred because the term "rectangular coordinates" is customarily associated with two-dimensional geometry.

The position vector to the point $P(x_1, y_1, z_1)$ is

$$\overrightarrow{OP} = \mathbf{a}_x x_1 + \mathbf{a}_y y_1 + \mathbf{a}_z z_1. \tag{2–40}$$

A vector **A** in Cartesian coordinates can be written as

$$\boxed{\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z.} \tag{2–41}$$

The dot product of two vectors **A** and **B** is, from Eq. (2–26),

$$\boxed{\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z,} \tag{2–42}$$

and the cross product of **A** and **B** is, from Eq. (2–27),

$$\mathbf{A} \times \mathbf{B} = \mathbf{a}_x(A_y B_z - A_z B_y) + \mathbf{a}_y(A_z B_x - A_x B_z) + \mathbf{a}_z(A_x B_y - A_y B_x)$$
$$= \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \tag{2–43}$$

Since x, y, and z are lengths themselves, all three metric coefficients are unity; that is, $h_1 = h_2 = h_3 = 1$. The expressions for the differential length, differential area, and differential volume are—from Eqs. (2–31), (2–36), (2–37), (2–38), and (2–33)—respectively,

$$\boxed{d\ell = \mathbf{a}_x\,dx + \mathbf{a}_y\,dy + \mathbf{a}_z\,dz;} \tag{2–44}$$

$$ds_x = dy\,dz, \tag{2–45a}$$
$$ds_y = dx\,dz, \tag{2–45b}$$
$$ds_z = dx\,dy; \tag{2–45c}$$

and

$$\boxed{dv = dx\,dy\,dz.} \tag{2–46}$$

A typical differential volume element at a point (x, y, z) resulting from differential changes dx, dy, and dz is shown in Fig. 2–11. The differential surface areas ds_x, ds_y, and ds_z normal to the directions $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ are also indicated.

EXAMPLE 2–5 Given $\mathbf{A} = \mathbf{a}_x 5 - \mathbf{a}_y 2 + \mathbf{a}_z$, find the expression of a unit vector **B** such that

a) $\mathbf{B} \parallel \mathbf{A}$.

b) $\mathbf{B} \perp \mathbf{A}$, if **B** lies in the xy-plane.

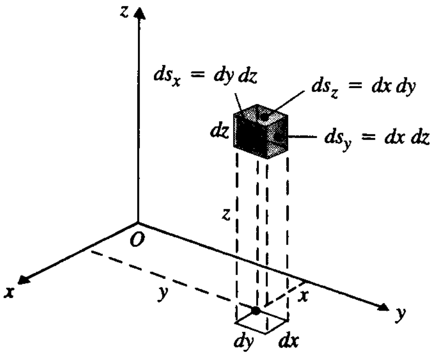


FIGURE 2-11
A differential volume in Cartesian coordinates.

Solution Let $\mathbf{B} = \mathbf{a}_x B_x + \mathbf{a}_y B_y + \mathbf{a}_z B_z$. We know that

$$|\mathbf{B}| = (B_x^2 + B_y^2 + B_z^2)^{1/2} = 1. \tag{2-47}$$

a) $\mathbf{B} \| \mathbf{A}$ requires $\mathbf{B} \times \mathbf{A} = 0$. From Eq. (2–43) we have

$$-2B_z - B_y = 0, \tag{2-48a}$$

$$B_x - 5B_z = 0, \tag{2-48b}$$

$$5B_y + 2B_x = 0. \tag{2-48c}$$

The above three equations are not all independent. For instance, subtracting Eq. (2–48c) from twice Eq. (2–48b) yields Eq. (2–48a). Solving Eqs. (2–47), (2–48a), and (2–48b) simultaneously, we obtain

$$B_x = \frac{5}{\sqrt{30}}, \qquad B_y = -\frac{2}{\sqrt{30}}, \qquad \text{and} \qquad B_z = \frac{1}{\sqrt{30}}.$$

Therefore,

$$\mathbf{B} = \frac{1}{\sqrt{30}}(\mathbf{a}_x 5 - \mathbf{a}_y 2 + \mathbf{a}_z).$$

b) $\mathbf{B} \perp \mathbf{A}$ requires $\mathbf{B} \cdot \mathbf{A} = 0$. From Eq. (2–42) we have

$$5B_x - 2B_y = 0, \tag{2-49}$$

where we have set $B_z = 0$, since $\mathbf{B}$ lies in the xy-plane. Solution of Eqs. (2–47) and (2–49) yields

$$B_x = \frac{2}{\sqrt{29}} \qquad \text{and} \qquad B_y = \frac{5}{\sqrt{29}}.$$

Hence,

$$\mathbf{B} = \frac{1}{\sqrt{29}}(\mathbf{a}_x 2 + \mathbf{a}_y 5).$$

EXAMPLE 2–6 (a) Write the expression of the vector going from point $P_1(1, 3, 2)$ to point $P_2(3, -2, 4)$ in Cartesian coordinates. (b) What is the length of this line?

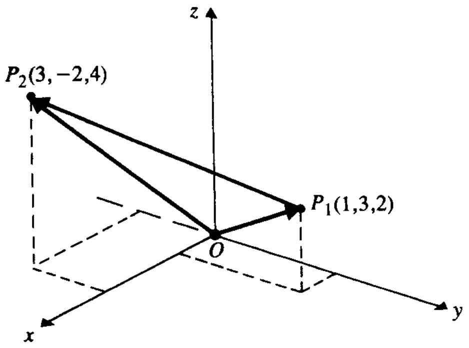


FIGURE 2–12
Illustrating Example 2–6.

Solution

a) From Fig. 2–12 we see that

$$\begin{aligned}\overrightarrow{P_1P_2} &= \overrightarrow{OP_2} - \overrightarrow{OP_1}\\ &= (\mathbf{a}_x 3 - \mathbf{a}_y 2 + \mathbf{a}_z 4) - (\mathbf{a}_x + \mathbf{a}_y 3 + \mathbf{a}_z 2)\\ &= \mathbf{a}_x 2 - \mathbf{a}_y 5 + \mathbf{a}_z 2.\end{aligned}$$

b) The length of the line is

$$\begin{aligned}P_1P_2 &= |\overrightarrow{P_1P_2}|\\ &= \sqrt{2^2 + (-5)^2 + 2^2}\\ &= \sqrt{33}.\end{aligned}$$

EXAMPLE 2–7 The equation of a straight line in the xy-plane is given by $2x + y = 4$.

a) Find the vector equation of a unit normal from the origin to the line.

b) Find the equation of a line passing through the point $P(0, 2)$ and perpendicular to the given line.

Solution It is clear that the given equation $y = -2x + 4$ represents a straight line having a slope -2 and a vertical intercept $+4$, shown as L_1 (solid line) in Fig. 2–13.

a) If the line is shifted down four units, we have the dashed parallel line L_1' passing through the origin whose equation is $2x + y = 0$. Let the position vector of a point on L_1' be

$$\mathbf{r} = \mathbf{a}_x x + \mathbf{a}_y y.$$

The vector $\mathbf{N} = \mathbf{a}_x 2 + \mathbf{a}_y$ is perpendicular to L_1' because

$$\mathbf{N} \cdot \mathbf{r} = 2x + y = 0.$$

Obviously, $\mathbf{N}$ is also perpendicular to L_1. Thus, the vector equation of the unit normal at the origin is

$$\mathbf{a}_N = \frac{\mathbf{N}}{|\mathbf{N}|} = \frac{1}{\sqrt{5}}(\mathbf{a}_x 2 + \mathbf{a}_y).$$

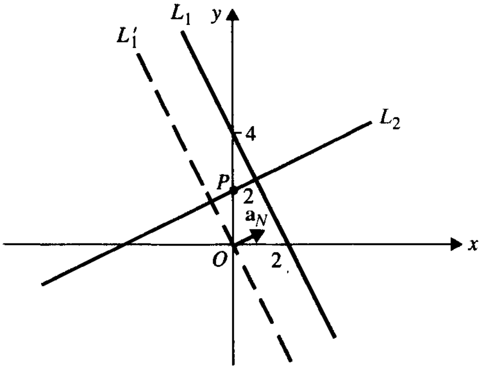


FIGURE 2–13
Illustrating Example 2–7.

Note that the slope of $\mathbf{a}_N$ $(=\frac{1}{2})$ is the negative reciprocal of that of lines L_1 and L_1' $(= -2)$.

b) Let the line passing through the point $P(0, 2)$ and perpendicular to L_1 be L_2. L_2 is parallel to and has the same slope as $\mathbf{a}_N$. The equation of L_2 is then

$$y = \frac{x}{2} + 2, \qquad \text{or} \qquad x - 2y = -4,$$

since L_2 is required to pass through the point $P(0, 2)$. ∎

2–4.2 CYLINDRICAL COORDINATES

$$(u_1, u_2, u_3) = (r, \phi, z)$$

In cylindrical coordinates a point $P(r_1, \phi_1, z_1)$ is the intersection of a circular cylindrical surface $r = r_1$, a half-plane containing the z-axis and making an angle $\phi = \phi_1$ with the xz-plane, and a plane parallel to the xy-plane at $z = z_1$. As indicated in Fig. 2–14, angle ϕ is measured from the positive x-axis, and the base vector $\mathbf{a}_\phi$ is

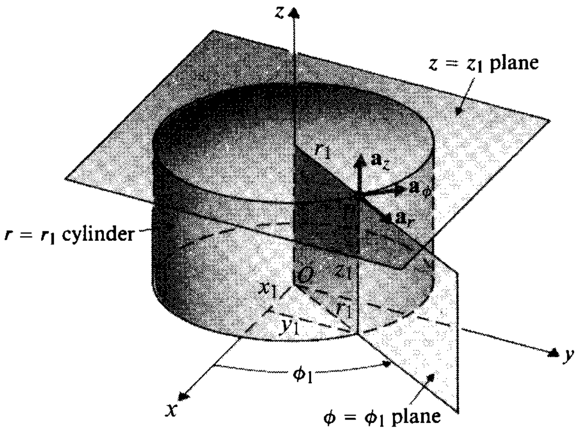


FIGURE 2–14
Cylindrical coordinates.

tangential to the cylindrical surface. The following right-hand relations apply:

$$\mathbf{a}_r \times \mathbf{a}_\phi = \mathbf{a}_z, \tag{2-50a}$$

$$\mathbf{a}_\phi \times \mathbf{a}_z = \mathbf{a}_r, \tag{2-50b}$$

$$\mathbf{a}_z \times \mathbf{a}_r = \mathbf{a}_\phi. \tag{2-50c}$$

Cylindrical coordinates are important for problems with long line charges or currents, and in places where cylindrical or circular boundaries exist. The two-dimensional polar coordinates are a special case at $z = 0$.

A vector in cylindrical coordinates is written as

$$\boxed{\mathbf{A} = \mathbf{a}_r A_r + \mathbf{a}_\phi A_\phi + \mathbf{a}_z A_z.} \tag{2-51}$$

The expressions for the dot and cross products of two vectors in cylindrical coordinates follow from Eqs. (2–26) and (2–27) directly.

Two of the three coordinates, r and z (u_1 and u_3), are themselves lengths; hence $h_1 = h_3 = 1$. However, ϕ is an angle requiring a metric coefficient $h_2 = r$ to convert $d\phi$ to $d\ell_2$. The general expression for a differential length in cylindrical coordinates is then, from Eq. (2–31),

$$\boxed{d\boldsymbol{\ell} = \mathbf{a}_r\,dr + \mathbf{a}_\phi r\,d\phi + \mathbf{a}_z\,dz.} \tag{2-52}$$

The expressions for differential areas and differential volume are

$$ds_r = r\,d\phi\,dz, \tag{2-53a}$$

$$ds_\phi = dr\,dz, \tag{2-53b}$$

$$ds_z = r\,dr\,d\phi, \tag{2-53c}$$

and

$$\boxed{dv = r\,dr\,d\phi\,dz.} \tag{2-54}$$

A typical differential volume element at a point (r, ϕ, z) resulting from differential changes dr, $d\phi$, and dz in the three orthogonal coordinate directions is shown in Fig. 2–15.

A vector given in cylindrical coordinates can be transformed into one in Cartesian coordinates, and vice versa. Suppose we want to express $\mathbf{A} = \mathbf{a}_r A_r + \mathbf{a}_\phi A_\phi + \mathbf{a}_z A_z$ in Cartesian coordinates; that is, we want to write $\mathbf{A}$ as $\mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z$ and determine A_x, A_y, and A_z. First of all, we note that A_z, the z-component of $\mathbf{A}$, is not changed by the transformation from cylindrical to Cartesian coordinates. To find A_x, we equate the dot products of both expressions of $\mathbf{A}$ with $\mathbf{a}_x$. Thus

$$\begin{aligned} A_x &= \mathbf{A} \cdot \mathbf{a}_x \\ &= A_r \mathbf{a}_r \cdot \mathbf{a}_x + A_\phi \mathbf{a}_\phi \cdot \mathbf{a}_x. \end{aligned}$$

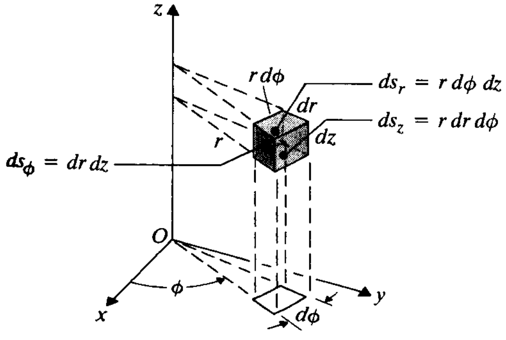


FIGURE 2–15
A differential volume element in cylindrical coordinates.

The term containing A_z disappears here because $\mathbf{a}_z \cdot \mathbf{a}_x = 0$. Referring to Fig. 2–16, which shows the relative positions of the base vectors $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_r$, and $\mathbf{a}_\phi$, we see that

$$\mathbf{a}_r \cdot \mathbf{a}_x = \cos\phi \tag{2–55}$$

and

$$\mathbf{a}_\phi \cdot \mathbf{a}_x = \cos\left(\frac{\pi}{2} + \phi\right) = -\sin\phi. \tag{2–56}$$

Hence,

$$A_x = A_r \cos\phi - A_\phi \sin\phi. \tag{2–57}$$

Similarly, to find A_y, we take the dot products of both expressions of $\mathbf{A}$ with $\mathbf{a}_y$:

$$\begin{aligned} A_y &= \mathbf{A} \cdot \mathbf{a}_y \\ &= A_r \mathbf{a}_r \cdot \mathbf{a}_y + A_\phi \mathbf{a}_\phi \cdot \mathbf{a}_y. \end{aligned}$$

From Fig. 2–16 we find that

$$\mathbf{a}_r \cdot \mathbf{a}_y = \cos\left(\frac{\pi}{2} - \phi\right) = \sin\phi \tag{2–58}$$

and

$$\mathbf{a}_\phi \cdot \mathbf{a}_y = \cos\phi. \tag{2–59}$$

It follows that

$$A_y = A_r \sin\phi + A_\phi \cos\phi. \tag{2–60}$$

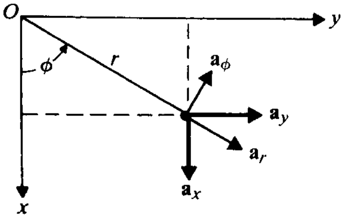


FIGURE 2–16
Relations between $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_r$, and $\mathbf{a}_\phi$.

It is convenient to write the relations between the components of a vector in Cartesian and cylindrical coordinates in a matrix form:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}. \tag{2-61}$$

Our problem is now solved except that the $\cos\phi$ and $\sin\phi$ in Eq. (2–61) should be converted into Cartesian coordinates. Moreover, A_r, A_ϕ, and A_z may themselves be functions of r, ϕ, and z. In that case, they too should be converted into functions of x, y, and z in the final answer. The following conversion formulas are obvious from Fig. 2–16. From cylindrical to Cartesian coordinates:

$$x = r\cos\phi, \tag{2-62a}$$
$$y = r\sin\phi, \tag{2-62b}$$
$$z = z. \tag{2-62c}$$

The inverse relations (from Cartesian to cylindrical coordinates) are

$$r = \sqrt{x^2 + y^2}, \tag{2-63a}$$
$$\phi = \tan^{-1}\frac{y}{x}, \tag{2-63b}$$
$$z = z. \tag{2-63c}$$

EXAMPLE 2–8 The cylindrical coordinates of an arbitrary point P in the $z = 0$ plane are $(r, \phi, 0)$. Find the unit vector that goes from a point $z = h$ on z-axis toward P.

Solution Referring to Fig. 2–17, we have

$$\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ}$$
$$= (\mathbf{a}_r r) - (\mathbf{a}_z h).$$

Hence,

$$\mathbf{a}_{QP} = \frac{\overrightarrow{QP}}{|\overrightarrow{QP}|} = \frac{1}{\sqrt{r^2 + h^2}}(\mathbf{a}_r r - \mathbf{a}_z h).$$

EXAMPLE 2–9 Express the vector

$$\mathbf{A} = \mathbf{a}_r(3\cos\phi) - \mathbf{a}_\phi 2r + \mathbf{a}_z 5$$

in Cartesian coordinates.

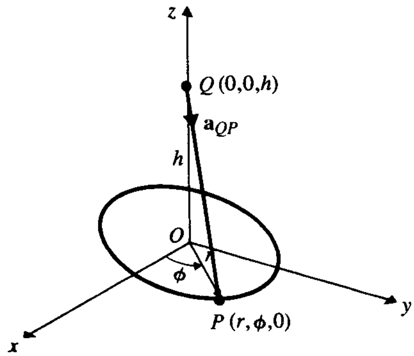


FIGURE 2–17
Illustrating Example 2–8.

Solution Using Eq. (2–61) directly, we have

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3\cos\phi \\ -2r \\ 5 \end{bmatrix}$$

or

$$\mathbf{A} = \mathbf{a}_x(3\cos^2\phi + 2r\sin\phi) + \mathbf{a}_y(3\sin\phi\cos\phi - 2r\cos\phi) + \mathbf{a}_z 5.$$

But, from Eqs. (2–62) and (2–63),

$$\cos\phi = \frac{x}{\sqrt{x^2 + y^2}}$$

and

$$\sin\phi = \frac{y}{\sqrt{x^2 + y^2}}.$$

Therefore,

$$\mathbf{A} = \mathbf{a}_x\left(\frac{3x^2}{x^2 + y^2} + 2y\right) + \mathbf{a}_y\left(\frac{3xy}{x^2 + y^2} - 2x\right) + \mathbf{a}_z 5,$$

which is the desired answer. ■

2–4.3 SPHERICAL COORDINATES

$$(u_1, u_2, u_3) = (R, \theta, \phi)$$

A point $P(R_1, \theta_1, \phi_1)$ in spherical coordinates is specified as the intersection of the following three surfaces: a spherical surface centered at the origin with a radius $R = R_1$; a right circular cone with its apex at the origin, its axis coinciding with the $+$ z-axis and having a half-angle $\theta = \theta_1$; and a half-plane containing the z-axis and making an angle $\phi = \phi_1$ with the xz-plane. *The base vector* $\mathbf{a}_R$ *at P is radial from the origin and is quite different from* $\mathbf{a}_r$ *in cylindrical coordinates, the latter being perpendicular to the z-axis.* The base vector $\mathbf{a}_\theta$ lies in the $\phi = \phi_1$ plane and is tangential to the

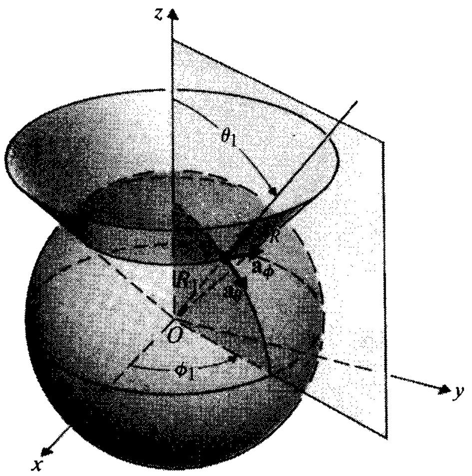


FIGURE 2–18
Spherical coordinates.

spherical surface, whereas the base vector $\mathbf{a}_\phi$ is the same as that in cylindrical coordinates. These are illustrated in Fig. 2–18. For a right-handed system we have

$$\mathbf{a}_R \times \mathbf{a}_\theta = \mathbf{a}_\phi, \tag{2–64a}$$

$$\mathbf{a}_\theta \times \mathbf{a}_\phi = \mathbf{a}_R, \tag{2–64b}$$

$$\mathbf{a}_\phi \times \mathbf{a}_R = \mathbf{a}_\theta. \tag{2–64c}$$

Spherical coordinates are important for problems involving point sources and regions with spherical boundaries. When an observer is very far from the source region of a finite extent, the latter could be considered as the origin of a spherical coordinate system; and, as a result, suitable simplifying approximations could be made. This is the reason that spherical coordinates are used in solving antenna problems in the far field.

A vector in spherical coordinates is written as

$$\boxed{\mathbf{A} = \mathbf{a}_R A_R + \mathbf{a}_\theta A_\theta + \mathbf{a}_\phi A_\phi.} \tag{2–65}$$

The expressions for the dot and cross products of two vectors in spherical coordinates can be obtained from Eqs. (2–26) and (2–27).

In spherical coordinates, only $R(u_1)$ is a length. The other two coordinates, θ and ϕ (u_2 and u_3), are angles. Referring to Fig. 2–19, in which a typical differential volume element is shown, we see that metric coefficients $h_2 = R$ and $h_3 = R \sin\theta$ are required to convert $d\theta$ and $d\phi$ into $d\ell_2$ and $d\ell_3$, respectively. The general expression for a differential length is, from Eq. (2–31),

$$\boxed{d\boldsymbol{\ell} = \mathbf{a}_R\, dR + \mathbf{a}_\theta R\, d\theta + \mathbf{a}_\phi R \sin\theta\, d\phi.} \tag{2–66}$$

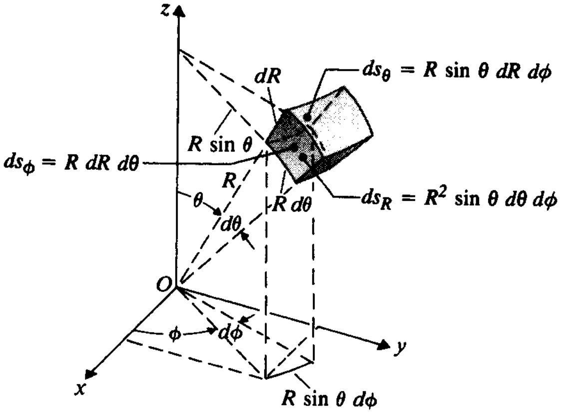


FIGURE 2–19
A differential volume element in spherical coordinates.

The expressions for differential areas and differential volume resulting from differential changes dR, $d\theta$, and $d\phi$ in the three coordinate directions are

$$ds_R = R^2 \sin\theta \, d\theta \, d\phi, \tag{2–67a}$$

$$ds_\theta = R \sin\theta \, dR \, d\phi, \tag{2–67b}$$

$$ds_\phi = R \, dR \, d\theta, \tag{2–67c}$$

and

$$dv = R^2 \sin\theta \, dR \, d\theta \, d\phi. \tag{2–68}$$

For convenience the base vectors, metric coefficients, and expressions for the differential volume are tabulated in Table 2–1.

TABLE 2–1
Three Basic Orthogonal Coordinate Systems

Coordinate System Relations		Cartesian Coordinates (x, y, z)	Cylindrical Coordinates (r, ϕ, z)	Spherical Coordinates (R, θ, ϕ)
Base vectors	$\mathbf{a}_{u_1}$	$\mathbf{a}_x$	$\mathbf{a}_r$	$\mathbf{a}_R$
	$\mathbf{a}_{u_2}$	$\mathbf{a}_y$	$\mathbf{a}_\phi$	$\mathbf{a}_\theta$
	$\mathbf{a}_{u_3}$	$\mathbf{a}_z$	$\mathbf{a}_z$	$\mathbf{a}_\phi$
Metric coefficients	h_1	1	1	1
	h_2	1	r	R
	h_3	1	1	$R \sin\theta$
Differential volume	dv	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2 \sin\theta \, dR \, d\theta \, d\phi$

A vector given in spherical coordinates can be transformed into one in Cartesian or cylindrical coordinates, and vice versa. From Fig. 2–19 it is easily seen that

$$x = R \sin\theta \cos\phi, \tag{2–69a}$$
$$y = R \sin\theta \sin\phi, \tag{2–69b}$$
$$z = R \cos\theta. \tag{2–69c}$$

Conversely, measurements in Cartesian coordinates can be transformed into those in spherical coordinates:

$$R = \sqrt{x^2 + y^2 + z^2}, \tag{2–70a}$$
$$\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z}, \tag{2–70b}$$
$$\phi = \tan^{-1} \frac{y}{x}. \tag{2–70c}$$

EXAMPLE 2–10 The position of a point P in spherical coordinates is (8, 120°, 330°). Specify its location (a) in Cartesian coordinates, and (b) in cylindrical coordinates.

Solution The spherical coordinates of the given point are $R = 8$, $\theta = 120°$, and $\phi = 330°$.

a) *In Cartesian coordinates.* We use Eqs. (2–69a, b, c):

$$x = 8 \sin 120° \cos 330° = 6,$$
$$y = 8 \sin 120° \sin 330° = -2\sqrt{3},$$
$$z = 8 \cos 120° = -4.$$

Hence the location of the point is $P(6, -2\sqrt{3}, -4)$, and the ***position vector*** (the vector going from the origin to the point) is

$$\overrightarrow{OP} = \mathbf{a}_x 6 - \mathbf{a}_y 2\sqrt{3} - \mathbf{a}_z 4.$$

b) *In cylindrical coordinates.* The cylindrical coordinates of point P can be obtained by applying Eqs. (2–63a, b, c) to the results in part (a), but they can be calculated directly from the given spherical coordinates by the following relations, which can be verified by comparing Figs. 2–14 and 2–18:

$$r = R \sin\theta, \tag{2–71a}$$
$$\phi = \phi, \tag{2–71b}$$
$$z = R \cos\theta. \tag{2–71c}$$

We have $P(4\sqrt{3}, 330°, -4)$; and its position vector in cylindrical coordinates is

$$\overrightarrow{OP} = \mathbf{a}_r 4\sqrt{3} - \mathbf{a}_z 4.$$

We note here that the position vector of a point in cylindrical coordinates does not contain the angle $\phi = 330°$ explicitly. However, the exact direction of $\mathbf{a}_r$ depends on ϕ. In terms of spherical coordinates the position vector (the vector from the origin to the point P) consists of only a single term:

$$\overrightarrow{OP} = \mathbf{a}_R 8.$$

Here the direction of $\mathbf{a}_R$ changes with the θ and ϕ coordinates of the point P.

EXAMPLE 2–11 Convert the vector $\mathbf{A} = \mathbf{a}_R A_R + \mathbf{a}_\theta A_\theta + \mathbf{a}_\phi A_\phi$ into Cartesian coordinates.

Solution In this problem we want to write $\mathbf{A}$ in the form of $\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z$. This is very different from the preceding problem of converting the coordinates of a point. First of all, we assume that the expression of the given vector $\mathbf{A}$ holds *for all points* of interest and that all three given components A_R, A_θ, and A_ϕ may be functions of coordinate variables. Second, at a given point, A_R, A_θ, and A_ϕ will have definite numerical values, but these values that determine the direction of $\mathbf{A}$ will, in general, be entirely different from the coordinate values of the point. Taking dot product of $\mathbf{A}$ with $\mathbf{a}_x$, we have

$$\begin{aligned} A_x &= \mathbf{A} \cdot \mathbf{a}_x \\ &= A_R \mathbf{a}_R \cdot \mathbf{a}_x + A_\theta \mathbf{a}_\theta \cdot \mathbf{a}_x + A_\phi \mathbf{a}_\phi \cdot \mathbf{a}_x. \end{aligned}$$

Recalling that $\mathbf{a}_R \cdot \mathbf{a}_x$, $\mathbf{a}_\theta \cdot \mathbf{a}_x$, and $\mathbf{a}_\phi \cdot \mathbf{a}_x$ yield, respectively, the component of unit vectors $\mathbf{a}_R$, $\mathbf{a}_\theta$, and $\mathbf{a}_\phi$ in the direction of $\mathbf{a}_x$, we find, from Fig. 2–19 and Eqs. (2–69a, b, c):

$$\mathbf{a}_R \cdot \mathbf{a}_x = \sin\theta\cos\phi = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \tag{2–72}$$

$$\mathbf{a}_\theta \cdot \mathbf{a}_x = \cos\theta\cos\phi = \frac{xz}{\sqrt{(x^2 + y^2)(x^2 + y^2 + z^2)}}, \tag{2–73}$$

$$\mathbf{a}_\phi \cdot \mathbf{a}_x = -\sin\phi = -\frac{y}{\sqrt{x^2 + y^2}}. \tag{2–74}$$

Thus,

$$\begin{aligned} A_x &= A_R \sin\theta\cos\phi + A_\theta \cos\theta\cos\phi - A_\phi \sin\phi \\ &= \frac{A_R x}{\sqrt{x^2 + y^2 + z^2}} + \frac{A_\theta xz}{\sqrt{(x^2 + y^2)(x^2 + y^2 + z^2)}} - \frac{A_\phi y}{\sqrt{x^2 + y^2}}. \end{aligned} \tag{2–75}$$

Similarly,

$$\begin{aligned} A_y &= A_R \sin\theta\sin\phi + A_\theta \cos\theta\sin\phi + A_\phi \cos\phi \\ &= \frac{A_R y}{\sqrt{x^2 + y^2 + z^2}} + \frac{A_\theta yz}{\sqrt{(x^2 + y^2)(x^2 + y^2 + z^2)}} + \frac{A_\phi x}{\sqrt{x^2 + y^2}}. \end{aligned} \tag{2–76}$$

and

$$A_z = A_R \cos\theta - A_\theta \sin\theta = \frac{A_R z}{\sqrt{x^2+y^2+z^2}} - \frac{A_\theta\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}. \quad (2\text{–}77)$$

If A_R, A_θ, and A_ϕ are themselves functions of R, θ, and ϕ, they too need to be converted into functions of x, y, and z by the use of Eqs. (2–70a, b, c). Equations (2–75), (2–76), and (2–77) disclose the fact that when a vector has a simple form in one coordinate system, its conversion into another coordinate system usually results in a more complicated expression.

EXAMPLE 2–12 Assuming that a cloud of electrons confined in a region between two spheres of radii 2 and 5 (cm) has a charge density of

$$\frac{-3\times 10^{-8}}{R^4}\cos^2\phi \qquad (\text{C/m}^3),$$

find the total charge contained in the region.

Solution We have

$$\rho = -\frac{3\times 10^{-8}}{R^4}\cos^2\phi,$$

$$Q = \int \rho\, dv.$$

The given conditions of the problem obviously point to the use of spherical coordinates. Using the expression for dv in Eq. (2–68), we perform a triple integration:

$$Q = \int_0^{2\pi}\int_0^{\pi}\int_{0.02}^{0.05} \rho R^2 \sin\theta\, dR\, d\theta\, d\phi.$$

Two things are of importance here. First, since ρ is given in units of coulombs per cubic meter, the limits of integration for R must be converted to meters. Second, the full range of integration for θ is from 0 to π radians, *not* from 0 to 2π radians. A little reflection will convince us that a half-circle (not a full-circle) rotated about the z-axis through 2π radians (ϕ from 0 to 2π) generates a sphere. We have

$$\begin{aligned}
Q &= -3\times 10^{-8}\int_0^{2\pi}\int_0^{\pi}\int_{0.02}^{0.05}\frac{1}{R^2}\cos^2\phi\sin\theta\, dR\, d\theta\, d\phi \\
&= -3\times 10^{-8}\int_0^{2\pi}\int_0^{\pi}\left(-\frac{1}{0.05}+\frac{1}{0.02}\right)\sin\theta\, d\theta\cos^2\phi\, d\phi \\
&= -0.9\times 10^{-6}\int_0^{2\pi}(-\cos\theta)\Big|_0^{\pi}\cos^2\phi\, d\phi \\
&= -1.8\times 10^{-6}\left(\frac{\phi}{2}+\frac{\sin 2\phi}{4}\right)\Big|_0^{2\pi} = -1.8\pi \qquad (\mu\text{C}).
\end{aligned}$$

2–5 Integrals Containing Vector Functions

In electromagnetics work we have occasion to encounter integrals that contain vector functions such as

$$\int_V \mathbf{F}\, dv, \tag{2–78}$$

$$\int_C V\, d\boldsymbol{\ell}, \tag{2–79}$$

$$\int_C \mathbf{F} \cdot d\boldsymbol{\ell}, \tag{2–80}$$

$$\int_S \mathbf{A} \cdot d\mathbf{s}. \tag{2–81}$$

The volume integral in (2–78) can be evaluated as the sum of three scalar integrals by first resolving the vector **F** into its three components in the appropriate coordinate system. If dv denotes a differential volume, then (2–78) is actually a shorthand way of representing a triple integral over three dimensions.

In the second integral, in (2–79), V is a scalar function of space, $d\boldsymbol{\ell}$ represents a differential increment of length, and C is the path of integration. If the integration is to be carried out from a point P_1 to another point P_2, we write $\int_{P_1}^{P_2} V\, d\boldsymbol{\ell}$. If the integration is to be evaluated around a closed path C, we denote it by $\oint_C V\, d\boldsymbol{\ell}$. In Cartesian coordinates, (2–79) can be written as

$$\int_C V\, d\boldsymbol{\ell} = \int_C V(x, y, z)[\mathbf{a}_x\, dx + \mathbf{a}_y\, dy + \mathbf{a}_z\, dz], \tag{2–82}$$

in view of Eq. (2–44). Since the Cartesian unit vectors are constant in both magnitude and direction, they can be taken out of the integral sign, and Eq. (2–82) becomes

$$\int_C V\, d\boldsymbol{\ell} = \mathbf{a}_x \int_C V(x, y, z)\, dx + \mathbf{a}_y \int_C V(x, y, z)\, dy + \mathbf{a}_z \int_C V(x, y, z)\, dz. \tag{2–83}$$

The three integrals on the right-hand side of Eq. (2–83) are ordinary scalar integrals; they can be evaluated for a given $V(x, y, z)$ around a path C.

EXAMPLE 2–13 Evaluate the integral $\int_O^P r^2\, d\mathbf{r}$, where $r^2 = x^2 + y^2$, from the origin to the point $P(1, 1)$: (a) along the direct path OP, (b) along the path OP_1P, and (c) along the path OP_2P in Fig. 2–20.

Solution

a) Along the direct path OP:

$$\int_O^P r^2\, d\mathbf{r} = \mathbf{a}_r \int_0^{\sqrt{2}} r^2\, dr = \mathbf{a}_r \frac{2\sqrt{2}}{3}$$

$$= \frac{2\sqrt{2}}{3}(\mathbf{a}_x \cos 45^\circ + \mathbf{a}_y \sin 45^\circ)$$

$$= \mathbf{a}_x \tfrac{2}{3} + \mathbf{a}_y \tfrac{2}{3}.$$

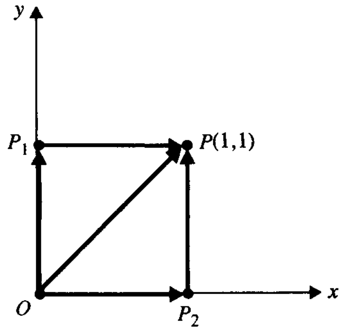


FIGURE 2–20
Illustrating Example 2–13.

b) Along the path OP_1P:

$$\int_O^P (x^2 + y^2)\,d\mathbf{r} = \mathbf{a}_y \int_O^{P_1} y^2\,dy + \mathbf{a}_x \int_{P_1}^P (x^2 + 1)\,dx$$
$$= \mathbf{a}_y \tfrac{1}{3}y^3\Big|_0^1 + \mathbf{a}_x(\tfrac{1}{3}x^3 + x)\Big|_0^1$$
$$= \mathbf{a}_x\tfrac{4}{3} + \mathbf{a}_y\tfrac{1}{3}.$$

c) Along the path OP_2P:

$$\int_O^P (x^2 + y^2)\,d\mathbf{r} = \mathbf{a}_x \int_O^{P_2} x^2\,dx + \mathbf{a}_y \int_{P_2}^P (1 + y^2)\,dy$$
$$= \mathbf{a}_x \tfrac{1}{3}x^3\Big|_0^1 + \mathbf{a}_y(y + \tfrac{1}{3}y^3)\Big|_0^1$$
$$= \mathbf{a}_x\tfrac{1}{3} + \mathbf{a}_y\tfrac{4}{3}.$$

Obviously, the value of the integral depends on the path of integration, since the results in parts (a), (b), and (c) are all different. ▬

The integrals in (2–80) and (2–81) are mathematically of the same form; they both lead to a scalar result. The expression in (2–80) is a line integral, in which the integrand represents the component of the vector **F** along the path of integration. This type of scalar line integral is of considerable importance in both physics and electromagnetics. (If **F** is a force, the integral is the work done by the force in moving an object from an initial point P_1 to a final point P_2 along a specified path C; if **F** is replaced by **E**, the electric field intensity, then the integral represents the work done by the electric field in moving a unit charge from P_1 to P_2.) We will encounter it again later in this chapter and in many other parts of this book.

EXAMPLE 2–14 Given $\mathbf{F} = \mathbf{a}_x xy - \mathbf{a}_y 2x$, evaluate the scalar line integral

$$\int_A^B \mathbf{F} \cdot d\boldsymbol{\ell}$$

along the quarter-circle shown in Fig. 2–21.

Solution We shall solve this problem in two ways: first in Cartesian coordinates, then in cylindrical coordinates.

a) *In Cartesian coordinates.* From the given **F** and the expression for $d\ell$ in Eq. (2–44) we have

$$\mathbf{F} \cdot d\ell = xy\,dx - 2x\,dy.$$

The equation of the quarter-circle is $x^2 + y^2 = 9$ $(0 \le x, y \le 3)$. Therefore,

$$\begin{aligned}\int_A^B \mathbf{F} \cdot d\ell &= \int_3^0 x\sqrt{9 - x^2}\,dx - 2\int_0^3 \sqrt{9 - y^2}\,dy \\ &= -\frac{1}{3}(9 - x^2)^{3/2}\Big|_3^0 - \left[y\sqrt{9 - y^2} + 9\sin^{-1}\frac{y}{3}\right]_0^3 \\ &= -9\left(1 + \frac{\pi}{2}\right).\end{aligned}$$

b) *In cylindrical coordinates.* Here we first transform **F** into cylindrical coordinates. Inverting Eq. (2–61), we have

$$\begin{aligned}\begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} &= \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \\ &= \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}.\end{aligned} \tag{2–84}$$

With the given **F**, Eq. (2–84) gives

$$\begin{bmatrix} F_r \\ F_\phi \\ F_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} xy \\ -2x \\ 0 \end{bmatrix},$$

which leads to

$$\mathbf{F} = \mathbf{a}_r(xy\cos\phi - 2x\sin\phi) - \mathbf{a}_\phi(xy\sin\phi + 2x\cos\phi).$$

For the present problem the path of integration is along a quarter-circle of a radius 3. There is no change in r or z along the path ($dr = 0$ and $dz = 0$); hence

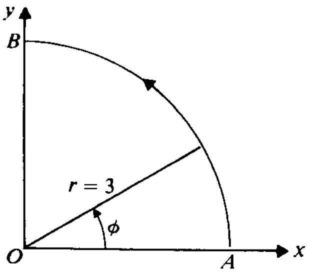


FIGURE 2–21
Path for line integral (Example 2–14).

Eq. (2–52) simplifies to

$$d\boldsymbol{\ell} = \mathbf{a}_\phi 3\, d\phi$$

and

$$\mathbf{F} \cdot d\boldsymbol{\ell} = -3(xy \sin \phi + 2x \cos \phi)\, d\phi.$$

Because of the circular path, F_r is immaterial to the present integration. Along the path, $x = 3 \cos \phi$ and $y = 3 \sin \phi$. Therefore

$$\begin{aligned} \int_A^B \mathbf{F} \cdot d\boldsymbol{\ell} &= \int_0^{\pi/2} -3(9 \sin^2 \phi \cos \phi + 6 \cos^2 \phi)\, d\phi \\ &= -9(\sin^3 \phi + \phi + \sin \phi \cos \phi)\Big|_0^{\pi/2} \\ &= -9\left(1 + \frac{\pi}{2}\right), \end{aligned}$$

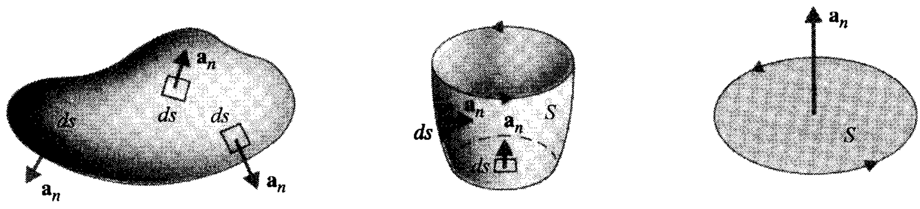
which is the same as before. ▬

In this particular example, **F** is given in Cartesian coordinates, and the path is circular. There is no compelling reason to solve the problem in one or the other coordinates. We have shown the conversion of vectors and the procedure of solution in both coordinates.

The expression in (2–81), $\int_S \mathbf{A} \cdot d\mathbf{s}$, is a surface integral. It is actually a double integral over two dimensions; but it is written with a single integral sign for simplicity. The integral measures the flux of the vector field **A** flowing through the area S. In the integral the vector differential surface element $d\mathbf{s} = \mathbf{a}_n\, ds$ has a magnitude ds and a direction denoted by the unit vector $\mathbf{a}_n$. The conventions for the positive direction of $d\mathbf{s}$ or $\mathbf{a}_n$ are as follows:

1. If the surface of integration, S, is a *closed surface* enclosing a volume, then the positive direction for $\mathbf{a}_n$ is always in the *outward* direction from the volume. This is illustrated in Fig. 2–22(a). We see that the positive direction of $\mathbf{a}_n$ depends on the location of ds. A small circle is added over the integral sign if the integration is to be performed over an enclosed surface:

$$\oint_S \mathbf{A} \cdot d\mathbf{s} = \oint_S \mathbf{A} \cdot \mathbf{a}_n\, ds.$$



(a) A closed surface. (b) An open surface. (c) A disk.

FIGURE 2–22
Illustrating the positive direction of $\mathbf{a}_n$ in scalar surface integral.

2. If S is an open surface, the positive direction for $\mathbf{a}_n$ depends on the direction in which the perimeter of the open surface is traversed. This is illustrated in Fig. 2–22(b), in which a cup-shaped surface (with no lid) is shown. We apply the right-hand rule: If the fingers of the right hand follows the direction of travel around the perimeter, then the thumb points in the direction of positive $\mathbf{a}_n$. Here again, the positive direction of $\mathbf{a}_n$ depends on the location of ds. A plane, such as the disk in Fig. 2–22(c), is a special case of an open surface where $\mathbf{a}_n$ is a constant.

EXAMPLE 2–15 Given $\mathbf{F} = \mathbf{a}_r k_1/r + \mathbf{a}_z k_2 z$, evaluate the scalar surface integral

$$\oint_S \mathbf{F} \cdot d\mathbf{s}$$

over the surface of a closed cylinder about the z-axis specified by $z = \pm 3$ and $r = 2$.

Solution The specified surface of integration S is that of a closed cylinder shown in Fig. 2–23. The cylinder has three surfaces: the top face, the bottom face, and the side wall. We write

$$\oint_S \mathbf{F} \cdot d\mathbf{s} = \oint_S \mathbf{F} \cdot \mathbf{a}_n \, ds$$

$$= \int_{\substack{\text{top}\\\text{face}}} \mathbf{F} \cdot \mathbf{a}_n \, ds + \int_{\substack{\text{bottom}\\\text{face}}} \mathbf{F} \cdot \mathbf{a}_n \, ds + \int_{\substack{\text{side}\\\text{wall}}} \mathbf{F} \cdot \mathbf{a}_n \, ds,$$

where $\mathbf{a}_n$ is the unit normal *outward* from the respective surfaces. The three integrals on the right side can be evaluated separately.

a) *Top face.* $z = 3$, $\mathbf{a}_n = \mathbf{a}_z$,
$\mathbf{F} \cdot \mathbf{a}_n = k_2 z = 3k_2$,
$ds = r\,dr\,d\phi$ (from Eq. 2–53c);

$$\int_{\substack{\text{top}\\\text{face}}} \mathbf{F} \cdot \mathbf{a}_n \, ds = \int_0^{2\pi} \int_0^2 3k_2 r \, dr \, d\phi = 12\pi k_2.$$

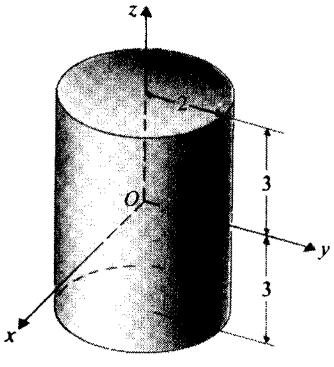


FIGURE 2–23
A cylindrical surface (Example 2–15).

b) *Bottom face.* $z = -3$, $\mathbf{a}_n = -\mathbf{a}_z$,
$\mathbf{F} \cdot \mathbf{a}_n = -k_2 z = 3k_2$,
$ds = r\,dr\,d\phi$;

$$\int_{\substack{\text{bottom}\\\text{face}}} \mathbf{F} \cdot \mathbf{a}_n\, ds = 12\pi k_2,$$

which is exactly the same as the integral over the top face.

c) *Side wall.* $r = 2$, $\mathbf{a}_n = \mathbf{a}_r$,

$$\mathbf{F} \cdot \mathbf{a}_n = \frac{k_1}{r} = \frac{k_1}{2},$$

$ds = r\,d\phi\,dz = 2\,d\phi\,dz$ (from Eq. 2–53a);

$$\int_{\substack{\text{side}\\\text{wall}}} \mathbf{F} \cdot \mathbf{a}_n\, ds = \int_{-3}^{3} \int_{0}^{2\pi} k_1\, d\phi\, dz = 12\pi k_1.$$

Therefore,

$$\oint_S \mathbf{F} \cdot d\mathbf{s} = 12\pi k_2 + 12\pi k_2 + 12\pi k_1$$
$$= 12\pi(k_1 + 2k_2).$$

This surface integral gives the net *outward flux* of the vector **F** through the closed cylindrical surface. ∎

2–6 Gradient of a Scalar Field

In electromagnetics we have to deal with quantities that depend on both time and position. Since three coordinate variables are involved in a three-dimensional space, we expect to encounter scalar and vector fields that are functions of four variables: (t, u_1, u_2, u_3). In general, the fields may change as any one of the four variables changes. We now address the method for describing the space rate of change of a scalar field at a given time. Partial derivatives with respect to the three space-coordinate variables are involved, and, inasmuch as the rate of change may be different in different directions, a vector is needed to define the space rate of change of a scalar field at a given point and at a given time.

Let us consider a scalar function of space coordinates $V(u_1, u_2, u_3)$, which may represent, say, the temperature distribution in a building, the altitude of a mountainous terrain, or the electric potential in a region. The magnitude of V, in general, depends on the position of the point in space, but it may be constant along certain lines or surfaces. Figure 2–24 shows two surfaces on which the magnitude of V is constant and has the values V_1 and $V_1 + dV$, respectively, where dV indicates a small change in V. We should note that constant-V surfaces need not coincide with any of the surfaces that define a particular coordinate system. Point P_1 is on surface V_1; P_2 is the corresponding point on surface $V_1 + dV$ along the normal vector $d\mathbf{n}$; and P_3 is a point close to P_2 along another vector $d\boldsymbol{\ell} \neq d\mathbf{n}$. For the same change dV in V, the space rate of change, $dV/d\ell$, is obviously greatest along $d\mathbf{n}$ because dn is the

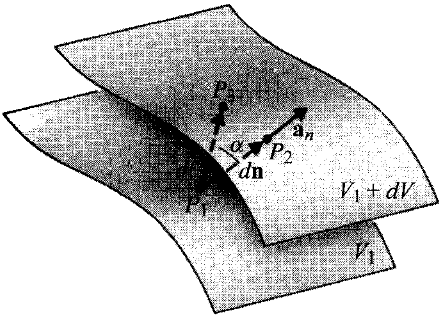


FIGURE 2–24
Concerning gradient of a scalar.

shortest distance between the two surfaces.† Since the magnitude of $dV/d\ell$ depends on the direction of $d\ell$, $dV/d\ell$ is a directional derivative. ***We define the vector that represents both the magnitude and the direction of the maximum space rate of increase of a scalar as the gradient of that scalar.*** We write

$$\boxed{\mathbf{grad}\, V \triangleq \mathbf{a}_n \frac{dV}{dn}.} \tag{2–85}$$

For brevity it is customary to employ the operator *del*, represented by the symbol ∇ and write ∇V in place of **grad** V. Thus,

$$\boxed{\nabla V \triangleq \mathbf{a}_n \frac{dV}{dn}.} \tag{2–86}$$

We have assumed that dV is positive (an increase in V); if dV is negative (a decrease in V from P_1 to P_2), ∇V will be negative in the $\mathbf{a}_n$ direction.

The directional derivative along $d\ell$ is

$$\begin{aligned} \frac{dV}{d\ell} &= \frac{dV}{dn}\frac{dn}{d\ell} = \frac{dV}{dn}\cos\alpha \\ &= \frac{dV}{dn}\mathbf{a}_n \cdot \mathbf{a}_\ell = (\nabla V)\cdot \mathbf{a}_\ell. \end{aligned} \tag{2–87}$$

Equation (2–87) states that the space rate of increase of V in the $\mathbf{a}_\ell$ direction is equal to the projection (the component) of the gradient of V in that direction. We can also write Eq. (2–87) as

$$\boxed{dV = (\nabla V)\cdot d\boldsymbol{\ell},} \tag{2–88}$$

† In a more formal treatment, changes ΔV and $\Delta\ell$ would be used, and the ratio $\Delta V/\Delta\ell$ would become the derivative $dV/d\ell$ as $\Delta\ell$ approaches zero. We avoid this formality in favor of simplicity.

where $d\boldsymbol{\ell} = \mathbf{a}_\ell d\ell$. Now, dV in Eq. (2–88) is the total differential of V as a result of a change in position (from P_1 to P_3 in Fig. 2–24); it can be expressed in terms of the differential changes in coordinates:

$$dV = \frac{\partial V}{\partial \ell_1} d\ell_1 + \frac{\partial V}{\partial \ell_2} d\ell_2 + \frac{\partial V}{\partial \ell_3} d\ell_3, \tag{2–89}$$

where $d\ell_1$, $d\ell_2$, and $d\ell_3$ are the components of the vector differential displacement $d\boldsymbol{\ell}$ in a chosen coordinate system. In terms of general orthogonal curvilinear coordinates (u_1, u_2, u_3), $d\boldsymbol{\ell}$ is (from Eq. 2–31),

$$\begin{aligned} d\boldsymbol{\ell} &= \mathbf{a}_{u_1} d\ell_1 + \mathbf{a}_{u_2} d\ell_2 + \mathbf{a}_{u_3} d\ell_3 \\ &= \mathbf{a}_{u_1}(h_1\, du_1) + \mathbf{a}_{u_2}(h_2\, du_2) + \mathbf{a}_{u_3}(h_3\, du_3). \end{aligned} \tag{2–90}$$

We can write dV in Eq. (2–89) as the dot product of two vectors, as follows:

$$\begin{aligned} dV &= \left(\mathbf{a}_{u_1} \frac{\partial V}{\partial \ell_1} + \mathbf{a}_{u_2} \frac{\partial V}{d\ell_2} + \mathbf{a}_{u_3} \frac{\partial V}{\partial \ell_3}\right) \cdot (\mathbf{a}_{u_1} d\ell_1 + \mathbf{a}_{u_2} d\ell_2 + \mathbf{a}_{u_3} d\ell_3) \\ &= \left(\mathbf{a}_{u_1} \frac{\partial V}{\partial \ell_1} + \mathbf{a}_{u_2} \frac{\partial V}{\partial \ell_2} + \mathbf{a}_{u_3} \frac{\partial V}{\partial \ell_3}\right) \cdot d\boldsymbol{\ell}. \end{aligned} \tag{2–91}$$

Comparing Eq. (2–91) with Eq. (2–88), we obtain

$$\boldsymbol{\nabla} V = \mathbf{a}_{u_1} \frac{\partial V}{\partial \ell_1} + \mathbf{a}_{u_2} \frac{\partial V}{\partial \ell_2} + \mathbf{a}_{u_3} \frac{\partial V}{d\ell_3} \tag{2–92}$$

or

$$\boxed{\boldsymbol{\nabla} V = \mathbf{a}_{u_1} \frac{\partial V}{h_1\, \partial u_1} + \mathbf{a}_{u_2} \frac{\partial V}{h_2\, \partial u_2} + \mathbf{a}_{u_3} \frac{\partial V}{h_3\, \partial u_3}.} \tag{2–93}$$

Equation (2–93) is a useful formula for computing the gradient of a scalar, when the scalar is given as a function of space coordinates.

In Cartesian coordinates, $(u_1, u_2, u_3) = (x, y, z)$ and $h_1 = h_2 = h_3 = 1$, we have

$$\boxed{\boldsymbol{\nabla} V = \mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}} \tag{2–94}$$

or

$$\boldsymbol{\nabla} V = \left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}\right) V. \tag{2–95}$$

In view of Eq. (2–95), it is convenient to consider $\boldsymbol{\nabla}$ *in Cartesian coordinates* as a vector differential operator.

$$\boxed{\boldsymbol{\nabla} \equiv \mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}.} \tag{2–96}$$

From Eq. (2–93), we see that we can define ∇ as

$$\nabla \equiv \left(\mathbf{a}_{u_1}\frac{\partial}{h_1\,\partial u_1} + \mathbf{a}_{u_2}\frac{\partial}{h_2\,\partial u_2} + \mathbf{a}_{u_3}\frac{\partial}{h_3\,\partial u_3}\right) \tag{2–97}$$

in general orthogonal coordinates. As we shall see later in this chapter, the same vector differential operator is also used to signify *divergence* ($\nabla\cdot$) and *curl* ($\nabla\times$) operations on a vector. In these cases it is important to remember that the differentiation of a base vector in a curvilinear coordinate system may lead to a new vector in a different direction. (For instance, $\partial\mathbf{a}_r/\partial\phi = \mathbf{a}_\phi$ and $\partial\mathbf{a}_\phi/\partial\phi = -\mathbf{a}_r$.) Proper care must be exercised when the ∇ defined in Eq. (2–97) is used to operate on vectors in curvilinear coordinate systems.

EXAMPLE 2–16 The electrostatic field intensity $\mathbf{E}$ is derivable as the negative gradient of a scalar electric potential V; that is, $\mathbf{E} = -\nabla V$. Determine $\mathbf{E}$ at the point (1, 1, 0) if

a) $V = V_0 e^{-x}\sin\dfrac{\pi y}{4}$,

b) $V = E_0 R\cos\theta$.

Solution We use Eq. (2–93) to evaluate $\mathbf{E} = -\nabla V$ in Cartesian coordinates for part (a) and in spherical coordinates for part (b).

a) $$\mathbf{E} = -\left[\mathbf{a}_x\frac{\partial}{\partial x} + \mathbf{a}_y\frac{\partial}{\partial y} + \mathbf{a}_z\frac{\partial}{\partial z}\right]E_0 e^{-x}\sin\frac{\pi y}{4}$$
$$= \left(\mathbf{a}_x\sin\frac{\pi y}{4} - \mathbf{a}_y\frac{\pi}{4}\cos\frac{\pi y}{4}\right)E_0 e^{-x}.$$

Thus, $\mathbf{E}(1, 1, 0) = \left(\mathbf{a}_x - \mathbf{a}_y\dfrac{\pi}{4}\right)\dfrac{E_0}{\sqrt{2}} = \mathbf{a}_E E,$

where

$$E = E_0\sqrt{\frac{1}{2}\left(1 + \frac{\pi^2}{16}\right)},$$
$$\mathbf{a}_E = \frac{1}{\sqrt{1 + (\pi^2/16)}}\left(\mathbf{a}_x - \mathbf{a}_y\frac{\pi}{4}\right).$$

b) $$\mathbf{E} = -\left[\mathbf{a}_R\frac{\partial}{\partial R} + \mathbf{a}_\theta\frac{\partial}{R\partial\theta} + \mathbf{a}_\phi\frac{\partial}{R\sin\theta\,\partial\phi}\right]E_0 R\cos\theta$$
$$= -(\mathbf{a}_R\cos\theta - \mathbf{a}_\theta\sin\theta)E_0.$$

In view of Eq. (2–77), the result above converts very simply to $\mathbf{E} = -\mathbf{a}_z E_0$ in Cartesian coordinates. This is not surprising, since a careful examination of the given V reveals that $E_0 R\cos\theta$ is, in fact, equal to $E_0 z$. In Cartesian coordinates,

$$\mathbf{E} = -\nabla V = -\mathbf{a}_z\frac{\partial}{\partial z}(E_0 z) = -\mathbf{a}_z E_0.$$

2–7 Divergence of a Vector Field

In the preceding section we considered the spatial derivatives of a scalar field, which led to the definition of the gradient. We now turn our attention to the spatial derivatives of a vector field. This will lead to the definitions of the ***divergence*** and the ***curl*** of a vector. We discuss the meaning of divergence in this section and that of curl in Section 2–9. Both are very important in the study of electromagnetism.

In the study of vector fields it is convenient to represent field variations graphically by directed field lines, which are called ***flux lines*** or ***streamlines***. They are directed lines or curves that indicate at each point the direction of the vector field, as illustrated in Fig. 2–25. The magnitude of the field at a point is depicted either by the density or by the length of the directed lines in the vicinity of the point. Figure 2–25(a) shows that the field in region A is stronger than that in region B because there is a higher density of equal-length directed lines in region A. In Fig. 2–25(b), the decreasing arrow lengths away from the point q indicate a radial field that is strongest in the region closest to q. Figure 2–25(c) depicts a uniform field.

The vector field strength in Fig. 2–25(a) is measured by the number of flux lines passing through a unit surface normal to the vector. The flux of a vector field is analogous to the flow of an incompressible fluid such as water. For a volume with an enclosed surface there will be an excess of outward or inward flow through the surface only when the volume contains a source or a sink, respectively; that is, a net positive divergence indicates the presence of a source of fluid inside the volume, and a net negative divergence indicates the presence of a sink. The net outward flow of the fluid per unit volume is therefore a measure of the strength of the enclosed source. In the uniform field shown in Fig. 2–25(c) there is an equal amount of inward and outward flux going through any closed volume containing no sources or sinks, resulting in a zero divergence.

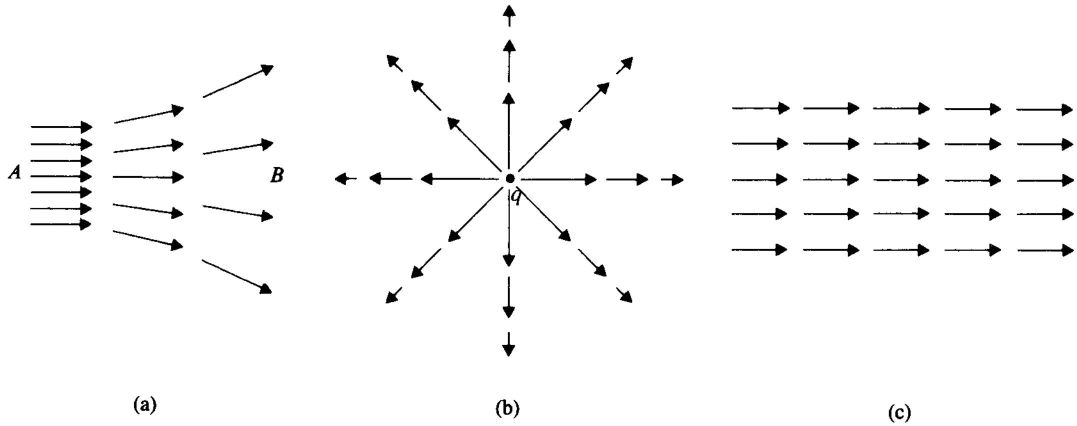


FIGURE 2–25
Flux lines of vector fields.

We define the divergence of a vector field* A *at a point, abbreviated div* A*, as the net outward flux of* A *per unit volume as the volume about the point tends to zero:

$$\boxed{\operatorname{div} \mathbf{A} \triangleq \lim_{\Delta v \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{s}}{\Delta v}.} \qquad (2\text{–}98)$$

The numerator in Eq. (2–98), representing the net outward flux, is an integral over the *entire* surface S that bounds the volume. We were exposed to this type of surface integral in Example 2–15. Equation (2–98) is the general definition of div **A** which is a *scalar quantity* whose magnitude may vary from point to point as **A** itself varies. This definition holds for any coordinate system; the expression for div **A**, like that for **A**, will, of course, depend on the choice of the coordinate system.

At the beginning of this section we intimated that the divergence of a vector is a type of spatial derivative. The reader might perhaps wonder about the presence of an integral in the expression given by Eq. (2–98); but a two-dimensional surface integral divided by a three-dimensional volume will lead to spatial derivatives as the volume approaches zero. We shall now derive the expression for div **A** in Cartesian coordinates.

Consider a differential volume of sides Δx, Δy, and Δz centered about a point $P(x_0, y_0, z_0)$ in the field of a vector **A**, as shown in Fig. 2–26. In Cartesian coordinates, $\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z$. We wish to find div **A** at the point (x_0, y_0, z_0). Since the differential volume has six faces, the surface integral in the numerator of Eq. (2–98) can be decomposed into six parts:

$$\oint_S \mathbf{A} \cdot d\mathbf{s} = \left[\int_{\substack{\text{front}\\\text{face}}} + \int_{\substack{\text{back}\\\text{face}}} + \int_{\substack{\text{right}\\\text{face}}} + \int_{\substack{\text{left}\\\text{face}}} + \int_{\substack{\text{top}\\\text{face}}} + \int_{\substack{\text{bottom}\\\text{face}}}\right] \mathbf{A} \cdot d\mathbf{s}. \qquad (2\text{–}99)$$

On the front face,

$$\begin{aligned}\int_{\substack{\text{front}\\\text{face}}} \mathbf{A} \cdot d\mathbf{s} &= \mathbf{A}_{\substack{\text{front}\\\text{face}}} \cdot \Delta\mathbf{s}_{\substack{\text{front}\\\text{face}}} = \mathbf{A}_{\substack{\text{front}\\\text{face}}} \cdot \mathbf{a}_x(\Delta y\, \Delta z)\\ &= A_x\left(x_0 + \frac{\Delta x}{2}, y_0, z_0\right) \Delta y\, \Delta z.\end{aligned} \qquad (2\text{–}100)$$

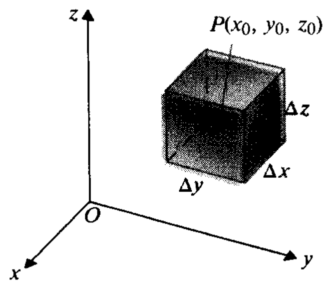


FIGURE 2–26
A differential volume in Cartesian coordinates.

The quantity $A_x([x_0 + (\Delta x/2), y_0, z_0])$ can be expanded as a Taylor series about its value at (x_0, y_0, z_0), as follows:

$$A_x\left(x_0 + \frac{\Delta x}{2}, y_0, z_0\right) = A_x(x_0, y_0, z_0) + \frac{\Delta x}{2}\frac{\partial A_x}{\partial x}\bigg|_{(x_0,\, y_0,\, z_0)} + \text{higher-order terms,} \tag{2-101}$$

where the higher-order terms (H.O.T.) contain the factors $(\Delta x/2)^2$, $(\Delta x/2)^3$, etc. Similarly, on the back face,

$$\begin{aligned}\int_{\substack{\text{back}\\ \text{face}}} \mathbf{A}\cdot d\mathbf{s} &= \mathbf{A}_{\substack{\text{back}\\ \text{face}}}\cdot \Delta\mathbf{s}_{\substack{\text{back}\\ \text{face}}} = \mathbf{A}_{\substack{\text{back}\\ \text{face}}}\cdot(-\mathbf{a}_x\,\Delta y\,\Delta z)\\ &= -A_x\left(x_0 - \frac{\Delta x}{2}, y_0, z_0\right)\Delta y\,\Delta z.\end{aligned} \tag{2-102}$$

The Taylor-series expansion of $A_x\left(x_0 - \frac{\Delta x}{2}, y_0, z_0\right)$ is

$$A_x\left(x_0 - \frac{\Delta x}{2}, y_0, z_0\right) = A_x(x_0, y_0, z_0) - \frac{\Delta x}{2}\frac{\partial A_x}{\partial x}\bigg|_{(x_0,\, y_0,\, z_0)} + \text{H.O.T.} \tag{2-103}$$

Substituting Eq. (2–101) in Eq. (2–100) and Eq. (2–103) in Eq. (2–102) and adding the contributions, we have

$$\left[\int_{\substack{\text{front}\\ \text{face}}} + \int_{\substack{\text{back}\\ \text{face}}}\right]\mathbf{A}\cdot d\mathbf{s} = \left(\frac{\partial A_x}{\partial x} + \text{H.O.T.}\right)\bigg|_{(x_0,\, y_0,\, z_0)} \Delta x\,\Delta y\,\Delta z. \tag{2-104}$$

Here a Δx has been factored out from the H.O.T. in Eqs. (2–101) and (2–103), but all terms of the H.O.T. in Eq. (2–104) still contain powers of Δx.

Following the same procedure for the right and left faces, where the coordinate changes are $+\Delta y/2$ and $-\Delta y/2$, respectively, and $\Delta s = \Delta x\,\Delta z$, we find

$$\left[\int_{\substack{\text{right}\\ \text{face}}} + \int_{\substack{\text{left}\\ \text{face}}}\right]\mathbf{A}\cdot d\mathbf{s} = \left(\frac{\partial A_y}{\partial y} + \text{H.O.T.}\right)\bigg|_{(x_0,\, y_0,\, z_0)} \Delta x\,\Delta y\,\Delta z. \tag{2-105}$$

Here the higher-order terms contain the factors Δy, $(\Delta y)^2$, etc. For the top and bottom faces we have

$$\left[\int_{\substack{\text{top}\\ \text{face}}} + \int_{\substack{\text{bottom}\\ \text{face}}}\right]\mathbf{A}\cdot d\mathbf{s} = \left(\frac{\partial A_z}{\partial z} + \text{H.O.T.}\right)\bigg|_{(x_0,\, y_0,\, z_0)} \Delta x\,\Delta y\,\Delta z, \tag{2-106}$$

where the higher-order terms contain the factors Δz, $(\Delta z)^2$, etc. Now the results from Eqs. (2–104), (2–105), and (2–106) are combined in Eq. (2–99) to obtain

$$\begin{aligned}\oint_S \mathbf{A}\cdot d\mathbf{s} = &\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\bigg|_{(x_0,\, y_0,\, z_0)} \Delta x\,\Delta y\,\Delta z\\ &+ \text{higher-order terms in } \Delta x, \Delta y, \Delta z.\end{aligned} \tag{2-107}$$

Since $\Delta v = \Delta x\,\Delta y\,\Delta z$, substitution of Eq. (2–107) in Eq. (2–98) yields the expression of div **A** in Cartesian coordinates:

$$\boxed{\text{div } \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.} \tag{2–108}$$

The higher-order terms vanish as the differential volume $\Delta x \Delta y \Delta z$ approaches zero. The value of div **A**, in general, depends on the position of the point at which it is evaluated. We have dropped the notation (x_0, y_0, z_0) in Eq. (2–108) because it applies to any point at which **A** and its partial derivatives are defined.

With the vector differential operator del, ∇, defined in Eq. (2–96) we can write Eq. (2–108) alternatively as $\nabla \cdot \mathbf{A}$; that is,

$$\boxed{\nabla \cdot \mathbf{A} \equiv \text{div } \mathbf{A}.} \tag{2–109}$$

In general orthogonal curvilinear coordinates (u_1, u_2, u_3), Eq. (2–98) will lead to

$$\boxed{\nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_1 h_3 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right].} \tag{2–110}$$

EXAMPLE 2–17 Find the divergence of the position vector to an arbitrary point.

Solution We will find the solution in Cartesian as well as in spherical coordinates.

a) *Cartesian coordinates.* The expression for the position vector to an arbitrary point (x, y, z) is

$$\overrightarrow{OP} = \mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z. \tag{2–111}$$

Using Eq. (2–108), we have

$$\nabla \cdot (\overrightarrow{OP}) = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$

b) *Spherical coordinates.* Here the position vector is simply

$$\overrightarrow{OP} = \mathbf{a}_R R. \tag{2–112}$$

Its divergence in spherical coordinates (R, θ, ϕ) can be obtained from Eq. (2–110) by using Table 2–1 as follows:

$$\boxed{\nabla \cdot \mathbf{A} = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R \sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{R\sin\theta}\frac{\partial A_\phi}{\partial \phi}.} \tag{2–113}$$

Substituting Eq. (2–112) in Eq. (2–113), we also obtain $\nabla \cdot (\overrightarrow{OP}) = 3$, as expected.

EXAMPLE 2–18 The magnetic flux density **B** outside a very long current-carrying wire is circumferential and is inversely proportional to the distance to the axis of the wire. Find $\nabla \cdot \mathbf{B}$.

Solution Let the long wire be coincident with the z-axis in a cylindrical coordinate system. The problem states that

$$\mathbf{B} = \mathbf{a}_\phi \frac{k}{r}.$$

The divergence of a vector field in cylindrical coordinates (r, ϕ, z) can be found from Eq. (2–110):

$$\boxed{\nabla \cdot \mathbf{B} = \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z}.} \qquad (2\text{–}114)$$

Now $B_\phi = k/r$, and $B_r = B_z = 0$. Equation (2–114) gives

$$\nabla \cdot \mathbf{B} = 0.$$

We have here a vector that is not a constant, but whose divergence is zero. This property indicates that the magnetic flux lines close upon themselves and that there are no magnetic sources or sinks. A divergenceless field is called a ***solenoidal field.*** More will be said about this type of field later in the book.

2–8 Divergence Theorem

In the preceding section we defined the divergence of a vector field as the net outward flux per unit volume. We may expect intuitively that ***the volume integral of the divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume***; that is,

$$\boxed{\int_V \nabla \cdot \mathbf{A}\, dv = \oint_S \mathbf{A} \cdot d\mathbf{s}.} \qquad (2\text{–}115)$$

This identity, which will be proved in the following paragraph, is called the ***divergence theorem.***† It applies to any volume V that is bounded by surface S. The direction of $d\mathbf{s}$ is always that of the *outward normal*, perpendicular to the surface ds and directed away from the volume.

For a very small differential volume element Δv_j bounded by a surface s_j, the definition of $\nabla \cdot \mathbf{A}$ in Eq. (2–98) gives directly

$$(\nabla \cdot \mathbf{A})_j \Delta v_j = \oint_{s_j} \mathbf{A} \cdot d\mathbf{s}. \qquad (2\text{–}116)$$

† It is also known as ***Gauss's theorem.***

In case of an arbitrary volume V, we can subdivide it into many, say N, small differential volumes, of which Δv_j is typical. This is depicted in Fig. 2-27. Let us now combine the contributions of all these differential volumes to both sides of Eq. (2-116). We have

$$\lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} (\nabla \cdot \mathbf{A})_j \Delta v_j \right] = \lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} \oint_{S_j} \mathbf{A} \cdot d\mathbf{s} \right]. \tag{2-117}$$

The left side of Eq. (2-117) is, by definition, the volume integral of $\nabla \cdot \mathbf{A}$:

$$\lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} (\nabla \cdot \mathbf{A})_j \Delta v_j \right] = \int_V (\nabla \cdot \mathbf{A})\, dv. \tag{2-118}$$

The surface integrals on the right side of Eq. (2-117) are summed over all the faces of all the differential volume elements. The contributions from the internal surfaces of adjacent elements will, however, cancel each other, because at a common internal surface the outward normals of the adjacent elements point in opposite directions. Hence the net contribution of the right side of Eq. (2-117) is due only to that of the external surface S bounding the volume V; that is,

$$\lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} \int_{S_j} \mathbf{A} \cdot d\mathbf{s} \right] = \oint_S \mathbf{A} \cdot d\mathbf{s}. \tag{2-119}$$

The substitution of Eqs. (2-118) and (2-119) in Eq. (2-117) yields the divergence theorem in Eq. (2-115).

The validity of the limiting processes leading to the proof of the divergence theorem requires that the vector field **A**, as well as its first derivatives, exist and be continuous both in V and on S. The divergence theorem is an important identity in vector analysis. *It converts a volume integral of the divergence of a vector to a closed surface integral of the vector, and vice versa.* We use it frequently in establishing other theorems and relations in electromagnetics. We emphasize that, although a single integral sign is used on both sides of Eq. (2-115) for simplicity, the volume and surface integrals represent triple and double integrations, respectively.

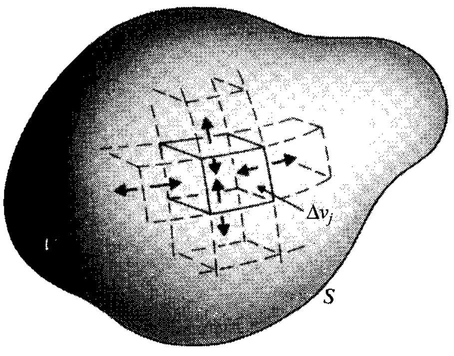


FIGURE 2-27
Subdivided volume for proof of divergence theorem.

EXAMPLE 2–19 Given $\mathbf{A} = \mathbf{a}_x x^2 + \mathbf{a}_y xy + \mathbf{a}_z yz$, verify the divergence theorem over a cube one unit on each side. The cube is situated in the first octant of the Cartesian coordinate system with one corner at the origin.

Solution Refer to Fig. 2–28. We first evaluate the surface integral over the six faces.

1. Front face: $x = 1$, $d\mathbf{s} = \mathbf{a}_x\, dy\, dz$;
$$\int_{\substack{\text{front}\\\text{face}}} \mathbf{A} \cdot d\mathbf{s} = \int_0^1 \int_0^1 dy\, dz = 1.$$
2. Back face: $x = 0$, $d\mathbf{s} = -\mathbf{a}_x\, dy\, dz$;
$$\int_{\substack{\text{back}\\\text{face}}} \mathbf{A} \cdot d\mathbf{s} = 0.$$
3. Left face: $y = 0$, $d\mathbf{s} = -\mathbf{a}_y\, dx\, dz$;
$$\int_{\substack{\text{left}\\\text{face}}} \mathbf{A} \cdot d\mathbf{s} = 0.$$
4. Right face: $y = 1$, $d\mathbf{s} = \mathbf{a}_y\, dx\, dz$;
$$\int_{\substack{\text{right}\\\text{face}}} \mathbf{A} \cdot d\mathbf{s} = \int_0^1 \int_0^1 x\, dx\, dz = \tfrac{1}{2}.$$
5. Top face: $z = 1$, $d\mathbf{s} = \mathbf{a}_z\, dx\, dy$;
$$\int_{\substack{\text{top}\\\text{face}}} \mathbf{A} \cdot d\mathbf{s} = \int_0^1 \int_0^1 y\, dx\, dy = \tfrac{1}{2}.$$
6. Bottom face: $z = 0$, $d\mathbf{s} = -\mathbf{a}_z\, dx\, dy$;
$$\int_{\substack{\text{bottom}\\\text{face}}} \mathbf{A} \cdot d\mathbf{s} = 0.$$

Adding the above six values, we have

$$\oint_S \mathbf{A} \cdot d\mathbf{s} = 1 + 0 + 0 + \tfrac{1}{2} + \tfrac{1}{2} + 0 = 2. \tag{2–120}$$

Now the divergence of $\mathbf{A}$ is

$$\nabla \cdot \mathbf{A} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial z}(yz) = 3x + y.$$

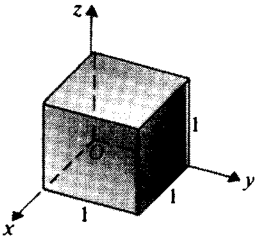


FIGURE 2–28
A unit cube (Example 2–19).

Hence,

$$\int_V \nabla \cdot \mathbf{A}\, dv = \int_0^1 \int_0^1 \int_0^1 (3x + y)\, dx\, dy\, dz = 2, \tag{2–121}$$

which is the same as the result of the closed surface integral in (2–120). The divergence theorem is therefore verified. ■

EXAMPLE 2–20 Given $\mathbf{F} = \mathbf{a}_R kR$, determine whether the divergence theorem holds for the shell region enclosed by spherical surfaces at $R = R_1$ and $R = R_2 (R_2 > R_1)$ centered at the origin, as shown in Fig. 2–29.

Solution Here the specified region has two surfaces, at $R = R_1$ and $R = R_2$.

At the outer surface: $R = R_2$, $d\mathbf{s} = \mathbf{a}_R R_2^2 \sin\theta\, d\theta\, d\phi$;

$$\int_{\substack{\text{outer}\\ \text{surface}}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \int_0^{\pi} (kR_2) R_2^2 \sin\theta\, d\theta\, d\phi = 4\pi k R_2^3.$$

At the inner surface: $R = R_1$, $d\mathbf{s} = -\mathbf{a}_R R_1^2 \sin\theta\, d\theta\, d\phi$;

$$\int_{\substack{\text{inner}\\ \text{surface}}} \mathbf{F} \cdot d\mathbf{s} = -\int_0^{2\pi} \int_0^{\pi} (kR_1) R_1^2 \sin\theta\, d\theta\, d\phi = -4\pi k R_1^3.$$

Actually, since the integrand is independent of θ or ϕ in both cases, the integral of a constant over a spherical surface is simply the constant multiplied by the area of the surface ($4\pi R_2^2$ for the outer surface and $4\pi R_1^2$ for the inner surface), and no integration is necessary. Adding the two results, we have

$$\oint_S \mathbf{F} \cdot d\mathbf{s} = 4\pi k (R_2^3 - R_1^3). \tag{2–122}$$

To find the volume integral, we first determine $\nabla \cdot \mathbf{F}$ for an $\mathbf{F}$ that has only an F_R component. From Eq. (2–113), we have

$$\nabla \cdot \mathbf{F} = \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 F_R) = \frac{1}{R^2} \frac{\partial}{\partial R} (kR^3) = 3k.$$

Since $\nabla \cdot \mathbf{F}$ is a constant, its volume integral equals the product of the constant and the volume. The volume of the shell region between the two spherical surfaces with

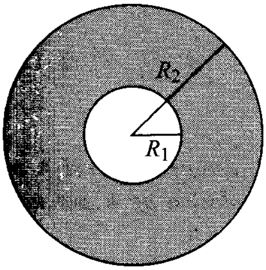


FIGURE 2–29
A spherical shell region (Example 2–20).

radii R_1 and R_2 is $4\pi(R_2^3 - R_1^3)/3$. Therefore,

$$\int_V \nabla \cdot \mathbf{F}\, dv = (\nabla \cdot \mathbf{F})V = 4\pi k(R_2^3 - R_1^3), \tag{2-123}$$

which is the same as the result in Eq. (2–122).

This example shows that the divergence theorem holds even when the volume has holes inside—that is, even when the volume is enclosed by a multiply connected surface. ▬

2–9 Curl of a Vector Field

In Section 2–7 we stated that a net outward flux of a vector **A** through a surface bounding a volume indicates the presence of a source. This source may be called a ***flow source***, and div **A** is a measure of the strength of the flow source. There is another kind of source, called ***vortex source***, which causes a circulation of a vector field around it. The ***net circulation*** (or simply ***circulation***) of a vector field around a *closed path* is defined as the scalar line integral of the vector over the path. We have

$$\text{Circulation of } \mathbf{A} \text{ around contour } C \triangleq \oint_C \mathbf{A} \cdot d\boldsymbol{\ell}. \tag{2-124}$$

Equation (2–124) is a mathematical definition. The physical meaning of circulation depends on what kind of field the vector **A** represents. If **A** is a force acting on an object, its circulation will be the work done by the force in moving the object once around the contour; if **A** represents an electric field intensity, then the circulation will be an electromotive force around the closed path, as we shall see later in the book. The familiar phenomenon of water whirling down a sink drain is an example of a *vortex sink* causing a circulation of fluid velocity. A circulation of **A** may exist even when div **A** = 0 (when there is no flow source).

Since circulation as defined in Eq. (2–124) is a line integral of a dot product, its value obviously depends on the orientation of the contour C relative to the vector **A**. In order to define a point function, which is a measure of the strength of a vortex source, we must make C very small and orient it in such a way that the circulation is a maximum. We define[†]

$$\boxed{\begin{aligned} \text{curl } \mathbf{A} &\equiv \nabla \times \mathbf{A} \\ &\triangleq \lim_{\Delta s \to 0} \frac{1}{\Delta s}\left[\mathbf{a}_n \oint_C \mathbf{A} \cdot d\boldsymbol{\ell}\right]_{\max}. \end{aligned}} \tag{2-125}$$

In words, Eq. (2–125) states that ***the curl of a vector field* A, *denoted by curl* A *or* ∇ × A, *is a vector whose magnitude is the maximum net circulation of* A *per unit***

[†] In books published in Europe, the curl of **A** is often called the rotation of **A** and written as rot **A**.

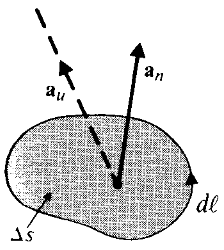


FIGURE 2–30
Relation between $\mathbf{a}_n$ and $d\ell$ in defining curl.

area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented to make the net circulation maximum. Because the normal to an area can point in two opposite directions, we adhere to the right-hand rule that when the fingers of the right hand follow the direction of $d\ell$, the thumb points to the $\mathbf{a}_n$ direction. This is illustrated in Fig. 2–30. Curl **A** is a vector point function and is conventionally written as $\nabla \times \mathbf{A}$ (del cross **A**). The component of $\nabla \times \mathbf{A}$ in any other direction $\mathbf{a}_u$ is $\mathbf{a}_u \cdot (\nabla \times \mathbf{A})$, which can be determined from the circulation per unit area normal to $\mathbf{a}_u$ as the area approaches zero.

$$\boxed{(\nabla \times \mathbf{A})_u = \mathbf{a}_u \cdot (\nabla \times \mathbf{A}) = \lim_{\Delta s_u \to 0} \frac{1}{\Delta s_u}\left(\oint_{C_u} \mathbf{A} \cdot d\ell\right),} \qquad (2\text{–}126)$$

where the direction of the line integration around the contour C_u bounding area Δs_u and the direction $\mathbf{a}_u$ follow the right-hand rule.

We now use Eq. (2–126) to find the three components of $\nabla \times \mathbf{A}$ in Cartesian coordinates. Refer to Fig. 2–31, in which a differential rectangular area parallel to the yz-plane and having sides Δy and Δz is drawn about a typical point $P(x_0, y_0, z_0)$. We have $\mathbf{a}_u = \mathbf{a}_x$ and $\Delta s_u = \Delta y\, \Delta z$, and the contour C_u consists of the four sides 1, 2, 3,

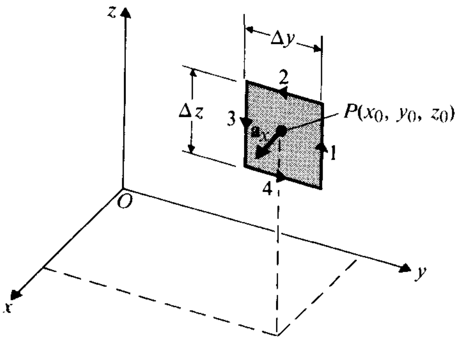


FIGURE 2–31
Determining $(\nabla \times \mathbf{A})_x$.

and 4. Thus,

$$(\nabla \times \mathbf{A})_x = \lim_{\Delta y \Delta z \to 0} \frac{1}{\Delta y \Delta z}\left(\oint_{\substack{\text{sides}\\1,2,3,4}} \mathbf{A} \cdot d\boldsymbol{\ell}\right). \tag{2-127}$$

In Cartesian coordinates, $\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z$. The contributions of the four sides to the line integral are as follows.

Side 1: $d\boldsymbol{\ell} = \mathbf{a}_z \Delta z$, $\mathbf{A} \cdot d\boldsymbol{\ell} = A_z\left(x_0, y_0 + \frac{\Delta y}{2}, z_0\right)\Delta z$,

where $A_z\left(x_0, y_0 + \frac{\Delta y}{2}, z_0\right)$ can be expanded as a Taylor series:

$$A_z\left(x_0, y_0 + \frac{\Delta y}{2}, z_0\right) = A_z(x_0, y_0, z_0) + \frac{\Delta y}{2}\frac{\partial A_z}{\partial y}\bigg|_{(x_0, y_0, z_0)} + \text{H.O.T.}, \tag{2-128}$$

where H.O.T. (higher-order terms) contain the factors $(\Delta y)^2$, $(\Delta y)^3$, etc. Thus,

$$\int_{\text{side 1}} \mathbf{A} \cdot d\boldsymbol{\ell} = \left\{A_z(x_0, y_0, z_0) + \frac{\Delta y}{2}\frac{\partial A_z}{\partial y}\bigg|_{(x_0, y_0, z_0)} + \text{H.O.T.}\right\}\Delta z. \tag{2-129}$$

Side 3: $d\boldsymbol{\ell} = -\mathbf{a}_z \Delta z$, $\mathbf{A} \cdot d\boldsymbol{\ell} = A_z\left(x_0, y_0 - \frac{\Delta y}{2}, z_0\right)\Delta z$,

where

$$A_z\left(x_0, y_0 - \frac{\Delta y}{2}, z_0\right) = A_z(x_0, y_0, z_0) - \frac{\Delta y}{2}\frac{\partial A_z}{\partial y}\bigg|_{(x_0, y_0, z_0)} + \text{H.O.T.}; \tag{2-130}$$

$$\int_{\text{side 3}} \mathbf{A} \cdot d\boldsymbol{\ell} = \left\{A_z(x_0, y_0, z_0) - \frac{\Delta y}{2}\frac{\partial A_z}{\partial y}\bigg|_{(x_0, y_0, z_0)} + \text{H.O.T.}\right\}(-\Delta z). \tag{2-131}$$

Combining Eqs. (2–129) and (2–131), we have

$$\int_{\substack{\text{sides}\\1\,\&\,3}} \mathbf{A} \cdot d\boldsymbol{\ell} = \left(\frac{\partial A_z}{\partial y} + \text{H.O.T.}\right)\bigg|_{(x_0, y_0, z_0)} \Delta y \Delta z. \tag{2-132}$$

The H.O.T. in Eq. (2–132) still contain powers of Δy. Similarly, it may be shown that

$$\int_{\substack{\text{sides}\\2\,\&\,4}} \mathbf{A} \cdot d\boldsymbol{\ell} = \left(-\frac{\partial A_y}{\partial z} + \text{H.O.T.}\right)\bigg|_{(x_0, y_0, z_0)} \Delta y \Delta z. \tag{2-133}$$

Substituting Eqs. (2–132) and (2–133) in Eq. (2–127) and noting that the higher-order terms tend to zero as $\Delta y \to 0$, we obtain the x-component of $\nabla \times \mathbf{A}$:

$$(\nabla \times \mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}. \tag{2-134}$$

A close examination of Eq. (2–134) will reveal a cyclic order in x, y, and z and enable us to write down the y- and z-components of $\nabla \times \mathbf{A}$. The entire expression for the curl of $\mathbf{A}$ in Cartesian coordinates is

$$\nabla \times \mathbf{A} = \mathbf{a}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{a}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{a}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right). \quad (2\text{–}135)$$

Compared to the expression for $\nabla \cdot \mathbf{A}$ in Eq. (2–108), that for $\nabla \times \mathbf{A}$ in Eq. (2–135) is more complicated, as it is expected to be, because it is a vector with three components, whereas $\nabla \cdot \mathbf{A}$ is a scalar. Fortunately, Eq. (2–135) can be remembered rather easily by arranging it in a determinantal form in the manner of the cross product exhibited in Eq. (2–43).

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}. \quad (2\text{–}136)$$

The derivation of $\nabla \times \mathbf{A}$ in other coordinate systems follows the same procedure. However, it is more involved because in curvilinear coordinates not only $\mathbf{A}$ but also $d\boldsymbol{\ell}$ changes in magnitude as the integration of $\mathbf{A} \cdot d\boldsymbol{\ell}$ is carried out on opposite sides of a curvilinear rectangle. The expression for $\nabla \times \mathbf{A}$ in general orthogonal curvilinear coordinates (u_1, u_2, u_3) is given below:

$$\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} \mathbf{a}_{u_1} h_1 & \mathbf{a}_{u_2} h_2 & \mathbf{a}_{u_3} h_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}. \quad (2\text{–}137)$$

The expressions of $\nabla \times \mathbf{A}$ in cylindrical and spherical coordinates can be easily obtained from Eq. (2–137) by using the appropriate u_1, u_2, and u_3 and their metric coefficients h_1, h_2, and h_3 listed in Table 2–1.

EXAMPLE 2–21 Show that $\nabla \times \mathbf{A} = 0$ if

a) $\mathbf{A} = \mathbf{a}_\phi(k/r)$ in cylindrical coordinates, where k is a constant, *or*

b) $\mathbf{A} = \mathbf{a}_R f(R)$ in spherical coordinates, where $f(R)$ is any function of the radial distance R.

Solution

a) In cylindrical coordinates the following apply: $(u_1, u_2, u_3) = (r, \phi, z)$; $h_1 = 1$, $h_2 = r$, and $h_3 = 1$. We have, from Eq. (2–137),

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \mathbf{a}_r & \mathbf{a}_\phi r & \mathbf{a}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix}, \tag{2–138}$$

which yields, for the given **A**,

$$\nabla \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \mathbf{a}_r & \mathbf{a}_\phi r & \mathbf{a}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ 0 & k & 0 \end{vmatrix} = 0.$$

b) In spherical coordinates the following apply: $(u_1, u_2, u_3) = (R, \theta, \phi)$; $h_1 = 1, h_2 = R$, and $h_3 = R \sin\theta$. Hence,

$$\nabla \times \mathbf{A} = \frac{1}{R^2 \sin\theta}\begin{vmatrix} \mathbf{a}_R & \mathbf{a}_\theta R & \mathbf{a}_\phi R\sin\theta \\ \dfrac{\partial}{\partial R} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_R & RA_\theta & R\sin\theta A_\phi \end{vmatrix}, \tag{2–139}$$

and, for the given **A**,

$$\nabla \times \mathbf{A} = \frac{1}{R^2 \sin\theta}\begin{vmatrix} \mathbf{a}_R & \mathbf{a}_\theta R & \mathbf{a}_\phi R\sin\theta \\ \dfrac{\partial}{\partial R} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ f(R) & 0 & 0 \end{vmatrix} = 0.$$

A curl-free vector field is called an ***irrotational*** or a ***conservative field***. We will see in the next chapter that an electrostatic field is irrotational (or conservative). The expressions for $\nabla \times \mathbf{A}$ given in Eqs. (2–138) and (2–139) for cylindrical and spherical coordinates, respectively, will be useful for later reference.

2–10 Stokes's Theorem

For a very small differential area Δs_j bounded by a contour C_j, the definition of $\nabla \times \mathbf{A}$ in Eq. (2–125) leads to

$$(\nabla \times \mathbf{A})_j \cdot (\Delta \mathbf{s}_j) = \oint_{C_j} \mathbf{A} \cdot d\boldsymbol{\ell}. \tag{2–140}$$

In obtaining Eq. (2–140), we have taken the dot product of both sides of Eq. (2–125) with $\mathbf{a}_n \Delta s_j$ or $\Delta \mathbf{s}_j$. For an arbitrary surface S, we can subdivide it into many, say N, small differential areas. Figure 2–32 shows such a scheme with Δs_j as a typical differential element. The left side of Eq. (2–140) is the flux of the vector $\nabla \times \mathbf{A}$ through the area $\Delta \mathbf{s}_j$. Adding the contributions of all the differential areas to the flux, we have

$$\lim_{\Delta s_j \to 0} \sum_{j=1}^{N} (\nabla \times \mathbf{A})_j \cdot (\Delta \mathbf{s}_j) = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s}. \tag{2–141}$$

Now we sum up the line integrals around the contours of all the differential elements represented by the right side of Eq. (2–140). Since the common part of the contours of two adjacent elements is traversed in opposite directions by two contours, the net contribution of all the common parts in the interior to the total line integral is zero, and only the contribution from the external contour C bounding the entire area S remains after the summation:

$$\lim_{\Delta s_j \to 0} \sum_{j=1}^{N} \left(\oint_{C_j} \mathbf{A} \cdot d\boldsymbol{\ell} \right) = \oint_C \mathbf{A} \cdot d\boldsymbol{\ell}. \tag{2–142}$$

Combining Eqs. (2–141) and (2–142), we obtain ***Stokes's theorem***:

$$\boxed{\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\boldsymbol{\ell},} \tag{2–143}$$

which states that ***the surface integral of the curl of a vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface.***

As with the divergence theorem, the validity of the limiting processes leading to Stokes's theorem requires that the vector field $\mathbf{A}$, as well as its first derivatives, exist and be continuous both on S and along C. Stokes's theorem converts a surface integral of the curl of a vector to a line integral of the vector, and vice versa. Like the divergence theorem, Stokes's theorem is an important identity in vector analysis, and we will use it frequently in establishing other theorems and relations in electromagnetics.

If the surface integral of $\nabla \times \mathbf{A}$ is carried over a closed surface, there will be no surface-bounding external contour, and Eq. (2–143) tells us that

$$\oint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = 0 \tag{2–144}$$

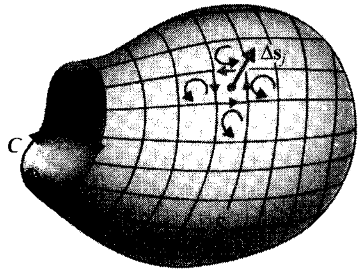


FIGURE 2–32
Subdivided area for proof of Stokes's theorem.

for any closed surface S. The geometry in Fig. 2–32 is chosen deliberately to emphasize the fact that a nontrivial application of Stokes's theorem always implies *an open surface with a rim.* The simplest open surface would be a two-dimensional plane or disk with its circumference as the contour. We remind ourselves here that the directions of $d\ell$ and $ds(\mathbf{a}_n)$ follow the right-hand rule.

EXAMPLE 2–22 Given $\mathbf{F} = \mathbf{a}_x xy - \mathbf{a}_y 2x$, verify Stokes's theorem over a quarter-circular disk with a radius 3 in the first quadrant, as was shown in Fig. 2–21 (Example 2–14, page 39).

Solution Let us first find the surface integral of $\nabla \times \mathbf{F}$. From Eq. (2–136),

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & -2x & 0 \end{vmatrix} = -\mathbf{a}_z(2 + x).$$

Therefore,

$$\begin{aligned} \int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{s} &= \int_0^3 \int_0^{\sqrt{9-y^2}} (\nabla \times \mathbf{F}) \cdot (\mathbf{a}_z\, dx\, dy) \\ &= \int_0^3 \left[\int_0^{\sqrt{9-y^2}} -(2 + x)\, dx \right] dy \\ &= -\int_0^3 \left[2\sqrt{9 - y^2} + \tfrac{1}{2}(9 - y^2)\right] dy \\ &= -\left[y\sqrt{9 - y^2} + 9 \sin^{-1}\frac{y}{3} + \frac{9}{2}y - \frac{y^3}{6} \right]_0^3 \\ &= -9\left(1 + \frac{\pi}{2}\right). \end{aligned}$$

It is *important* to use the proper limits for the two variables of integration. We can interchange the order of integration as

$$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{s} = \int_0^3 \left[\int_0^{\sqrt{9-x^2}} -(2 + x)\, dy \right] dx$$

and get the same result. But it would be quite wrong if the 0 to 3 range were used as the range of integration for both x and y. (Do you know why?)

For the line integral around $ABOA$ we have already evaluated the part around the arc from A to B in Example 2–14.

From B to O: $x = 0$, and $\mathbf{F} \cdot d\ell = \mathbf{F} \cdot (-\mathbf{a}_y\, dy) = 2x\, dy = 0$.

From O to A: $y = 0$, and $\mathbf{F} \cdot d\ell = \mathbf{F} \cdot (\mathbf{a}_x\, dx) = xy\, dx = 0$. Hence

$$\oint_{ABOA} \mathbf{F} \cdot d\ell = \int_A^B \mathbf{F} \cdot d\ell = -9\left(1 + \frac{\pi}{2}\right),$$

from Example 2–14, and Stokes's theorem is verified.

Of course, Stokes's theorem has been established in Eq. (2–143) as a general identity; there is no need to use a particular example to prove it. We worked out the example above for practice on surface and line integrals. (We note here that both the vector field and its first spatial derivatives are finite and continuous on the surface as well as on the contour of interest.)

2–11 Two Null Identities

Two identities involving repeated del operations are of considerable importance in the study of electromagnetism, especially when we introduce potential functions. We shall discuss them separately below.

2–11.1 IDENTITY I

$$\boxed{\nabla \times (\nabla V) \equiv 0} \tag{2–145}$$

In words, ***the curl of the gradient of any scalar field is identically zero.*** (The existence of V and its first derivatives everywhere is implied here.)

Equation (2–145) can be proved readily in Cartesian coordinates by using Eq. (2–96) for ∇ and performing the indicated operations. In general, if we take the surface integral of $\nabla \times (\nabla V)$ over any surface, the result is equal to the line integral of ∇V around the closed path bounding the surface, as asserted by Stokes's theorem:

$$\int_S [\nabla \times (\nabla V)] \cdot d\mathbf{s} = \oint_C (\nabla V) \cdot d\boldsymbol{\ell}. \tag{2–146}$$

However, from Eq. (2–88),

$$\oint_C (\nabla V) \cdot d\boldsymbol{\ell} = \oint_C dV = 0. \tag{2–147}$$

The combination of Eqs. (2–146) and (2–147) states that the surface integral of $\nabla \times (\nabla V)$ over *any* surface is zero. The integrand itself must therefore vanish, which leads to the identity in Eq. (2–145). Since a coordinate system is not specified in the derivation, the identity is a general one and is invariant with the choices of coordinate systems.

A converse statement of Identity I can be made as follows: ***If a vector field is curl-free, then it can be expressed as the gradient of a scalar field.*** Let a vector field be $\mathbf{E}$. Then, if $\nabla \times \mathbf{E} = 0$, we can define a scalar field V such that

$$\mathbf{E} = -\nabla V. \tag{2–148}$$

The negative sign here is unimportant as far as Identity I is concerned. (It is included in Eq. (2–148) because this relation conforms with a basic relation between *electric field intensity* $\mathbf{E}$ and *electric scalar potential* V in electrostatics, which we will take up in the next chapter. At this stage it is immaterial what $\mathbf{E}$ and V represent.) We know from Section 2–9 that a curl-free vector field is a conservative field; hence ***an irrotational (a conservative) vector field can always be expressed as the gradient of a scalar field.***

2–11.2 IDENTITY II

$$\boxed{\nabla \cdot (\nabla \times \mathbf{A}) \equiv 0} \tag{2–149}$$

In words, ***the divergence of the curl of any vector field is identically zero.***

Equation (2–149), too, can be proved easily in Cartesian coordinates by using Eq. (2–96) for ∇ and performing the indicated operations. We can prove it in general without regard to a coordinate system by taking the volume integral of $\nabla \cdot (\nabla \times \mathbf{A})$ on the left side. Applying the divergence theorem, we have

$$\int_V \nabla \cdot (\nabla \times \mathbf{A})\, dv = \oint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s}. \tag{2–150}$$

Let us choose, for example, the arbitrary volume V enclosed by a surface S in Fig. 2–33. The closed surface S can be split into two open surfaces, S_1 and S_2, connected by a common boundary that has been drawn twice as C_1 and C_2. We then apply Stokes's theorem to surface S_1 bounded by C_1, and surface S_2 bounded by C_2, and we write the right side of Eq. (2–150) as

$$\begin{aligned}\oint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} &= \int_{S_1} (\nabla \times \mathbf{A}) \cdot \mathbf{a}_{n1}\, ds + \int_{S_2} (\nabla \times \mathbf{A}) \cdot \mathbf{a}_{n2}\, ds \\ &= \oint_{C_1} \mathbf{A} \cdot d\boldsymbol{\ell} + \oint_{C_2} \mathbf{A} \cdot d\boldsymbol{\ell}.\end{aligned} \tag{2–151}$$

The normals $\mathbf{a}_{n1}$ and $\mathbf{a}_{n2}$ to surfaces S_1 and S_2 are *outward* normals, and their relations with the path directions of C_1 and C_2 follow the right-hand rule. Since the contours C_1 and C_2 are, in fact, one and the same common boundary between S_1 and S_2, the two line integrals on the right side of Eq. (2–151) traverse the same path in opposite directions. Their sum is therefore zero, and the volume integral of $\nabla \cdot (\nabla \times \mathbf{A})$ on the left side of Eq. (2–150) vanishes. Because this is true for any arbitrary volume, the integrand itself must be zero, as indicated by the identity in Eq. (2–149).

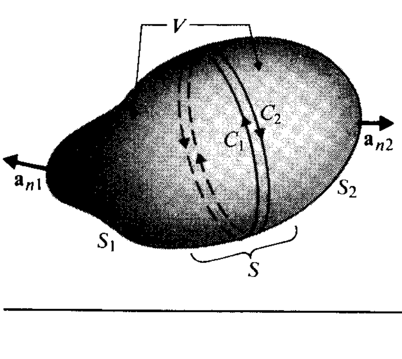


FIGURE 2–33
An arbitrary volume V enclosed by surface S.

A converse statement of Identity II is as follows: ***If a vector field is divergenceless, then it can be expressed as the curl of another vector field.*** Let a vector field be **B**. This converse statement asserts that if $\nabla \cdot \mathbf{B} = 0$, we can define a vector field **A** such that

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{2-152}$$

In Section 2–7 we mentioned that a divergenceless field is also called a solenoidal field. Solenoidal fields are not associated with flow sources or sinks. The net outward flux of a solenoidal field through any closed surface is zero, and the flux lines close upon themselves. We are reminded of the circling magnetic flux lines of a solenoid or an inductor. As we will see in Chapter 6, *magnetic flux density* **B** is solenoidal and can be expressed as the curl of another vector field called *magnetic vector potential* **A**.

2–12 Helmholtz's Theorem

In previous sections we mentioned that *a divergenceless field is solenoidal* and *a curl-free field is irrotational.* We may classify vector fields in accordance with their being solenoidal and/or irrotational. A vector field **F** is

1. Solenoidal and irrotational if

$$\nabla \cdot \mathbf{F} = 0 \qquad \text{and} \qquad \nabla \times \mathbf{F} = 0.$$

EXAMPLE: A static electric field in a charge-free region.

2. Solenoidal but not irrotational if

$$\nabla \cdot \mathbf{F} = 0 \qquad \text{and} \qquad \nabla \times \mathbf{F} \neq 0.$$

EXAMPLE: A steady magnetic field in a current-carrying conductor.

3. Irrotational but not solenoidal if

$$\nabla \times \mathbf{F} = 0 \qquad \text{and} \qquad \nabla \cdot \mathbf{F} \neq 0.$$

EXAMPLE: A static electric field in a charged region.

4. Neither solenoidal nor irrotational if

$$\nabla \cdot \mathbf{F} \neq 0 \qquad \text{and} \qquad \nabla \times \mathbf{F} \neq 0.$$

EXAMPLE: An electric field in a charged medium with a time-varying magnetic field.

The most general vector field then has both a nonzero divergence and a nonzero curl, and can be considered as the sum of a solenoidal field and an irrotational field.

Helmholtz's Theorem: A vector field (vector point function) is determined to within an additive constant if both its divergence and its curl are specified everywhere. In an unbounded region we assume that both the divergence and the curl of the vector field vanish at infinity. If the vector field is confined within a region bounded by a surface, then it is determined if its divergence and curl throughout the region, as well as the normal component of the vector over the bounding surface, are given.

Here we assume that the vector function is single-valued and that its derivatives are finite and continuous.

Helmholtz's theorem can be proved as a mathematical theorem in a general way.† For our purposes, we remind ourselves (see Section 2–9) that the divergence of a vector is a measure of the strength of the flow source and that the curl of a vector is a measure of the strength of the vortex source. When the strengths of both the flow source and the vortex source are specified, we expect that the vector field will be determined. Thus, we can decompose a general vector field $\mathbf{F}$ into an irrotational part $\mathbf{F}_i$ and a solenoidal part $\mathbf{F}_s$:

$$\mathbf{F} = \mathbf{F}_i + \mathbf{F}_s, \tag{2–153}$$

with

$$\begin{cases} \nabla \times \mathbf{F}_i = 0 & (2\text{–}154\text{a}) \\ \nabla \cdot \mathbf{F}_i = g & (2\text{–}154\text{b}) \end{cases}$$

and

$$\begin{cases} \nabla \cdot \mathbf{F}_s = 0 & (2\text{–}155\text{a}) \\ \nabla \times \mathbf{F}_s = \mathbf{G}, & (2\text{–}155\text{b}) \end{cases}$$

where g and $\mathbf{G}$ are assumed to be known. We have

$$\nabla \cdot \mathbf{F} = \nabla \cdot \mathbf{F}_i = g \tag{2–156}$$

and

$$\nabla \times \mathbf{F} = \nabla \times \mathbf{F}_s = \mathbf{G}. \tag{2–157}$$

Helmholtz's theorem asserts that when g and $\mathbf{G}$ are specified, the vector function $\mathbf{F}$ is determined. Since $\nabla\cdot$ and $\nabla\times$ are differential operators, $\mathbf{F}$ must be obtained by integrating g and $\mathbf{G}$ in some manner, which will lead to constants of integration. The determination of these additive constants requires the knowledge of some boundary conditions. The procedure for obtaining $\mathbf{F}$ from given g and $\mathbf{G}$ is not obvious at this time; it will be developed in stages in later chapters.

The fact that $\mathbf{F}_i$ is irrotational enables us to define a scalar (potential) function V, in view of identity (2–145), such that

$$\mathbf{F}_i = -\nabla V. \tag{2–158}$$

Similarly, identity (2–149) and Eq. (2–155a) allow the definition of a vector (potential) function $\mathbf{A}$ such that

$$\mathbf{F}_s = \nabla \times \mathbf{A}. \tag{2–159}$$

Helmholtz's theorem states that a general vector function $\mathbf{F}$ can be written as the sum of the gradient of a scalar function and the curl of a vector function. Thus

$$\mathbf{F} = -\nabla V + \nabla \times \mathbf{A}. \tag{2–160}$$

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† See, for instance, G. Arfken, *Mathematical Methods for Physicists,* Section 1.15, Academic Press, New York, 1966.

In following chapters we will rely on Helmholtz's theorem as a basic element in the axiomatic development of electromagnetism.

EXAMPLE 2–23 Given a vector function

$$\mathbf{F} = \mathbf{a}_x(3y - c_1 z) + \mathbf{a}_y(c_2 x - 2z) - \mathbf{a}_z(c_3 y + z).$$

a) Determine the constants c_1, c_2, and c_3 if $\mathbf{F}$ is irrotational.

b) Determine the scalar potential function V whose negative gradient equals $\mathbf{F}$.

Solution

a) For $\mathbf{F}$ to be irrotational, $\nabla \times \mathbf{F} = 0$; that is,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3y - c_1 z & c_2 x - 2z & -(c_3 y + z) \end{vmatrix}$$

$$= \mathbf{a}_x(-c_3 + 2) - \mathbf{a}_y c_1 + \mathbf{a}_z(c_2 - 3) = 0.$$

Each component of $\nabla \times \mathbf{F}$ must vanish. Hence $c_1 = 0$, $c_2 = 3$, and $c_3 = 2$.

b) Since $\mathbf{F}$ is irrotational, it can be expressed as the negative gradient of a scalar function V; that is,

$$\mathbf{F} = -\nabla V = -\mathbf{a}_x \frac{\partial V}{\partial x} - \mathbf{a}_y \frac{\partial V}{\partial y} - \mathbf{a}_z \frac{\partial V}{\partial z}$$

$$= \mathbf{a}_x 3y + \mathbf{a}_y(3x - 2z) - \mathbf{a}_z(2y + z).$$

Three equations are obtained:

$$\frac{\partial V}{\partial x} = -3y, \tag{2–161}$$

$$\frac{\partial V}{\partial y} = -3x + 2z, \tag{2–162}$$

$$\frac{\partial V}{\partial z} = 2y + z. \tag{2–163}$$

Integrating Eq. (2–161) with respect to x, we have

$$V = -3xy + f_1(y, z), \tag{2–164}$$

where $f_1(y, z)$ is a function of y and z yet to be determined. Similarly, integrating Eq. (2–162) with respect to y and Eq. (2–163) with respect to z leads to

$$V = -3xy + 2yz + f_2(x, z) \tag{2–165}$$

and

$$V = 2yz + \frac{z^2}{2} + f_3(x, y). \tag{2–166}$$

Examination of Eqs. (2–164), (2–165), and (2–166) enables us to write the scalar potential function as

$$V = -3xy + 2yz + \frac{z^2}{2}. \qquad (2\text{–}167)$$

Any constant added to Eq. (2–167) would still make V an answer. The constant is to be determined by a boundary condition or the condition at infinity. ▬

Review Questions

R.2–1 Three vectors **A**, **B**, and **C**, drawn in a head-to-tail fashion, form three sides of a triangle. What is $\mathbf{A} + \mathbf{B} + \mathbf{C}$? What is $\mathbf{A} + \mathbf{B} - \mathbf{C}$?

R.2–2 Under what conditions can the dot product of two vectors be negative?

R.2–3 Write down the results of $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{A} \times \mathbf{B}$ if (a) $\mathbf{A} \| \mathbf{B}$, and (b) $\mathbf{A} \perp \mathbf{B}$.

R.2–4 Which of the following products of vectors do not make sense? Explain.

a) $(\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C}$ **b)** $\mathbf{A}(\mathbf{B} \cdot \mathbf{C})$ **c)** $\mathbf{A} \times \mathbf{B} \times \mathbf{C}$
d) $\mathbf{A}/\mathbf{B}$ **e)** $\mathbf{A}/\mathbf{a}_A$ **f)** $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$

R.2–5 Is $(\mathbf{A} \cdot \mathbf{B})\mathbf{C}$ equal to $\mathbf{A}(\mathbf{B} \cdot \mathbf{C})$?

R.2–6 Does $\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{C}$ imply $\mathbf{B} = \mathbf{C}$? Explain.

R.2–7 Does $\mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{C}$ imply $\mathbf{B} = \mathbf{C}$? Explain.

R.2–8 Given two vectors **A** and **B**, how do you find (a) the component of **A** in the direction of **B**, and (b) the component of **B** in the direction of **A**?

R.2–9 What makes a coordinate system (a) orthogonal? (b) curvilinear? and (c) right-handed?

R.2–10 Given a vector **F** in orthogonal curvilinear coordinates (u_1, u_2, u_3), explain how to determine (a) F, and (b) $\mathbf{a}_F$.

R.2–11 What are metric coefficients?

R.2–12 Given two points $P_1(1, 2, 3)$ and $P_2(-1, 0, 2)$ in Cartesian coordinates, write the expressions of the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_2P_1}$.

R.2–13 What are the expressions for $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{A} \times \mathbf{B}$ in Cartesian coordinates?

R.2–14 What is the difference between a scalar quantity and a scalar field? Between a vector quantity and a vector field?

R.2–15 What is the physical definition of the gradient of a scalar field?

R.2–16 Express the space rate of change of a scalar in a given direction in terms of its gradient.

R.2–17 What does the del operator ∇ stand for in Cartesian coordinates?

R.2–18 What is the physical definition of the divergence of a vector field?

R.2–19 A vector field with only radial flux lines cannot be solenoidal. True or false? Explain.

R.2–20 A vector field with only curved flux lines can have a nonzero divergence. True or false? Explain.

R.2–21 State the divergence theorem in words.

R.2–22 What is the physical definition of the curl of a vector field?

R.2–23 A vector field with only curved flux lines cannot be irrotational. True or false? Explain.

R.2–24 A vector field with only straight flux lines can be solenoidal. True or false? Explain.

R.2–25 State Stokes's theorem in words.

R.2–26 What is the difference between an irrotational field and a solenoidal field?

R.2–27 State Helmholtz's theorem in words.

R.2–28 Explain how a general vector function can be expressed in terms of a scalar potential function and a vector potential function.

Problems

P.2–1 Given three vectors **A**, **B**, and **C** as follows,

$$\mathbf{A} = \mathbf{a}_x + \mathbf{a}_y 2 - \mathbf{a}_z 3,$$
$$\mathbf{B} = -\mathbf{a}_y 4 + \mathbf{a}_z,$$
$$\mathbf{C} = \mathbf{a}_x 5 - \mathbf{a}_z 2,$$

find

- **a)** $\mathbf{a}_A$
- **b)** $|\mathbf{A} - \mathbf{B}|$
- **c)** $\mathbf{A} \cdot \mathbf{B}$
- **d)** θ_{AB}
- **e)** the component of **A** in the direction of **C**
- **f)** $\mathbf{A} \times \mathbf{C}$
- **g)** $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ and $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$
- **h)** $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ and $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$

P.2–2 Given

$$\mathbf{A} = \mathbf{a}_x - \mathbf{a}_y 2 + \mathbf{a}_z 3,$$
$$\mathbf{B} = \mathbf{a}_x + \mathbf{a}_y - \mathbf{a}_z 2,$$

find the expression for a unit vector **C** that is perpendicular to both **A** and **B**.

P.2–3 Two vector fields represented by $\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z$ and $\mathbf{B} = \mathbf{a}_x B_x + \mathbf{a}_y B_y + \mathbf{a}_z B_z$, where all components may be functions of space coordinates. If these two fields are parallel to each other everywhere, what must be the relations between their components?

P.2–4 Show that, if $\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{C}$ *and* $\mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{C}$, where **A** is not a null vector, then $\mathbf{B} = \mathbf{C}$.

P.2–5 An unknown vector can be determined if both its scalar product and its vector product with a known vector are given. Assuming that **A** is a known vector, determine the unknown vector **X** if both p and **B** are given, where $p = \mathbf{A} \cdot \mathbf{X}$ and $\mathbf{B} = \mathbf{A} \times \mathbf{X}$.

P.2–6 The three corners of a triangle are at $P_1(0, 1, -2)$, $P_2(4, 1, -3)$, and $P_3(6, 2, 5)$.

- **a)** Determine whether $\triangle P_1 P_2 P_3$ is a right triangle.
- **b)** Find the area of the triangle.

P.2–7 Show that the two diagonals of a rhombus are perpendicular to each other. (A rhombus is an equilateral parallelogram.)

P.2–8 Prove that the line joining the midpoints of two sides of a triangle is parallel to and half as long as the third side.

P.2–9 Unit vectors $\mathbf{a}_A$ and $\mathbf{a}_B$ denote the directions of two-dimensional vectors **A** and **B** that make angles α and β, respectively, with a reference x-axis, as shown in Fig. 2–34. a) Obtain a formula for the expansion of the cosine of the difference of two angles, $\cos(\alpha - \beta)$, by taking the scalar product $\mathbf{a}_A \cdot \mathbf{a}_B$. b) Obtain a formula for $\sin(\alpha - \beta)$.

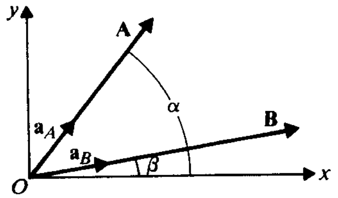


FIGURE 2–34
Graph for Problem P.2–9.

P.2–10 Prove the law of sines for a triangle.

P.2–11 Prove that an angle inscribed in a semicircle is a right angle.

P.2–12 Verify the back-cab rule of the vector triple product of three vectors, as expressed in Eq. (2–20) in Cartesian coordinates.

P.2–13 Prove by vector relations that two lines in the xy-plane (L_1: $b_1x + b_2y = c$; L_2: $b'_1x + b'_2y = c'$) are perpendicular if their slopes are the negative reciprocals of each other.

P.2–14

a) Prove that the equation of any plane in space can be written in the form $b_1x + b_2y + b_3z = c$. (*Hint:* Prove that the dot product of the position vector to any point in the plane and a normal vector is a constant.)
b) Find the expression for the unit normal passing through the origin.
c) For the plane $3x - 2y + 6z = 5$, find the perpendicular distance from the origin to the plane.

P.2–15 Find the component of the vector $\mathbf{A} = -\mathbf{a}_y z + \mathbf{a}_z y$ at the point $P_1(0, -2, 3)$, which is directed toward the point $P_2(\sqrt{3}, -60°, 1)$.

P.2–16 The position of a point in cylindrical coordinates is specified by $(4, 2\pi/3, 3)$. What is the location of the point

a) in Cartesian coordinates?
b) in spherical coordinates?

P.2–17 A field is expressed in spherical coordinates by $\mathbf{E} = \mathbf{a}_R(25/R^2)$.

a) Find $|\mathbf{E}|$ and E_x at the point $P(-3, 4, -5)$.
b) Find the angle that $\mathbf{E}$ makes with the vector $\mathbf{B} = \mathbf{a}_x 2 - \mathbf{a}_y 2 + \mathbf{a}_z$ at point P.

P.2–18 Express the base vectors $\mathbf{a}_R$, $\mathbf{a}_\theta$, and $\mathbf{a}_\phi$ of a spherical coordinate system in Cartesian coordinates.

P.2–19 Determine the values of the following products of base vectors:

a) $\mathbf{a}_x \cdot \mathbf{a}_\phi$	b) $\mathbf{a}_\theta \cdot \mathbf{a}_y$	c) $\mathbf{a}_r \times \mathbf{a}_x$
d) $\mathbf{a}_R \cdot \mathbf{a}_r$	e) $\mathbf{a}_y \cdot \mathbf{a}_R$	f) $\mathbf{a}_R \cdot \mathbf{a}_z$
g) $\mathbf{a}_R \times \mathbf{a}_z$	h) $\mathbf{a}_\theta \cdot \mathbf{a}_z$	i) $\mathbf{a}_z \times \mathbf{a}_\theta$.

P.2–20 Given a vector function $\mathbf{F} = \mathbf{a}_x xy + \mathbf{a}_y(3x - y^2)$, evaluate the integral $\int \mathbf{F} \cdot d\boldsymbol{\ell}$ from $P_1(5, 6)$ to $P_2(3, 3)$ in Fig. 2–35

a) along the direct path P_1P_2,
b) along path P_1AP_2.

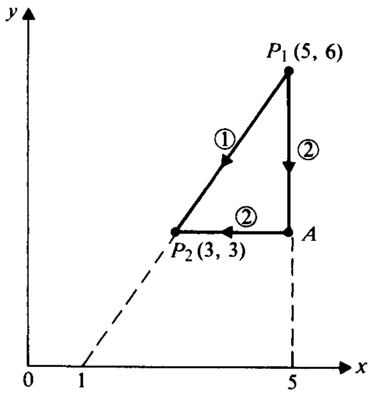


FIGURE 2–35
Paths of integration for Problem P.2–20.

P.2–21 Given a vector function $\mathbf{E} = \mathbf{a}_x y + \mathbf{a}_y x$, evaluate the scalar line integral $\int \mathbf{E} \cdot d\boldsymbol{\ell}$ from $P_1(2, 1, -1)$ to $P_2(8, 2, -1)$

a) along the parabola $x = 2y^2$,

b) along the straight line joining the two points.

Is this **E** a conservative field?

P.2–22 For the **E** of Problem P.2–21, evaluate $\int \mathbf{E} \cdot d\boldsymbol{\ell}$ from $P_3(3, 4, -1)$ to $P_4(4, -3, -1)$ by converting both **E** and the positions of P_3 and P_4 into cylindrical coordinates.

P.2–23 Given a scalar function

$$V = \left(\sin \frac{\pi}{2} x\right)\left(\sin \frac{\pi}{3} y\right)e^{-z},$$

determine

a) the magnitude and the direction of the maximum rate of increase of V at the point $P(1, 2, 3)$,

b) the rate of increase of V at P in the direction of the origin.

P.2–24 Evaluate

$$\oint_S (\mathbf{a}_R 3 \sin \theta) \cdot d\mathbf{s}$$

over the surface of a sphere of a radius 5 centered at the origin.

P.2–25 The equation in space of a plane containing the point (x_1, y_1, z_1) can be written as

$$\ell(x - x_1) + m(y - y_1) + p(z - z_1) = 0,$$

where ℓ, m, and p are direction cosines of a unit normal to the plane:

$$\mathbf{a}_n = \mathbf{a}_x \ell + \mathbf{a}_y m + \mathbf{a}_z p.$$

Given a vector field $\mathbf{F} = \mathbf{a}_x + \mathbf{a}_y 2 + \mathbf{a}_z 3$, evaluate the integral $\int_S \mathbf{F} \cdot d\mathbf{s}$ over the square plane surface whose corners are at (0, 0, 2), (2, 0, 2), (2, 2, 0), and (0, 2, 0).

P.2–26 Find the divergence of the following radial vector fields:

a) $f_1(\mathbf{R}) = \mathbf{a}_R R^n$,

b) $f_2(\mathbf{R}) = \mathbf{a}_R \dfrac{k}{R^2}$.

P.2–27 Show that $\frac{1}{3} \oint_S \mathbf{R} \cdot d\mathbf{s} = V$, where **R** is the radial vector and V is the volume of the region enclosed by surface S.

P.2–28 For a scalar function f and a vector function $\mathbf{A}$, prove that

$$\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f$$

in Cartesian coordinates.

P.2–29 For vector function $\mathbf{A} = \mathbf{a}_r r^2 + \mathbf{a}_z 2z$, verify the divergence theorem for the circular cylindrical region enclosed by $r = 5$, $z = 0$, and $z = 4$.

P.2–30 For the vector function $\mathbf{F} = \mathbf{a}_r k_1/r + \mathbf{a}_z k_2 z$ given in Example 2–15 (page 41) evaluate $\int \nabla \cdot \mathbf{F}\, dv$ over the volume specified in that example. Explain why the divergence theorem fails here.

P.2–31 Use the definition in Eq. (2–98) to derive the expression of $\nabla \cdot \mathbf{A}$ for a vector field $\mathbf{A} = \mathbf{a}_r A_r + \mathbf{a}_\phi A_\phi + \mathbf{a}_z A_z$ in cylindrical coordinates.

P.2–32 A vector field $\mathbf{D} = \mathbf{a}_R(\cos^2 \phi)/R^3$ exists in the region between two spherical shells defined by $R = 1$ and $R = 2$. Evaluate

a) $\oint \mathbf{D} \cdot d\mathbf{s}$,
b) $\int \nabla \cdot \mathbf{D}\, dv$.

P.2–33 For two differentiable vector functions $\mathbf{E}$ and $\mathbf{H}$, prove that

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}).$$

P.2–34 Assume the vector function $\mathbf{A} = \mathbf{a}_x 3x^2y^3 - \mathbf{a}_y x^3y^2$.

a) Find $\oint \mathbf{A} \cdot d\boldsymbol{\ell}$ around the triangular contour shown in Fig. 2–36.
b) Evaluate $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{s}$ over the triangular area.
c) Can $\mathbf{A}$ be expressed as the gradient of a scalar? Explain.

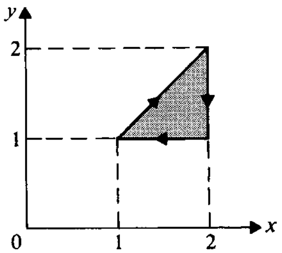


FIGURE 2–36
Graph for Problem P.2–34.

P.2–35 Use the definition in Eq. (2–126) to derive the expression of the $\mathbf{a}_R$-component of $\nabla \times \mathbf{A}$ in spherical coordinates for a vector field $\mathbf{A} = \mathbf{a}_R A_R + \mathbf{a}_\theta A_\theta + \mathbf{a}_\phi A_\phi$.

P.2–36 Given the vector function $\mathbf{A} = \mathbf{a}_\phi \sin(\phi/2)$, verify Stokes's theorem over the hemispherical surface and its circular contour that are shown in Fig. 2–37.

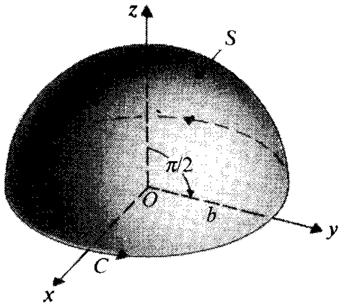


FIGURE 2–37
Graph for Problem P.2–36.

P.2–37 For a scalar function f and a vector function $\mathbf{G}$, prove that

$$\nabla \times (f\mathbf{G}) = f\nabla \times \mathbf{G} + (\nabla f) \times \mathbf{G}$$

in Cartesian coordinates.

P.2–38 Verify the null identities:

a) $\nabla \times (\nabla V) \equiv 0$

b) $\nabla \cdot (\nabla \times \mathbf{A}) \equiv 0$

by expansion in general orthogonal curvilinear coordinates.

P.2–39 Given a vector function $\mathbf{F} = \mathbf{a}_x(x + c_1 z) + \mathbf{a}_y(c_2 x - 3z) + \mathbf{a}_z(x + c_3 y + c_4 z)$.

a) Determine the constants c_1, c_2, and c_3 if $\mathbf{F}$ is irrotational.

b) Determine the constant c_4 if $\mathbf{F}$ is also solenoidal.

c) Determine the scalar potential function V whose negative gradient equals $\mathbf{F}$.

3

Static Electric Fields

3–1 Introduction

In Section 1–2 we mentioned that three essential steps are involved in constructing a deductive theory for the study of a scientific subject. They are: the definition of basic quantities, the development of rules of operation, and the postulation of fundamental relations. We have defined the source and field quantities for the electromagnetic model in Chapter 1 and developed the fundamentals of vector algebra and vector calculus in Chapter 2. We are now ready to introduce the fundamental postulates for the study of source-field relationships in electrostatics.

A ***field*** is a spatial distribution of a scalar or vector quantity, which may or may not be a function of time. An example of a scalar is the altitude of a location on a mountain relative to the sea level. It is a scalar, which is not a function of time if long-term erosion and earthquake effects are neglected. Various locations on the mountain have different altitudes, constituting an altitude field. The gradient of altitude is a vector that gives both the direction and the magnitude of the maximum rate of increase (the upward slope) of altitude. On a flat mountaintop or flat land the altitude is constant, and its gradient vanishes. The gravitational field of the earth, representing the force of gravity on a unit mass, is a vector field directed toward the center of the earth, having a magnitude depending on the altitude of the mass. Electric and magnetic field intensities are vector fields.

In electrostatics, electric charges (the sources) are at rest, and electric fields do not change with time. There are no magnetic fields; hence we deal with a relatively simple situation. After we have studied the behavior of static electric fields and mastered the techniques for solving electrostatic boundary-value problems, we will go on to the subject of magnetic fields and time-varying electromagnetic fields. Although electrostatics is relatively simple in the electromagnetics scheme of things, its mastery is fundamental to the understanding of more complicated electromagnetic models. Moreover, the explanation of many natural phenomena (such as lightning, corona, St. Elmo's fire, and grain explosion) and the principles of some important industrial applications (such as oscilloscope, ink-jet printer, xerography, and electret microphone) are based on electrostatics. Many articles on special applications of electrostatics appear in the literature, and a number of books on this subject have also been published.†

The development of electrostatics in elementary physics usually begins with the experimental Coulomb's law (formulated in 1785) for the force between two point charges. This law states that the force between two charged bodies, q_1 and q_2, that are very small in comparison with the distance of separation, R_{12}, is proportional to the product of the charges and inversely proportional to the square of the distance, the direction of the force being along the line connecting the charges. In addition, Coulomb found that unlike charges attract and like charges repel each other. Using vector notation, ***Coulomb's law*** can be written mathematically as

$$\mathbf{F}_{12} = \mathbf{a}_{R_{12}} k \frac{q_1 q_2}{R_{12}^2}, \tag{3-1}$$

where $\mathbf{F}_{12}$ is the vector force exerted by q_1 on q_2, $\mathbf{a}_{R_{12}}$ is a unit vector in the direction from q_1 to q_2, and k is a proportionality constant depending on the medium and the system of units. Note that if q_1 and q_2 are of the same sign (both positive or both negative), $\mathbf{F}_{12}$ is positive (repulsive); and if q_1 and q_2 are of opposite signs, $\mathbf{F}_{12}$ is negative (attractive). Electrostatics can proceed from Coulomb's law to define electric field intensity $\mathbf{E}$, electric scalar potential V, and electric flux density $\mathbf{D}$, and then lead to Gauss's law and other relations. This approach has been accepted as "logical," perhaps because it begins with an experimental law observed in a laboratory and not with some abstract postulates.

We maintain, however, that Coulomb's law, though based on experimental evidence, is in fact also a postulate. Consider the two stipulations of Coulomb's law: that the charged bodies be very small in comparison with the distance of separation and that the force be inversely proportional to the square of the distance. The question arises regarding the first stipulation: How small must the charged bodies be in order to be considered "very small" in comparison with the distance? In practice the charged bodies cannot be of vanishing sizes (ideal point charges), and there is difficulty in determining the "true" distance between two bodies of finite dimensions. For given body sizes, the relative accuracy in distance measurements is better when the separation is larger. However, practical considerations (weakness of force, existence of extraneous charged bodies, etc.) restrict the usable distance of separation in the laboratory, and experimental inaccuracies cannot be entirely avoided. This leads to a more important question concerning the inverse-square relation of the second

† A. Klinkenberg and J. L. van der Minne, *Electrostatics in the Petroleum Industry*, Elsevier, Amsterdam, 1958. J. H. Dessauer and H. E. Clark, *Xerography and Related Processes*, Focal Press, London, 1965. A. D. Moore (Ed.), *Electrostatics and Its Applications*, John Wiley, New York, 1973. C. E. Jewett, *Electrostatics in the Electronics Environment*, John Wiley, New York, 1976. J.C. Crowley, *Fundamentals of Applied Electrostatics*, John Wiley, New York, 1986.

stipulation. Even if the charged bodies are of vanishing sizes, experimental measurements cannot be of infinite accuracy, no matter how skillful and careful an experimentor is. How then was it possible for Coulomb to know that the force was *exactly* inversely proportional to the *square* (not the 2.000001th or the 1.999999th power) of the distance of separation? This question cannot be answered from an experimental viewpoint because it is not likely that experiments could have been accurate to the seventh place during Coulomb's time.† We must therefore conclude that Coulomb's law is itself a postulate and that the exact relation stipulated by Eq. (3–1) is a law of nature discovered and assumed by Coulomb on the basis of his experiments of limited accuracy.

Instead of following the historical development of electrostatics, we introduce the subject by postulating both the divergence and the curl of the electric field intensity in free space. From Helmholtz's theorem in Section 2–12 we know that a vector field is determined if its divergence and curl are specified. We derive Gauss's law and Coulomb's law from the divergence and curl relations, and we do not present them as separate postulates. The concept of scalar potential follows naturally from a vector identity. Field behaviors in material media will be studied and expressions for electrostatic energy and forces will be developed.

3–2 Fundamental Postulates of Electrostatics in Free Space

We start the study of electromagnetism with the consideration of electric fields due to stationary (static) electric charges in free space. Electrostatics in free space is the simplest special case of electromagnetics. We need to consider only one of the four fundamental vector field quantities of the electromagnetic model discussed in Section 1–2, namely, the electric field intensity **E**. Furthermore, only the permittivity of free space ϵ_0, of the three universal constants mentioned in Section 1–3 enters into our formulation.

Electric field intensity is defined as the force per unit charge that a very small stationary test charge experiences when it is placed in a region where an electric field exists. That is,

$$\boxed{\mathbf{E} = \lim_{q \to 0} \frac{\mathbf{F}}{q} \qquad (\text{V/m}).} \tag{3–2}$$

The electric field intensity **E** is, then, proportional to and in the direction of the force **F**. If **F** is measured in newtons (N) and charge q in coulombs (C), then **E** is in newtons per coulomb (N/C), which is the same as volts per meter (V/m). The test charge

† The exponent on the distance in Coulomb's law has been verified by an indirect experiment to be 2 to within one part in 10^{15}. (See E. R. Williams, J. E. Faller, and H. A. Hall, *Phys. Rev. Letters*, vol. 26, 1971, p. 721.)

q, of course, cannot be zero in practice; as a matter of fact, it cannot be less than the charge on an electron. However, the finiteness of the test charge would not make the measured $\mathbf{E}$ differ appreciably from its calculated value if the test charge is small enough not to disturb the charge distribution of the source. An inverse relation of Eq. (3–2) gives the force $\mathbf{F}$ on a stationary charge q in an electric field $\mathbf{E}$:

$$\boxed{\mathbf{F} = q\mathbf{E} \qquad \text{(N)}.} \tag{3–3}$$

The two fundamental postulates of electrostatics in free space specify the divergence and curl of $\mathbf{E}$. They are

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}} \tag{3–4}$$

and

$$\boxed{\nabla \times \mathbf{E} = 0.} \tag{3–5}$$

In Eq. (3–4), ρ is the volume charge density of free charges (C/m^3), and ϵ_0 is the permittivity of free space, a universal constant.† Equation (3–5) asserts that ***static electric fields are irrotational***, whereas Eq. (3–4) implies that a static electric field is not solenoidal unless $\rho = 0$. These two postulates are concise, simple, and independent of any coordinate system; and they can be used to derive all other relations, laws, and theorems in electrostatics! Such is the beauty of the deductive, axiomatic approach.

Equations (3–4) and (3–5) are point relations; that is, they hold at every point in space. They are referred to as the differential form of the postulates of electrostatics, since both divergence and curl operations involve spatial derivatives. In practical applications we are usually interested in the total field of an aggregate or a distribution of charges. This is more conveniently obtained by an integral form of Eq. (3–4). Taking the volume integral of both sides of Eq. (3–4) over an arbitrary volume V, we have

$$\int_V \nabla \cdot \mathbf{E}\, dv = \frac{1}{\epsilon_0} \int_V \rho\, dv. \tag{3–6}$$

In view of the divergence theorem in Eq. (2–115), Eq. (3–6) becomes

$$\boxed{\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0},} \tag{3–7}$$

---

† The permittivity of free space $\epsilon_0 \cong \dfrac{1}{36\pi} \times 10^{-9}$ (F/m). See Eq. (1–11).

where Q is the total charge contained in volume V bounded by surface S. Equation (3–7) is a form of ***Gauss's law***, which states that ***the total outward flux of the electric field intensity over any closed surface in free space is equal to the total charge enclosed in the surface divided by*** ϵ_0. Gauss's law is one of the most important relations in electrostatics. We will discuss it further in Section 3–4, along with illustrative examples.

An integral form can also be obtained for the curl relation in Eq. (3–5) by integrating $\nabla \times \mathbf{E}$ over an open surface and invoking Stokes's theorem as expressed in Eq. (2–143). We have

$$\boxed{\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0.} \tag{3–8}$$

The line integral is performed over a closed contour C bounding an arbitrary surface; hence C is itself arbitrary. As a matter of fact, the surface does not even enter into Eq. (3–8), which asserts that ***the scalar line integral of the static electric field intensity around any closed path vanishes***. The scalar product $\mathbf{E} \cdot d\boldsymbol{\ell}$ integrated over any path is the voltage along that path. Thus Eq. (3–8) is an expression of ***Kirchhoff's voltage law*** in circuit theory that ***the algebraic sum of voltage drops around any closed circuit is zero***. This will be discussed again in Section 5–3.

Equation (3–8) is another way of saying that $\mathbf{E}$ is irrotational (conservative). Referring to Fig. 3–1, we see that if the scalar line integral of $\mathbf{E}$ over the arbitrary closed contour C_1C_2 is zero, then

$$\int_{C_1} \mathbf{E} \cdot d\boldsymbol{\ell} + \int_{C_2} \mathbf{E} \cdot d\boldsymbol{\ell} = 0 \tag{3–9}$$

or

$$\underset{\text{Along } C_1}{\int_{P_1}^{P_2}} \mathbf{E} \cdot d\boldsymbol{\ell} = -\underset{\text{Along } C_2}{\int_{P_2}^{P_1}} \mathbf{E} \cdot d\boldsymbol{\ell} \tag{3–10}$$

or

$$\underset{\text{Along } C_1}{\int_{P_1}^{P_2}} \mathbf{E} \cdot d\boldsymbol{\ell} = \underset{\text{Along } C_2}{\int_{P_1}^{P_2}} \mathbf{E} \cdot d\boldsymbol{\ell}. \tag{3–11}$$

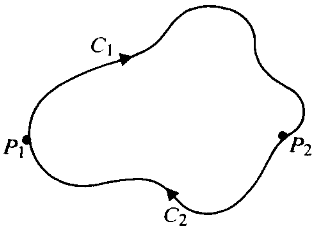


FIGURE 3–1
An arbitrary contour.

Equation (3–11) says that the scalar line integral of the irrotational **E** field is independent of the path; it depends only on the end points. As we shall see in Section 3–5, the integrals in Eq. (3–11) represent the work done *by* the electric field in moving a unit charge from point P_1 to point P_2; hence Eqs. (3–8) and (3–9) imply a statement of conservation of work or energy in an electrostatic field.

The two fundamental postulates of electrostatics in free space are repeated below because they form the foundation upon which we build the structure of electrostatics.

Postulates of Electrostatics in Free Space	
Differential Form	Integral Form
$\nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}$	$\oint_S \mathbf{E} \cdot d\mathbf{s} = \dfrac{Q}{\epsilon_0}$
$\nabla \times \mathbf{E} = 0$	$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0$

We consider these postulates, like the principle of conservation of charge, to be representations of laws of nature. In the following section we shall *derive* Coulomb's law.

3–3 Coulomb's Law

We consider the simplest possible electrostatic problem of a single point charge, q, at rest in a boundless free space. In order to find the electric field intensity due to q, we draw a hypothetical spherical surface of a radius R centered at q. Since a point charge has no preferred directions, its electric field must be everywhere radial and has the same intensity at all points on the spherical surface. Applying Eq. (3–7) to Fig. 3–2(a), we have

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \oint_S (\mathbf{a}_R E_R) \cdot \mathbf{a}_R \, ds = \frac{q}{\epsilon_0}$$

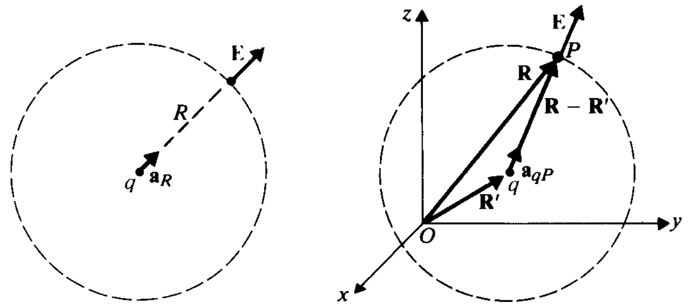
or

$$E_R \oint_S ds = E_R(4\pi R^2) = \frac{q}{\epsilon_0}.$$

Therefore,

$$\boxed{\mathbf{E} = \mathbf{a}_R E_R = \mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2} \qquad \text{(V/m).}} \tag{3–12}$$

Equation (3–12) tells us that ***the electric field intensity of a positive point charge is in the outward radial direction and has a magnitude proportional to the charge and inversely proportional to the square of the distance from the charge.*** This is a very important basic formula in electrostatics. Using Eq. (2–139), we can verify that



(a) Point charge at the origin. (b) Point charge not at the origin.

FIGURE 3–2
Electric field iFIGUREdue to a point charge.

$\nabla \times \mathbf{E} = 0$ for the $\mathbf{E}$ given in Eq. (3–12). A flux-line graph for the electric field intensity of a positive point charge q will look like Fig. 2–25(b).

If the charge q is not located at the origin of a chosen coordinate system, suitable changes should be made to the unit vector $\mathbf{a}_R$ and the distance R to reflect the locations of the charge and of the point at which $\mathbf{E}$ is to be determined. Let the position vector of q be $\mathbf{R}'$ and that of a field point P be $\mathbf{R}$, as shown in Fig. 3–2(b). Then, from Eq. (3–12),

$$\mathbf{E}_P = \mathbf{a}_{qP}\frac{q}{4\pi\epsilon_0|\mathbf{R}-\mathbf{R}'|^2}, \tag{3–13}$$

where $\mathbf{a}_{qP}$ is the unit vector drawn from q to P. Since

$$\mathbf{a}_{qP} = \frac{\mathbf{R}-\mathbf{R}'}{|\mathbf{R}-\mathbf{R}'|} \tag{3–14}$$

we have

$$\boxed{\mathbf{E}_p = \frac{q(\mathbf{R}-\mathbf{R}')}{4\pi\epsilon_0|\mathbf{R}-\mathbf{R}'|^3} \qquad \text{(V/m)}.} \tag{3–15}$$

EXAMPLE 3–1 Determine the electric field intensity at $P(-0.2, 0, -2.3)$ due to a point charge of $+5$ (nC) at $Q(0.2, 0.1, -2.5)$ in air. All dimensions are in meters.

Solution The position vector for the field point P

$$\mathbf{R} = \overrightarrow{OP} = -\mathbf{a}_x 0.2 - \mathbf{a}_z 2.3.$$

The position vector for the point charge Q is

$$\mathbf{R}' = \overrightarrow{OQ} = \mathbf{a}_x 0.2 + \mathbf{a}_y 0.1 - \mathbf{a}_z 2.5.$$

The difference is

$$\mathbf{R}-\mathbf{R}' = -\mathbf{a}_x 0.4 - \mathbf{a}_y 0.1 + \mathbf{a}_z 0.2,$$

which has a magnitude

$$|\mathbf{R}-\mathbf{R}'| = [(-0.4)^2 + (-0.1)^2 + (0.2)^2]^{1/2} = 0.458 \text{ (m)}.$$

Substituting in Eq. (3–15), we obtain

$$\begin{aligned}\mathbf{E}_P &= \left(\frac{1}{4\pi\epsilon_0}\right)\frac{q(\mathbf{R}-\mathbf{R}')}{|\mathbf{R}-\mathbf{R}'|^3}\\ &= (9\times 10^9)\frac{5\times 10^{-9}}{0.458^3}(-\mathbf{a}_x 0.4 - \mathbf{a}_y 0.1 + \mathbf{a}_z 0.2)\\ &= 214.5(-\mathbf{a}_x 0.873 - \mathbf{a}_y 0.218 + \mathbf{a}_z 0.437) \quad \text{(V/m)}.\end{aligned}$$

The quantity within the parentheses is the unit vector $\mathbf{a}_{QP} = (\mathbf{R}-\mathbf{R}')/|\mathbf{R}-\mathbf{R}'|$, and $\mathbf{E}_P$ has a magnitude of 214.5 (V/m). ∎

Note: The permittivity of air is essentially the same as that of the free space. The factor $1/(4\pi\epsilon_0)$ appears very frequently in electrostatics. From Eq. (1–11) we know that $\epsilon_0 = 1/(c^2\mu_0)$. But $\mu_0 = 4\pi \times 10^{-7}$ (H/m) in SI units; so

$$\frac{1}{4\pi\epsilon_0} = \frac{\mu_0 c^2}{4\pi} = 10^{-7}\, c^2 \qquad \text{(m/F)} \tag{3–16}$$

exactly. If we use the approximate value $c = 3\times 10^8$ (m/s), then $1/(4\pi\epsilon_0) = 9 \times 10^9$ (m/F).

When a point charge q_2 is placed in the field of another point charge q_1 at the origin, a force $\mathbf{F}_{12}$ is experienced by q_2 due to electric field intensity $\mathbf{E}_{12}$ of q_1 at q_2. Combining Eqs. (3–3) and (3–12), we have

$$\boxed{\mathbf{F}_{12} = q_2\mathbf{E}_{12} = \mathbf{a}_R\frac{q_1 q_2}{4\pi\epsilon_0 R^2} \qquad \text{(N)}.} \tag{3–17}$$

Equation (3–17) is a mathematical form of ***Coulomb's law*** already stated in Section 3–1 in conjunction with Eq. (3–1). Note that the exponent on R is *exactly* 2, which is a consequence of the fundamental postulate Eq. (3–4). In SI units the proportionality constant k equals $1/(4\pi\epsilon_0)$, and the force is in newtons (N).

EXAMPLE 3–2 A total charge Q is put on a thin spherical shell of radius b. Determine the electric field intensity at an arbitrary point inside the shell.

Solution We shall solve this problem in two ways.

a) At any point, such as P, inside the hollow shell shown in Fig. 3–3, an arbitrary hypothetical closed surface (a ***Gaussian surface***) may be drawn, over which we apply Gauss's law, Eq. (3–7). Since no charge exists inside the shell and the surface is arbitrary, we conclude readily that $\mathbf{E} = 0$ everywhere inside the shell.

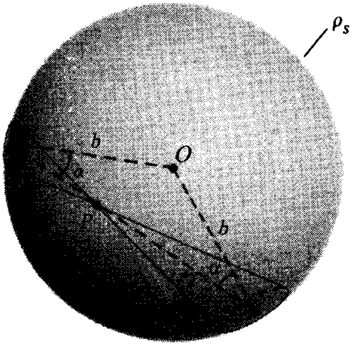


FIGURE 3–3
A charged shell (Example 3–2).

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b) Let us now examine the problem in more detail. Draw a pair of elementary cones of solid angle $d\Omega$ with vertex at an arbitrary point P. The cones extend in both directions, intersecting the shell in areas ds_1 and ds_2 at distances r_1 and r_2, respectively, from the point P. Since charge Q distributes uniformly over the spherical shell, there is a uniform surface charge density

$$\rho_s = \frac{Q}{4\pi b^2}. \tag{3-18}$$

The magnitude of the electric field intensity at P due to charges on the elementary surfaces ds_1 and ds_2 is, from Eq. (3–12),

$$dE = \frac{\rho_s}{4\pi\epsilon_0}\left(\frac{ds_1}{r_1^2} - \frac{ds_2}{r_2^2}\right). \tag{3-19}$$

But the solid angle $d\Omega$ equals

$$d\Omega = \frac{ds_1}{r_1^2}\cos\alpha = \frac{ds_2}{r_2^2}\cos\alpha. \tag{3-20}$$

Combining the expressions of dE and $d\Omega$, we find that

$$dE = \frac{\rho_S}{4\pi\epsilon_0}\left(\frac{d\Omega}{\cos\alpha} - \frac{d\Omega}{\cos\alpha}\right) = 0. \tag{3-21}$$

Since the above result applies to every pair of elementary cones, we conclude that $\mathbf{E} = 0$ everywhere inside the conducting shell, as before. ■

It will be noted that if Coulomb's law as expressed in Eq. (3–12) and used in Eq. (3–19) was slightly different from an inverse-square relation, the substitution of Eq. (3–20), which is a geometrical relation, in Eq. (3–19) would not yield the result $dE = 0$. Consequently, the electric field intensity inside the shell would not vanish; indeed, it would vary with the location of the point P. Coulomb originally used a torsion balance to conduct his experiments, which were necessarily of limited accuracy. Nevertheless, he was brilliant enough to *postulate* the inverse-square law. Many scientists subsequently made use of the vanishing field inside a spherical shell illustrated in this example to verify the inverse-square law. The field inside a charged shell, if it existed, could be detected to a very high accuracy by a probe through a small hole in the shell.

EXAMPLE 3–3 The electrostatic deflection system of a cathode-ray oscilloscope is depicted in Fig. 3–4. Electrons from a heated cathode are given an initial velocity $\mathbf{u}_0 = \mathbf{a}_z u_0$ by a positively charged anode (not shown). The electrons enter at $z = 0$ into a region of deflection plates where a uniform electric field $\mathbf{E}_d = -\mathbf{a}_y E_d$ is maintained over a width w. Ignoring gravitational effects, find the vertical deflection of the electrons on the fluorescent screen at $z = L$.

Solution Since there is no force in the z-direction in the $z > 0$ region, the horizontal velocity u_0 is maintained. The field $\mathbf{E}_d$ exerts a force on the electrons each carrying a charge $-e$, causing a deflection in the y-direction:

$$\mathbf{F} = (-e)\mathbf{E}_d = \mathbf{a}_y e E_d.$$

From Newton's second law of motion in the vertical direction we have

$$m \frac{du_y}{dt} = eE_d,$$

where m is the mass of an electron. Integrating both sides, we obtain

$$u_y = \frac{dy}{dt} = \frac{e}{m} E_d t,$$

where the constant of integration is set to zero because $u_y = 0$ at $t = 0$. Integrating again, we have

$$y = \frac{e}{2m} E_d t^2.$$

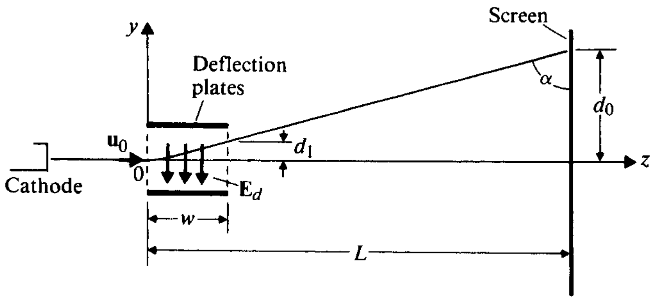


FIGURE 3–4
Electrostatic deflection system of a cathode-ray oscilloscope (Example 3–3)

The constant of integration is again zero because $y = 0$ at $t = 0$. Note that the electrons have a parabolic trajectory between the deflection plates. At the exit from the deflection plates, $t = w/u_0$,

$$d_1 = \frac{eE_d}{2m}\left(\frac{w}{u_0}\right)^2$$

and

$$u_{y1} = u_y\left(t = \frac{w}{u_0}\right) = \frac{eE_d}{m}\left(\frac{w}{u_0}\right).$$

When the electrons reach the screen, they have traveled a further horizontal distance of $(L - w)$ which takes $(L - w)/u_0$ seconds. During that time there is an additional vertical deflection

$$d_2 = u_{y1}\left(\frac{L - w}{u_0}\right) = \frac{eE_d}{m}\frac{w(L - w)}{u_0^2}.$$

Hence the deflection at the screen is

$$d_0 = d_1 + d_2 = \frac{eE_d}{mu_0^2}\, w\left(L - \frac{w}{2}\right).$$

■

Ink-jet printers used in computer output, like cathode-ray oscilloscopes, are devices based on the principle of electrostatic deflection of a stream of charged particles. Minute droplets of ink are forced through a vibrating nozzle controlled by a piezoelectric transducer. The output of the computer imparts variable amounts of charges on the ink droplets, which then pass through a pair of deflection plates where a uniform static electric field exists. The amount of droplet deflection depends on the charge it carries, causing the ink jet to strike the print surface and form an image as the print head moves in a horizontal direction.

3–3.1 ELECTRIC FIELD DUE TO A SYSTEM OF DISCRETE CHARGES

Suppose an electrostatic field is created by a group of n discrete point charges q_1, $q_2, \ldots, q_n$ located at different positions. Since electric field intensity is a linear function of (proportional to) $\mathbf{a}_R q/R^2$, the principle of superposition applies, and the total **E** field at a point is the *vector sum* of the fields caused by all the individual charges. From Eq. (3–15) we can write the electric intensity at a field point whose position vector is **R** as

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{n}\frac{q_k(\mathbf{R} - \mathbf{R}'_k)}{|\mathbf{R} - \mathbf{R}'_k|^3} \quad \text{(V/m).}} \qquad (3\text{–}22)$$

Although Eq. (3–22) is a succinct expression, it is somewhat inconvenient to use because of the need to add vectors of different magnitudes and directions.

Let us consider the simple case of an ***electric dipole*** that consists of a pair of equal and opposite charges $+q$ and $-q$, separated by a small distance, d, as shown in Fig. 3–5. Let the center of the dipole coincide with the origin of a spherical coordinate system. Then the E field at the point P is the sum of the contributions due to $+q$ and $-q$. Thus,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\frac{\mathbf{R}-\dfrac{\mathbf{d}}{2}}{\left|\mathbf{R}-\dfrac{\mathbf{d}}{2}\right|^3} - \frac{\mathbf{R}+\dfrac{\mathbf{d}}{2}}{\left|\mathbf{R}+\dfrac{\mathbf{d}}{2}\right|^3}\right\}. \tag{3–23}$$

The first term on the right side of Eq. (3–23) can be simplified if $d \ll R$. We write

$$\begin{aligned}\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^{-3} &= \left[\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)\cdot\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)\right]^{-3/2}\\ &= \left[R^2 - \mathbf{R}\cdot\mathbf{d} + \frac{d^2}{4}\right]^{-3/2}\\ &\cong R^{-3}\left[1 - \frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right]^{-3/2}\\ &\cong R^{-3}\left[1 + \frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right],\end{aligned} \tag{3–24}$$

where the binomial expansion has been used and all terms containing the second and higher powers of (d/R) have been neglected. Similarly, for the second term on the right side of Eq. (3–23) we have

$$\left|\mathbf{R}+\frac{\mathbf{d}}{2}\right|^{-3} \cong R^{-3}\left[1 - \frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right]. \tag{3–25}$$

Substitution of Eqs. (3–24) and (3–25) in Eq. (3–23) leads to

$$\mathbf{E} \cong \frac{q}{4\pi\epsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\mathbf{R} - \mathbf{d}\right]. \tag{3–26}$$

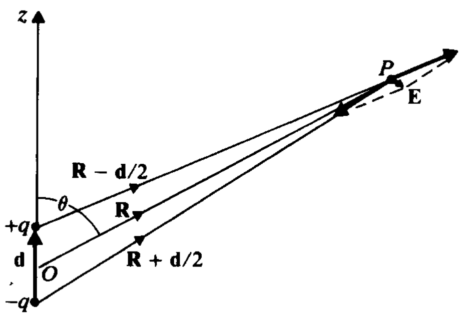


FIGURE 3–5
Electric field of a dipole.

The derivation and interpretation of Eq. (3–26) require the manipulation of vector quantities. We can appreciate that determining the electric field caused by three or more discrete charges will be even more tedious. In Section 3–5 we will introduce the concept of a scalar electric potential, with which the electric field intensity caused by a distribution of charges can be found more easily.

The electric dipole is an important entity in the study of the electric field in dielectric media. We define the product of the charge q and the vector $\mathbf{d}$ (going from $-q$ to $+q$) as the ***electric dipole moment***, $\mathbf{p}$:

$$\mathbf{p} = q\mathbf{d}. \tag{3–27}$$

Equation (3–26) can then be rewritten as

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{p}}{R^2}\mathbf{R} - \mathbf{p}\right], \tag{3–28}$$

where the approximate sign ($\sim$) over the equal sign has been left out for simplicity. If the dipole lies along the z-axis as in Fig. 3–5, then (see Eq. 2–77)

$$\mathbf{p} = \mathbf{a}_z p = p(\mathbf{a}_R \cos\theta - \mathbf{a}_\theta \sin\theta), \tag{3–29}$$

$$\mathbf{R}\cdot\mathbf{p} = Rp\cos\theta, \tag{3–30}$$

and Eq. (3–28) becomes

$$\boxed{\mathbf{E} = \frac{p}{4\pi\epsilon_0 R^3}(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta) \quad \text{(V/m)}.} \tag{3–31}$$

Equation (3–31) gives the electric field intensity of an electric dipole in spherical coordinates. We see that $\mathbf{E}$ of a dipole is inversely proportional to the cube of the distance R. This is reasonable because as R increases, the fields due to the closely spaced $+q$ and $-q$ tend to cancel each other more completely, thus decreasing more rapidly than that of a single point charge.

3–3.2 ELECTRIC FIELD DUE TO A CONTINUOUS DISTRIBUTION OF CHARGE

The electric field caused by a continuous distribution of charge can be obtained by integrating (superposing) the contribution of an element of charge over the charge distribution. Refer to Fig. 3–6, where a volume charge distribution is shown. The volume charge density ρ (C/m^3) is a function of the coordinates. Since a differential element of charge behaves like a point charge, the contribution of the charge $\rho\,dv'$ in a differential volume element dv' to the electric field intensity at the field point P is

$$d\mathbf{E} = \mathbf{a}_R \frac{\rho\,dv'}{4\pi\epsilon_0 R^2}. \tag{3–32}$$

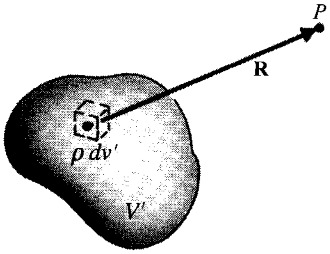


FIGURE 3–6
Electric field due to a continuous charge distribution.

We have

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{V'} \mathbf{a}_R \frac{\rho}{R^2}\, dv' \qquad \text{(V/m)}, \tag{3-33}$$

or, since $\mathbf{a}_R = \mathbf{R}/R$,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{V'} \rho \frac{\mathbf{R}}{R^3}\, dv' \qquad \text{(V/m)}. \tag{3-34}$$

Except for some especially simple cases, the vector triple integral in Eq. (3–33) or Eq. (3–34) is difficult to carry out because, in general, all three quantities in the integrand ($\mathbf{a}_R$, ρ, and R) change with the location of the differential volume dv'.

If the charge is distributed on a surface with a surface charge density ρ_s (C/m^2), then the integration is to be carried out over the surface (not necessarily flat). Thus,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{S'} \mathbf{a}_R \frac{\rho_s}{R^2}\, ds' \qquad \text{(V/m)}. \tag{3-35}$$

For a line charge we have

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{L'} \mathbf{a}_R \frac{\rho_\ell}{R^2}\, d\ell' \qquad \text{(V/m)}, \tag{3-36}$$

where ρ_ℓ (C/m) is the line charge density, and L' the line (not necessarily straight) along which the charge is distributed.

EXAMPLE 3–4 Determine the electric field intensity of an infinitely long, straight, line charge of a uniform density ρ_ℓ in air.

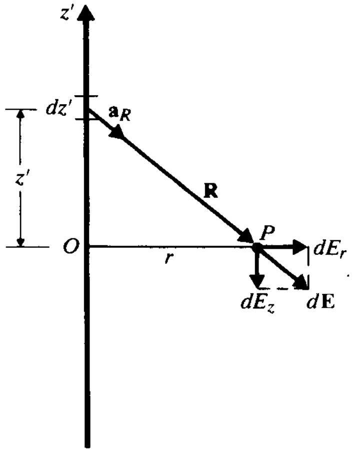


FIGURE 3–7
An infinitely long, straight, line charge.

Solution Let us assume that the line charge lies along the z'-axis as shown in Fig. 3–7. (We are perfectly free to do this because the field obviously does not depend on how we designate the line. *It is an accepted convention to use primed coordinates for source points and unprimed coordinates for field points when there is a possibility of confusion.*) The problem asks us to find the electric field intensity at a point P, which is at a distance r from the line. Since the problem has a cylindrical symmetry (that is, the electric field is independent of the azimuth angle ϕ), it would be most convenient to work with cylindrical coordinates. We rewrite Eq. (3–36) as

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{L'} \rho_\ell \frac{\mathbf{R}}{R^3}\,d\ell' \qquad (\text{V/m}).} \tag{3–37}$$

For the problem at hand, ρ_ℓ is constant, and a line element $d\ell' = dz'$ is chosen to be at an arbitrary distance z' from the origin. It is most important to remember that $\mathbf{R}$ is the distance vector directed *from the source to the field point*, not the other way around. We have

$$\mathbf{R} = \mathbf{a}_r r - \mathbf{a}_z z'. \tag{3–38}$$

The electric field, $d\mathbf{E}$, due to the differential line charge element $\rho_\ell\, d\ell' = \rho_\ell\, dz'$ is

$$\begin{aligned} d\mathbf{E} &= \frac{\rho_\ell\, dz'}{4\pi\epsilon_0}\frac{\mathbf{a}_r r - \mathbf{a}_z z'}{(r^2 + z'^2)^{3/2}} \\ &= \mathbf{a}_r\, dE_r + \mathbf{a}_z\, dE_z, \end{aligned} \tag{3–39}$$

where

$$dE_r = \frac{\rho_\ell r\, dz'}{4\pi\epsilon_0(r^2 + z'^2)^{3/2}} \tag{3–39a}$$

and

$$dE_z = \frac{-\rho_\ell z'\,dz'}{4\pi\epsilon_0(r^2 + z'^2)^{3/2}}. \tag{3–39b}$$

In Eq. (3–39) we have decomposed $d\mathbf{E}$ into its components in the $\mathbf{a}_r$ and $\mathbf{a}_z$ directions. It is easy to see that for every $\rho_\ell\,dz'$ at $+z'$ there is a charge element $\rho_\ell\,dz'$ at $-z'$, which will produce a $d\mathbf{E}$ with components dE_r and $-dE_z$. Hence the $\mathbf{a}_z$ components will cancel in the integration process, and we only need to integrate the dE_r in Eq. (3–39a):

$$\mathbf{E} = \mathbf{a}_r E_r = \mathbf{a}_r \frac{\rho_\ell r}{4\pi\epsilon_0}\int_{-\infty}^{\infty} \frac{dz'}{(r^2 + z'^2)^{3/2}}$$

or

$$\boxed{\mathbf{E} = \mathbf{a}_r \frac{\rho_\ell}{2\pi\epsilon_0 r} \qquad \text{(V/m)}.} \tag{3–40}$$

Equation (3–40) is an important result for an infinite line charge. Of course, no physical line charge is infinitely long; nevertheless, Eq. (3–40) gives the approximate $\mathbf{E}$ field of a long straight line charge at a point close to the line charge.

3–4 Gauss's Law and Applications

Gauss's law follows directly from the divergence postulate of electrostatics, Eq. (3–4), by the application of the divergence theorem. It was derived in Section 3–2 as Eq. (3–7) and is repeated here on account of its importance:

$$\boxed{\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}.} \tag{3–41}$$

Gauss's law asserts that the total outward flux of the* E*-field over any closed surface in free space is equal to the total charge enclosed in the surface divided by ϵ_0. We note that the surface S can be *any hypothetical (mathematical) closed surface chosen for convenience*; it does not have to be, and usually is not, a physical surface.

Gauss's law is particularly useful in determining the E-field of charge distributions with some symmetry conditions, such that *the normal component of the electric field intensity is constant over an enclosed surface.* In such cases the surface integral on the left side of Eq. (3–41) would be very easy to evaluate, and Gauss's law would be a much more efficient way for finding the electric field intensity than Eqs. (3–33) through (3–37). On the other hand, when symmetry conditions do not exist, Gauss's law would not be of much help. The essence of applying Gauss's law lies first in the recognition of symmetry conditions and second in the suitable choice of a surface over which the normal component of $\mathbf{E}$ resulting from a given charge distribution is a constant. Such a surface is referred to as a ***Gaussian surface***. This basic principle was used to obtain Eq. (3–12) for a point charge that possesses spherical symmetry; consequently, a proper Gaussian surface is the surface of a sphere centered at the point charge. Gauss's law could not help in the derivation of Eq. (3–26) or (3–31) for an electric dipole, since a surface about a separated pair of equal and opposite charges over which the normal component of **E** remains constant was not known.

EXAMPLE 3–5 Use Gauss's law to determine the electric field intensity of an infinitely long, straight, line charge of a uniform density ρ_ℓ in air.

Solution This problem was solved in Example 3–4 by using Eq. (3–36). Since the line charge is infinitely long, the resultant **E** field must be radial and perpendicular to the line charge ($\mathbf{E} = \mathbf{a}_r E_r$), and a component of **E** along the line cannot exist. With the obvious cylindrical symmetry we construct a cylindrical Gaussian surface of a radius r and an arbitrary length L with the line charge as its axis, as shown in Fig. 3–8. On this surface, E_r is constant, and $d\mathbf{s} = \mathbf{a}_r r\, d\phi\, dz$ (from Eq. 2–53a). We have

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \int_0^L \int_0^{2\pi} E_r r\, d\phi\, dz = 2\pi r L E_r.$$

There is no contribution from the top or the bottom face of the cylinder because on the top face $d\mathbf{s} = \mathbf{a}_z r\, dr\, d\phi$ but **E** has no z-component there, making $\mathbf{E} \cdot d\mathbf{s} = 0$. Similarly for the bottom face. The total charge enclosed in the cylinder is $Q = \rho_\ell L$. Substitution into Eq. (3–41) gives us immediately

$$2\pi r L E_r = \frac{\rho_\ell L}{\epsilon_0}$$

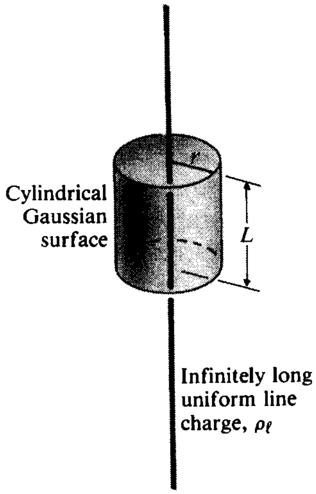


FIGURE 3–8
Applying Gauss's law to an infinitely long line charge (Example 3–5).

or

$$\mathbf{E} = \mathbf{a}_r E_r = \mathbf{a}_r \frac{\rho_\ell}{2\pi\epsilon_0 r}.$$

This result is, of course, the same as that given in Eq. (3–40), but it is obtained here in a much simpler way. We note that the length L of the cylindrical Gaussian surface does not appear in the final expression; hence we could have chosen a cylinder of a unit length.

EXAMPLE 3–6 Determine the electric field intensity of an infinite planar charge with a uniform surface charge density ρ_s.

Solution It is clear that the **E** field caused by a charged sheet of an infinite extent is normal to the sheet. Equation (3–35) could be used to find **E**, but this would involve a double integration between infinite limits of a general expression of $1/R^2$. Gauss's law can be used to much advantage here.

We choose as the Gaussian surface a rectangular box with top and bottom faces of an arbitrary area A equidistant from the planar charge, as shown in Fig. 3–9. The sides of the box are perpendicular to the charged sheet. If the charged sheet coincides with the xy-plane, then on the top face,

$$\mathbf{E} \cdot d\mathbf{s} = (\mathbf{a}_z E_z) \cdot (\mathbf{a}_z\, ds) = E_z\, ds.$$

On the bottom face,

$$\mathbf{E} \cdot d\mathbf{s} = (-\mathbf{a}_z E_z) \cdot (-\mathbf{a}_z\, ds) = E_z\, ds.$$

Since there is no contribution from the side faces, we have

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = 2E_z \int_A ds = 2E_z A.$$

The total charge enclosed in the box is $Q = \rho_s A$. Therefore,

$$2E_z A = \frac{\rho_s A}{\epsilon_0},$$

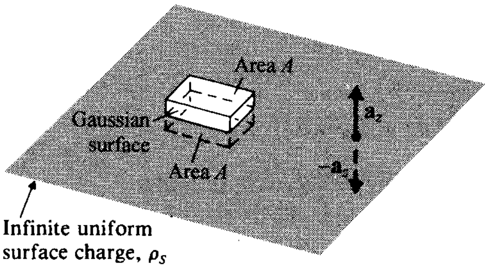


FIGURE 3–9
Applying Gauss's law to an infinite planar charge (Example 3–6).

from which we obtain

$$\mathbf{E} = \mathbf{a}_z E_z = \mathbf{a}_z \frac{\rho_s}{2\epsilon_0}, \qquad z > 0, \tag{3-42a}$$

and

$$\mathbf{E} = -\mathbf{a}_z E_z = -\mathbf{a}_z \frac{\rho_s}{2\epsilon_0}, \qquad z < 0. \tag{3-42b}$$

Of course, the charged sheet may not coincide with the xy-plane (in which case we do not speak in terms of above and below the plane), but the **E** field always points *away* from the sheet if ρ_s is *positive*. It is obvious that the Gaussian surface could have been a pillbox of any shape, not necessarily rectangular. ■

The lighting scheme of an office or a classroom may consist of incandescent bulbs, long fluorescent tubes, or ceiling panel lights. These correspond roughly to point sources, line sources, and planar sources, respectively. From Eqs. (3–12), (3–40), and (3–42) we can estimate that light intensity will fall off rapidly as the square of the distance from the source in the case of incandescent bulbs, less rapidly as the first power of the distance for long fluorescent tubes, and not at all for ceiling panel lights.

EXAMPLE 3–7 Determine the **E** field caused by a spherical cloud of electrons with a volume charge density $\rho = -\rho_o$ for $0 \leq R \leq b$ (both ρ_o and b are positive) and $\rho = 0$ for $R > b$.

Solution First we recognize that the given source condition has spherical symmetry. The proper Gaussian surfaces must therefore be concentric spherical surfaces. We must find the **E** field in two regions. Refer to Fig. 3–10.

a) $0 \leq R \leq b$

A hypothetical spherical Gaussian surface S_i with $R < b$ is constructed within the electron cloud. On this surface, **E** is radial and has a constant magnitude:

$$\mathbf{E} = \mathbf{a}_R E_R, \qquad d\mathbf{s} = \mathbf{a}_R\, ds.$$

The total outward E flux is

$$\oint_{S_i} \mathbf{E} \cdot d\mathbf{s} = E_R \int_{S_i} ds = E_R 4\pi R^2.$$

The total charge enclosed within the Gaussian surface is

$$\begin{aligned} Q &= \int_V \rho\, dv \\ &= -\rho_o \int_V dv = -\rho_o \frac{4\pi}{3} R^3. \end{aligned}$$

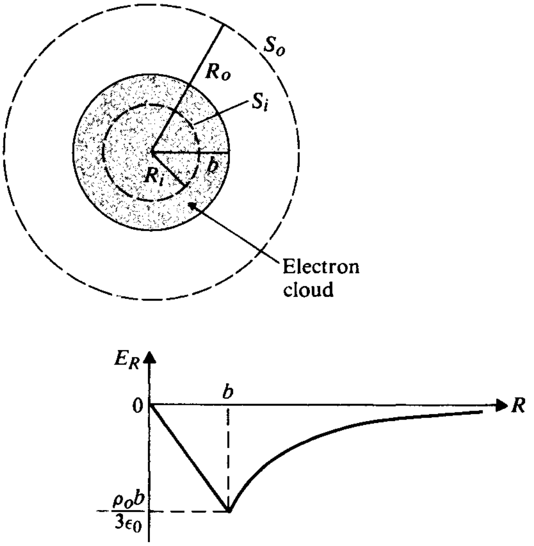


FIGURE 3–10
Electric field intensity of a spherical electron cloud (Example 3–7).

Substitution into Eq. (3–7) yields

$$\mathbf{E} = -\mathbf{a}_R \frac{\rho_o}{3\epsilon_0} R, \qquad 0 \leq R \leq b.$$

We see that within the uniform electron cloud the **E** field is directed toward the center and has a magnitude proportional to the distance from the center.

b) $R \geq b$
For this case we construct a spherical Gaussian surface S_o with $R > b$ outside the electron cloud. We obtain the same expression for $\oint_{S_o} \mathbf{E} \cdot d\mathbf{s}$ as in case (a). The total charge enclosed is

$$Q = -\rho_o \frac{4\pi}{3} b^3.$$

Consequently,

$$\mathbf{E} = -\mathbf{a}_R \frac{\rho_o b^3}{3\epsilon_0 R^2}, \qquad R \geq b,$$

which follows the inverse square law and could have been obtained directly from Eq. (3–12). We observe that *outside* the charged cloud the **E** field is exactly the same as though the total charge is concentrated on a single point charge at the center. This is true, in general, for a spherically symmetrical charged region even though ρ is a function of R. ■

The variation of E_R versus R is plotted in Fig. 3–10. Note that the formal solution of this problem requires only a few lines. If Gauss's law is not used, it is necessary (1) to choose a differential volume element arbitrarily located in the electron cloud, (2) to express its vector distance **R** to a field point in a chosen coordinate system, and (3) to perform a triple integration as indicated in Eq. (3–33). This is a hopelessly involved process. The moral is: *Try to apply Gauss's law if symmetry conditions exist for the given charge distribution.*

3–5 Electric Potential

In connection with the null identity in Eq. (2–145) we noted that a curl-free vector field could always be expressed as the gradient of a scalar field. This induces us to *define* a scalar ***electric potential*** V such that

$$\boxed{\mathbf{E} = -\nabla V} \qquad (3\text{–}43)$$

because scalar quantities are easier to handle than vector quantities. If we can determine V more easily, then **E** can be found by a gradient operation, which is a straightforward process in an orthogonal coordinate system. The reason for the inclusion of a negative sign in Eq. (3–43) will be explained presently.

Electric potential does have physical significance, and it is related to the work done in carrying a charge from one point to another. In Section 3–2 we defined the electric field intensity as the force acting on a unit test charge. Therefore in moving a unit charge from point P_1 to point P_2 in an electric field, work must be done *against the field* and is equal to

$$\frac{W}{q} = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\boldsymbol{\ell} \qquad \text{(J/C or V).} \qquad (3\text{–}44)$$

Many paths may be followed in going from P_1 to P_2. Two such paths are drawn in Fig. 3–11. Since the path between P_1 and P_2 is not specified in Eq. (3–44), the

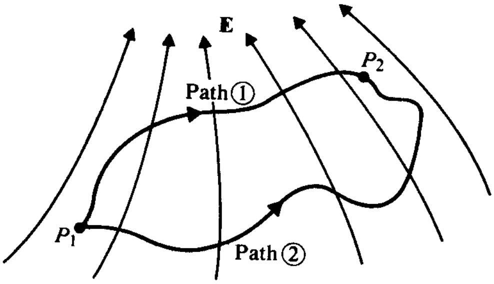


FIGURE 3–11
Two paths leading from P_1 to P_2 in an electric field.

question naturally arises, how does the work depend on the path taken? A little thought will lead us to conclude that W/q in Eq. (3–44) should not depend on the path; if it did, one would be able to go from P_1 to P_2 along a path for which W is smaller and then to come back to P_1 along another path, achieving a net gain in work or energy. This would be contrary to the principle of conservation of energy. We have already alluded to the path-independent nature of the scalar line integral of the irrotational (conservative) **E** field when we discussed Eq. (3–8).

Analogous to the concept of potential energy in mechanics, Eq. (3–44) represents the difference in electric potential energy of a unit charge between point P_2 and point P_1. Denoting the electric potential energy per unit charge by V, the ***electric potential***, we have

$$\boxed{V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E}\cdot d\ell \qquad \text{(V)}.} \tag{3–45}$$

Mathematically, Eq. (3–45) can be obtained by substituting Eq. (3–43) in Eq. (3–44). Thus, in view of Eq. (2–88),

$$\begin{aligned} -\int_{P_1}^{P_2} \mathbf{E}\cdot d\ell &= \int_{P_1}^{P_2} (\nabla V)\cdot(\mathbf{a}_\ell\, d\ell) \\ &= \int_{P_1}^{P_2} dV = V_2 - V_1. \end{aligned}$$

What we have defined in Eq. (3–45) is a ***potential difference (electrostatic voltage)*** between points P_2 and P_1. It makes no more sense to talk about the absolute potential of a point than about the absolute phase of a phasor or the absolute altitude of a geographical location; a reference zero-potential point, a reference zero phase (usually at $t = 0$), or a reference zero altitude (usually at sea level) must first be specified. In most (but not all) cases the zero-potential point is taken at infinity. When the reference zero-potential point is not at infinity, it should be specifically stated.

We want to make two more points about Eq. (3–43). First, the inclusion of the negative sign is necessary in order to conform with the convention that in going *against* the **E** field the electric potential V *increases*. For instance, when a d-c battery of a voltage V_0 is connected between two parallel conducting plates, as in Fig. 3–12, positive and negative charges cumulate on the top and bottom plates, respectively. The **E** field is directed from positive to negative charges, while the potential increases in the *opposite* direction. Second, we know from Section 2–6, when we defined the gradient of a scalar field, that the direction of ∇V is normal to the surfaces of constant

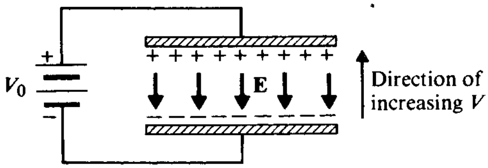


FIGURE 3–12
Relative directions of **E** and increasing V.

V. Hence if we use directed ***field lines*** or ***streamlines*** to indicate the direction of the **E** field, they are everywhere perpendicular to ***equipotential lines*** and ***equipotential surfaces.***

3-5.1 ELECTRIC POTENTIAL DUE TO A CHARGE DISTRIBUTION

The electric potential of a point at a distance R from a point charge q referred to that at infinity can be obtained readily from Eq. (3-45):

$$V = -\int_{\infty}^{R} \left(\mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2}\right) \cdot (\mathbf{a}_R\, dR), \tag{3-46}$$

which gives

$$\boxed{V = \frac{q}{4\pi\epsilon_0 R} \qquad \text{(V).}} \tag{3-47}$$

This is a scalar quantity and depends on, besides q, only the distance R. The potential difference between any two points P_2 and P_1 at distances R_2 and R_1, respectively, from q is

$$V_{21} = V_{P_2} - V_{P_1} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R_2} - \frac{1}{R_1}\right). \tag{3-48}$$

This result may appear a little surprising at first, since P_2 and P_1 may not lie on the same radial line through q, as illustrated in Fig. 3-13. However, the concentric circles (spheres) passing through P_2 and P_1 are equipotential lines (surfaces), and $V_{P_2} - V_{P_1}$ is the same as $V_{P_2} - V_{P_3}$. From the point of view of Eq. (3-45) we can choose the path of integration from P_1 to P_3 and then from P_3 to P_2. No work is done from P_1 to P_3 because **F** is perpendicular to $d\ell = \mathbf{a}_\phi R_1\, d\phi$ along the circular path ($\mathbf{E} \cdot d\ell = 0$).

The electric potential at **R** due to a system of n discrete point charges $q_1, q_2, \ldots, q_n$ located at $\mathbf{R}'_1, \mathbf{R}'_2, \ldots, \mathbf{R}'_n$ is, by superposition, the sum of the potentials due to

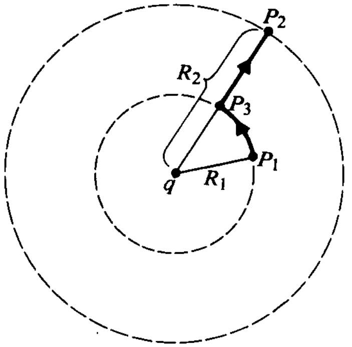


FIGURE 3-13
Path of integration about a point charge.

the individual charges:

$$\boxed{V = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{q_k}{|\mathbf{R} - \mathbf{R}'_k|} \qquad \text{(V)}.} \tag{3–49}$$

Since this is a scalar sum, it is, in general, easier to determine **E** by taking the negative gradient of V than from the vector sum in Eq. (3–22) directly.

As an example, let us again consider an electric dipole consisting of charges $+q$ and $-q$ with a small separation d. The distances from the charges to a field point P are designated R_+ and R_-, as shown in Fig. 3–14. The potential at P can be written down directly:

$$V = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right). \tag{3–50}$$

If $d \ll R$, we have

$$\frac{1}{R_+} \cong \left(R - \frac{d}{2}\cos\theta\right)^{-1} \cong R^{-1}\left(1 + \frac{d}{2R}\cos\theta\right) \tag{3–51}$$

and

$$\frac{1}{R_-} \cong \left(R + \frac{d}{2}\cos\theta\right)^{-1} \cong R^{-1}\left(1 - \frac{d}{2R}\cos\theta\right). \tag{3–52}$$

Substitution of Eqs. (3–51) and (3–52) in Eq. (3–50) gives

$$V = \frac{qd\cos\theta}{4\pi\epsilon_0 R^2} \tag{3–53a}$$

or

$$\boxed{V = \frac{\mathbf{p}\cdot\mathbf{a}_R}{4\pi\epsilon_0 R^2} \qquad \text{(V),}} \tag{3–53b}$$

where $\mathbf{p} = q\mathbf{d}$. (The "approximate" sign (~) has been dropped for simplicity.)

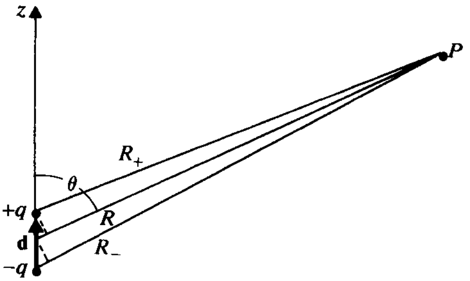


FIGURE 3–14
An electric dipole.

The E field can be obtained from $-\nabla V$. In spherical coordinates we have

$$\begin{aligned}\mathbf{E} = -\nabla V &= -\mathbf{a}_R \frac{\partial V}{\partial R} - \mathbf{a}_\theta \frac{\partial V}{R\,\partial \theta} \\ &= \frac{p}{4\pi\epsilon_0 R^3}(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta).\end{aligned} \tag{3-54}$$

Equation (3–54) is the same as Eq. (3–31) but has been obtained by a simpler procedure without manipulating position vectors.

EXAMPLE 3–8 Make a two-dimensional sketch of the equipotential lines and the electric field lines for an electric dipole.

Solution The equation of an equipotential surface of a charge distribution is obtained by setting the expression for V to equal a constant. Since q, d, and ϵ_0 in Eq. (3–53a) for an electric dipole are fixed quantities, a constant V requires a constant ratio $(\cos\theta/R^2)$. Hence the equation for an equipotential surface is

$$R = c_V\sqrt{\cos\theta}, \tag{3-55}$$

where c_V is a constant. By plotting R versus θ for various values of c_V we draw the solid equipotential lines in Fig. 3–15. In the range $0 \le \theta \le \pi/2$, V is positive; R is maximum at $\theta = 0$ and zero at $\theta = 90°$. A mirror image is obtained in the range $\pi/2 \le \theta \le \pi$ where V is negative.

The electric field lines or streamlines represent the direction of the **E** field in space. We set

$$d\boldsymbol{\ell} = k\mathbf{E}, \tag{3-56}$$

where k is a constant. In spherical coordinates, Eq. (3–56) becomes (see Eq. 2–66)

$$\mathbf{a}_R\,dR + \mathbf{a}_\theta R\,d\theta + \mathbf{a}_\phi R\sin\theta\,d\phi = k(\mathbf{a}_R E_R + \mathbf{a}_\theta E_\theta + \mathbf{a}_\phi E_\phi), \tag{3-57}$$

which can be written

$$\frac{dR}{E_R} = \frac{R\,d\theta}{E_\theta} = \frac{R\sin\theta\,d\phi}{E_\phi}. \tag{3-58}$$

For the electric dipole in Fig. 3–15 there is no E_ϕ component, and

$$\frac{dR}{2\cos\theta} = \frac{R\,d\theta}{\sin\theta}$$

or

$$\frac{dR}{R} = \frac{2\,d(\sin\theta)}{\sin\theta}. \tag{3-59}$$

Integrating Eq. (3–59), we obtain

$$R = c_E\sin^2\theta, \tag{3-60}$$

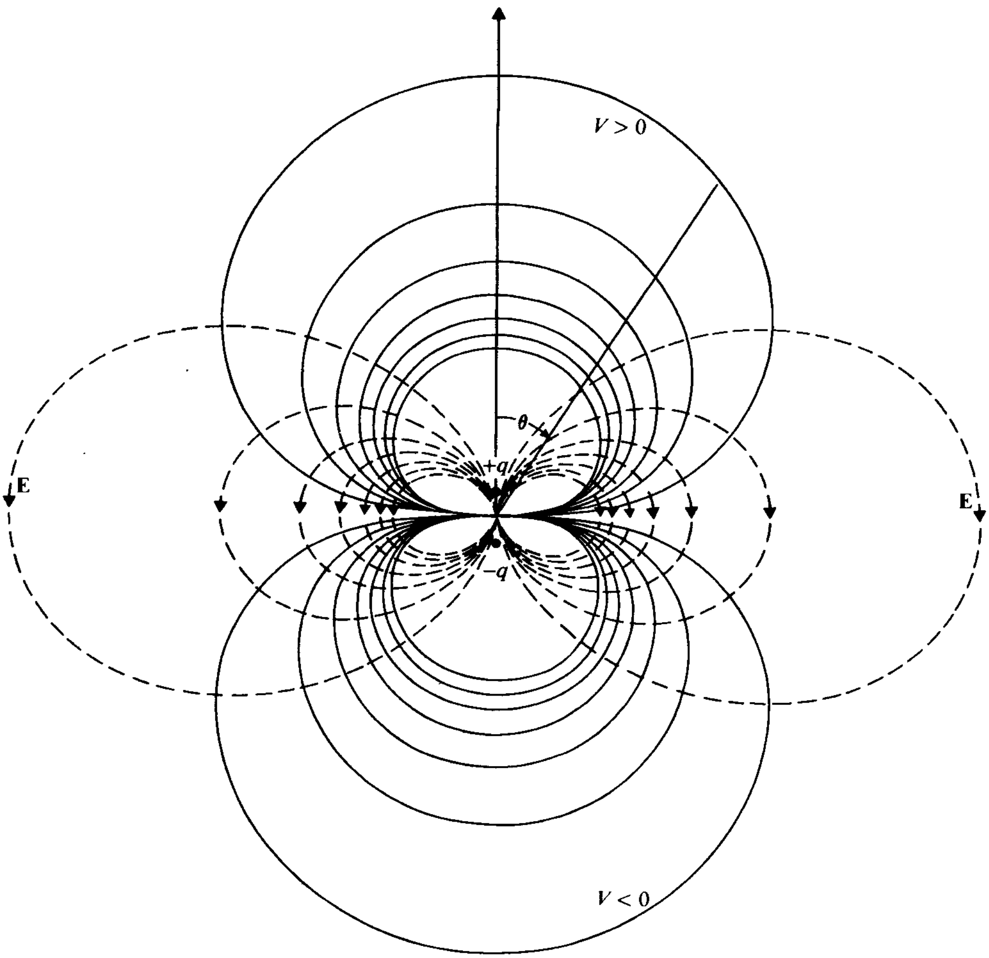


FIGURE 3–15
Equipotential and electric field lines of an electric dipole (Example 3–8).

where c_E is a constant. The electric field lines are drawn as dashed lines in Fig. 3–15. They are rotationally symmetrical about the z-axis (independent of ϕ) and are everywhere normal to the equipotential lines. ▬

The electric potential due to a continuous distribution of charge confined in a given region is obtained by integrating the contribution of an element of charge over the charged region. We have, for a volume charge distribution,

$$\boxed{V = \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{\rho}{R}\, dv' \qquad \text{(V).}} \tag{3-61}$$

For a surface charge distribution,

$$\boxed{V = \frac{1}{4\pi\epsilon_0}\int_{S'} \frac{\rho_s}{R}\, ds' \qquad \text{(V)};} \tag{3-62}$$

and for a line charge,

$$\boxed{V = \frac{1}{4\pi\epsilon_0}\int_{L'} \frac{\rho_\ell}{R}\, d\ell' \qquad \text{(V)}.} \tag{3-63}$$

We note here again that the integrals in Eqs. (3–61) and (3–62) represent integrations in three and two dimensions respectively.

EXAMPLE 3–9 Obtain a formula for the electric field intensity on the axis of a circular disk of radius b that carries a uniform surface charge density ρ_s.

Solution Although the disk has circular symmetry, we cannot visualize a surface around it over which the normal component of **E** has a constant magnitude; hence Gauss's law is not useful for the solution of this problem. We use Eq. (3–62). Working with cylindrical coordinates indicated in Fig. 3–16, we have

$$ds' = r'\, dr'\, d\phi'$$

and

$$R = \sqrt{z^2 + r'^2}.$$

The electric potential at the point $P(0, 0, z)$ referring to the point at infinity is

$$\begin{aligned} V &= \frac{\rho_s}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^b \frac{r'}{(z^2 + r'^2)^{1/2}}\, dr'\, d\phi' \\ &= \frac{\rho_s}{2\epsilon_0}\left[(z^2 + b^2)^{1/2} - |z|\right]. \end{aligned} \tag{3-64}$$

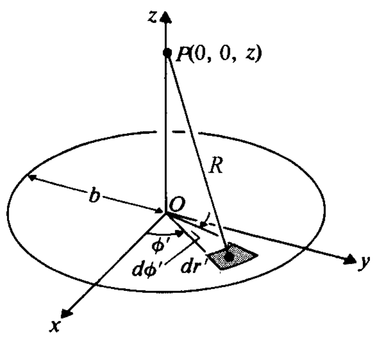


FIGURE 3–16
A uniformly charged disk (Example 3–9)

Therefore,

$$\mathbf{E} = -\nabla V = -\mathbf{a}_z \frac{\partial V}{\partial z}$$

$$= \begin{cases} \mathbf{a}_z \dfrac{\rho_s}{2\epsilon_0}[1 - z(z^2 + b^2)^{-1/2}], & z > 0 \quad (3\text{–}65a) \\ -\mathbf{a}_z \dfrac{\rho_s}{2\epsilon_0}[1 + z(z^2 + b^2)^{-1/2}], & z < 0. \quad (3\text{–}65b) \end{cases}$$

The determination of **E** field at an off-axis point would be a much more difficult problem. Do you know why?

For very large z, it is convenient to expand the second term in Eqs. (3–65a) and (3–65b) into a binomial series and neglect the second and all higher powers of the ratio (b^2/z^2). We have

$$z(z^2 + b^2)^{-1/2} = \left(1 + \frac{b^2}{z^2}\right)^{-1/2} \cong 1 - \frac{b^2}{2z^2}.$$

Substituting this into Eqs. (3–65a) and (3–65b), we obtain

$$\mathbf{E} = \mathbf{a}_z \frac{(\pi b^2 \rho_s)}{4\pi\epsilon_0 z^2}$$

$$= \begin{cases} \mathbf{a}_z \dfrac{Q}{4\pi\epsilon_0 z^2}, & z > 0 \quad (3\text{–}66a) \\ -\mathbf{a}_z \dfrac{Q}{4\pi\epsilon_0 z^2}, & z < 0, \quad (3\text{–}66b) \end{cases}$$

where Q is the total charge on the disk. Hence, when the point of observation is very far away from the charged disk, the **E** field approximately follows the inverse square law as if the total charge were concentrated at a point. ■

EXAMPLE 3–10 Obtain a formula for the electric field intensity along the axis of a uniform line charge of length L. The uniform line-charge density is ρ_ℓ.

Solution For an infinitely long line charge, the **E** field can be determined readily by applying Gauss's law, as in the solution to Example 3–5. However, for a line charge of finite length, as shown in Fig. 3–17, we cannot construct a Gaussian surface over which $\mathbf{E} \cdot d\mathbf{s}$ is constant. Gauss's law is therefore not useful here.

Instead, we use Eq. (3–63) by taking an element of charge $d\ell' = dz'$ at z'. The distance R from the charge element to the point $P(0, 0, z)$ along the axis of the line charge is

$$R = (z - z'), \qquad z > \frac{L}{2}.$$

Here it is extremely important to distinguish the position of the field point (unprimed coordinates) from the position of the source point (primed coordinates). We

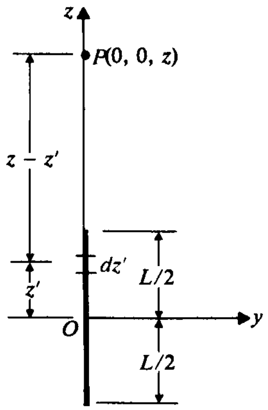


FIGURE 3–17
A finite line charge of a uniform line density ρ_ℓ (Example 3–10).

integrate over the source region:

$$\begin{aligned} V &= \frac{\rho_\ell}{4\pi\epsilon_0}\int_{-L/2}^{L/2}\frac{dz'}{z-z'} \\ &= \frac{\rho_\ell}{4\pi\epsilon_0}\ln\left[\frac{z+(L/2)}{z-(L/2)}\right], \qquad z > \frac{L}{2}. \end{aligned} \tag{3-67}$$

The **E** field at P is the negative gradient of V with respect to the *unprimed* field coordinates. For this problem,

$$\mathbf{E} = -\mathbf{a}_z\frac{dV}{dz} = \mathbf{a}_z\frac{\rho_\ell L}{4\pi\epsilon_0[z^2-(L/2)^2]}, \qquad z > \frac{L}{2}. \tag{3-68}$$

The preceding two examples illustrate the procedure for determining **E** by first finding V when Gauss's law cannot be conveniently applied. However, we emphasize that *if symmetry conditions exist such that a Gaussian surface can be constructed over which* $\mathbf{E}\cdot d\mathbf{s}$ *is constant, it is always easier to determine* **E** *directly.* The potential V, if desired, may be obtained from **E** by integration.

3–6 Conductors in Static Electric Field

So far we have discussed only the electric field of stationary charge distributions in free space or air. We now examine the field behavior in material media. In general, we classify materials according to their electrical properties into three types: ***conductors***, ***semiconductors***, and ***insulators*** (or ***dielectrics***). In terms of the crude atomic model of an atom consisting of a positively charged nucleus with orbiting electrons, the electrons in the outermost shells of the atoms of ***conductors*** are very loosely held and migrate easily from one atom to another. Most metals belong to this group. The electrons in the atoms of ***insulators*** or dielectrics, however, are confined to their orbits; they cannot be liberated in normal circumstances, even by the application of an external electric field. The electrical properties of ***semiconductors*** fall between those of conductors and insulators in that they possess a relatively small number of freely movable charges.

In terms of the band theory of solids we find that there are allowed energy bands for electrons, each band consisting of many closely spaced, discrete energy states. Between these energy bands there may be forbidden regions or gaps where no electrons of the solid's atom can reside. Conductors have an upper energy band partially filled with electrons or an upper pair of overlapping bands that are partially filled so that the electrons in these bands can move from one to another with only a small change in energy. Insulators or dielectrics are materials with a completely filled upper band, so conduction could not normally occur because of the existence of a large energy gap to the next higher band. If the energy gap of the forbidden region is relatively small, small amounts of external energy may be sufficient to excite the electrons in the filled upper band to jump into the next band, causing conduction. Such materials are semiconductors.

The macroscopic electrical property of a material medium is characterized by a constitutive parameter called ***conductivity***, which we will define in Chapter 5. The definition of conductivity is not important in this chapter because we are not dealing with current flow and are now interested only in the behavior of static electric fields in material media. In this section we examine the electric field and charge distribution both inside the bulk and on the surface of a conductor.

Assume for the present that some positive (or negative) charges are introduced in the interior of a conductor. An electric field will be set up in the conductor, the field exerting a force on the charges and making them move away from one another. This movement will continue until *all* the charges reach the conductor surface and redistribute themselves in such a way that both the charge and the field inside vanish. Hence,

Inside a Conductor *(Under Static Conditions)*	
$\rho = 0$	(3–69)
$\mathbf{E} = 0$	(3–70)

When there is no charge in the interior of a conductor ($\rho = 0$), **E** must be zero because, according to Gauss's law, the total outward electric flux through *any* closed surface constructed inside the conductor must vanish.

The charge distribution on the surface of a conductor depends on the shape of the surface. Obviously, the charges would not be in a state of equilibrium if there were a tangential component of the electric field intensity that produces a tangential

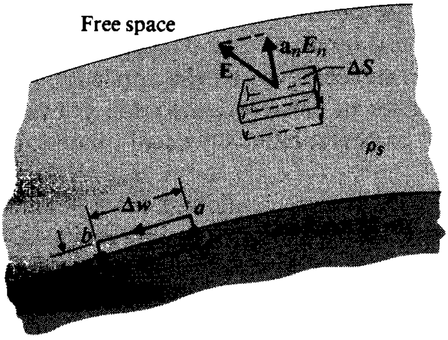


FIGURE 3–18
A conductor–free space interface.

force and moves the charges. Therefore, ***under static conditions the* E *field on a conductor surface is everywhere normal to the surface***. In other words, ***the surface of a conductor is an equipotential surface under static conditions***. As a matter of fact, since $\mathbf{E} = 0$ everywhere inside a conductor, the *whole* conductor has the same electrostatic potential. A finite time is required for the charges to redistribute on a conductor surface and reach the equilibrium state. This time depends on the conductivity of the material. For a good conductor such as copper this time is of the order of 10^{-19} (s), a very brief transient. (This point will be elaborated in Section 5–4.)

Figure 3–18 shows an interface between a conductor and free space. Consider the contour *abcda*, which has width $ab = cd = \Delta w$ and height $bc = da = \Delta h$. Sides *ab* and *cd* are parallel to the interface. Applying Eq. (3–8),† letting $\Delta h \to 0$, and noting that **E** in a conductor is zero, we obtain immediately

$$\oint_{abcda} \mathbf{E} \cdot d\boldsymbol{\ell} = E_t \Delta w = 0$$

or

$$E_t = 0, \tag{3–71}$$

which says that ***the tangential component of the* E *field on a conductor surface is zero***. In order to find E_n, the normal component of **E** at the surface of the conductor, we construct a Gaussian surface in the form of a thin pillbox with the top face in free space and the bottom face in the conductor where $\mathbf{E} = 0$. Using Eq. (3–7), we obtain

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = E_n \Delta S = \frac{\rho_s \Delta S}{\epsilon_0}$$

or

$$E_n = \frac{\rho_s}{\epsilon_0}. \tag{3–72}$$

† We assume that Eqs. (3–7) and (3–8) are valid for regions containing discontinuous media.

Hence, ***the normal component of the E field at a conductor/free space boundary is equal to the surface charge density on the conductor divided by the permittivity of free space.*** Summarizing the *boundary conditions* at the conductor surface, we have

Boundary Conditions at a Conductor/Free Space Interface
$E_t = 0$ (3–71)
$E_n = \dfrac{\rho_s}{\epsilon_0}$ (3–72)

When an uncharged conductor is placed in a static electric field, the external field will cause loosely held electrons inside the conductor to move in a direction opposite to that of the field and cause net positive charges to move in the direction of the field. These induced free charges will distribute on the conductor surface and create an *induced field* in such a way that they cancel the external field both inside the conductor and tangent to its surface. When the surface charge distribution reaches an equilibrium, all four relations, Eqs. (3–69) through (3–72), will hold; and the conductor is again an equipotential body.

EXAMPLE 3–11 A positive point charge Q is at the center of a spherical conducting shell of an inner radius R_i and an outer radius R_o. Determine $\mathbf{E}$ and V as functions of the radial distance R.

Solution The geometry of the problem is shown in Fig. 3–19(a). Since there is spherical symmetry, it is simplest to use Gauss's law to determine $\mathbf{E}$ and then find V by integration. There are three distinct regions: (a) $R > R_o$, (b) $R_i < R < R_o$, and (c) $R < R_i$. Suitable spherical Gaussian surfaces will be constructed in these regions. Obviously, $\mathbf{E} = \mathbf{a}_R E_R$ in all three regions.

a) $R > R_o$ (Gaussian surface S_1):

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = E_{R1} 4\pi R^2 = \frac{Q}{\epsilon_0}$$

or

$$E_{R1} = \frac{Q}{4\pi\epsilon_0 R^2}. \tag{3–73}$$

The $\mathbf{E}$ field is the same as that of a point charge Q without the presence of the shell. The potential referring to the point at infinity is

$$V_1 = -\int_\infty^R (E_{R1})\, dR = \frac{Q}{4\pi\epsilon_0 R}. \tag{3–74}$$

b) $R_i < R < R_o$ (Gaussian surface S_2): Because of Eq. (3–70), we know that

$$E_{R2} = 0. \tag{3–75}$$

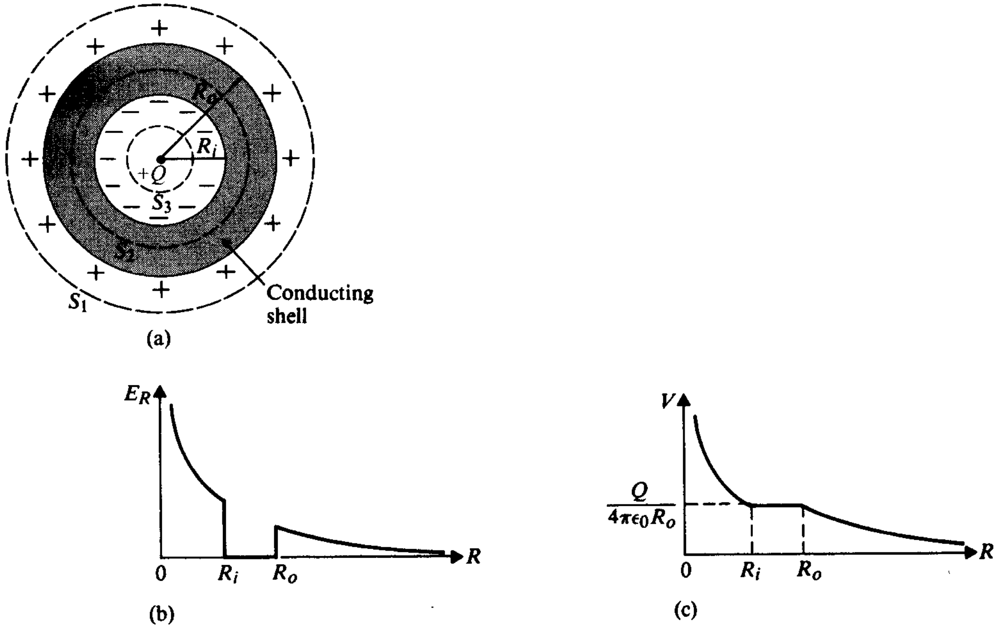


FIGURE 3–19
Electric field intensity and potential variations of a point charge $+Q$ at the center of a conducting shell (Example 3–11).

Since $\rho = 0$ in the conducting shell and since the total charge enclosed in surface S_2 must be zero, an amount of negative charge equal to $-Q$ must be induced on the inner shell surface at $R = R_i$. (This also means that an amount of positive charge equal to $+Q$ is induced on the outer shell surface at $R = R_o$.) The conducting shell is an equipotential body. Hence,

$$V_2 = V_1\Big|_{R=R_o} = \frac{Q}{4\pi\epsilon_0 R_o}. \tag{3–76}$$

c) $R < R_i$ (Gaussian surface S_3): Application of Gauss's law yields the same formula for E_{R3} as E_{R1} in Eq. (3–73) for the first region:

$$E_{R3} = \frac{Q}{4\pi\epsilon_0 R^2}. \tag{3–77}$$

The potential in this region is

$$V_3 = -\int E_{R3}\,dR + C = \frac{Q}{4\pi\epsilon_0 R} + C,$$

where the integration constant C is determined by requiring V_3 at $R = R_i$ to equal V_2 in Eq. (3–76). We have

$$C = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_o} - \frac{1}{R_i}\right)$$

and

$$V_3 = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R} + \frac{1}{R_o} - \frac{1}{R_i}\right). \tag{3-78}$$

The variations of E_R and V versus R in all three regions are plotted in Figs. 3–19(b) and 3–19(c). Note that while the electric intensity has discontinuous jumps, the potential remains continuous. A discontinuous jump in potential would mean an infinite electric field intensity. ▬

3–7 Dielectrics in Static Electric Field

Ideal dielectrics do not contain free charges. When a dielectric body is placed in an external electric field, there are no induced free charges that move to the surface and make the interior charge density and electric field vanish, as with conductors. However, since dielectrics contain ***bound charges***, we cannot conclude that they have no effect on the electric field in which they are placed.

All material media are composed of atoms with a positively charged nucleus surrounded by negatively charged electrons. Although the molecules of dielectrics are macroscopically neutral, the presence of an external electric field causes a force to be exerted on each charged particle and results in small displacements of positive and negative charges in opposite directions. These displacements, though small in comparison to atomic dimensions, nevertheless *polarize* a dielectric material and create electric dipoles. The situation is depicted in Fig. 3–20. Inasmuch as electric dipoles do have nonvanishing electric potential and electric field intensity, we expect that the ***induced electric dipoles*** will modify the electric field both inside and outside the dielectric material.

The molecules of some dielectrics possess permanent dipole moments, even in the absence of an external polarizing field. Such molecules usually consist of two or

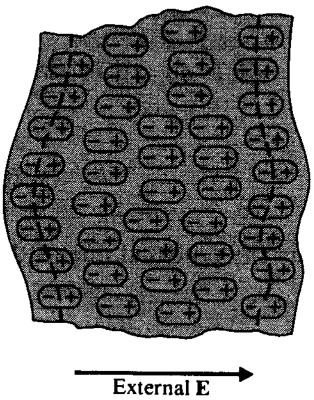


FIGURE 3–20
A cross section of a polarized dielectric medium.

more dissimilar atoms and are called ***polar molecules***, in contrast to ***nonpolar molecules***, which do not have permanent dipole moments. An example is the water molecule H_2O, which consists of two hydrogen atoms and one oxygen atom. The atoms do not arrange themselves in a manner that makes the molecule have a zero dipole moment; that is, the hydrogen atoms do not lie exactly on diametrically opposite sides of the oxygen atom.

The dipole moments of polar molecules are of the order of 10^{-30} (C·m). When there is no external field, the individual dipoles in a polar dielectric are randomly oriented, producing no net dipole moment macroscopically. An applied electric field will exert a torque on the individual dipoles and tend to align them with the field in a manner similar to that shown in Fig. 3–20.

Some dielectric materials can exhibit a permanent dipole moment even in the absence of an externally applied electric field. Such materials are called ***electrets***. Electrets can be made by heating (softening) certain waxes or plastics and placing them in an electric field. The polarized molecules in these materials tend to align with the applied field and to be frozen in their new positions after they return to normal temperatures. Permanent polarization remains without an external electric field. Electrets are the electrical equivalents of permanent magnets; they have found important applications in high fidelity electret microphones.†

3–7.1 EQUIVALENT CHARGE DISTRIBUTIONS OF POLARIZED DIELECTRICS

To analyze the macroscopic effect of induced dipoles we define a ***polarization vector***, **P**, as

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \mathbf{p}_k}{\Delta v} \qquad (\mathrm{C/m^2}), \tag{3–79}$$

where n is the number of molecules per unit volume and the numerator represents the vector sum of the induced dipole moments contained in a very small volume Δv. The vector **P**, a smoothed point function, is the *volume density of electric dipole moment*. The dipole moment $d\mathbf{p}$ of an elemental volume dv' is $d\mathbf{p} = \mathbf{P}\,dv'$, which produces an electrostatic potential (see Eq. 3–53b):

$$dV = \frac{\mathbf{P}\cdot\mathbf{a}_R}{4\pi\epsilon_0 R^2}\,dv'. \tag{3–80}$$

Integrating over the volume V' of the dielectric, we obtain the potential due to the polarized dielectric.

† See, for instance, J. M. Crowley, *Fundamentals of Applied Electrostatics*, Section 8–3, *Wiley, New York*, 1986.

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\mathbf{P} \cdot \mathbf{a}_R}{R^2}\, dv', \tag{3–81}$$ [†]

where R is the distance from the elemental volume dv' to a fixed field point. In Cartesian coordinates,

$$R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2, \tag{3–82}$$

and it is readily verified that the gradient of $1/R$ with respect to the *primed coordinates* is

$$\nabla'\left(\frac{1}{R}\right) = \frac{\mathbf{a}_R}{R^2}. \tag{3–83}$$

Hence Eq. (3–81) can be written as

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \mathbf{P} \cdot \nabla'\left(\frac{1}{R}\right) dv'. \tag{3–84}$$

Recalling the vector identity (Problem 2–28),

$$\nabla' \cdot (f\mathbf{A}) = f\nabla' \cdot \mathbf{A} + \mathbf{A} \cdot \nabla' f, \tag{3–85}$$

and letting $\mathbf{A} = \mathbf{P}$ and $f = 1/R$, we can rewrite Eq. (3–84) as

$$V = \frac{1}{4\pi\epsilon_0} \left[\int_{V'} \nabla' \cdot \left(\frac{\mathbf{P}}{R}\right) dv' - \int_{V'} \frac{\nabla' \cdot \mathbf{P}}{R}\, dv' \right]. \tag{3–86}$$

The first volume integral on the right side of Eq. (3–86) can be converted into a closed surface integral by the divergence theorem. We have

$$V = \frac{1}{4\pi\epsilon_0} \oint_{S'} \frac{\mathbf{P} \cdot \mathbf{a}'_n}{R}\, ds' + \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{(-\nabla' \cdot \mathbf{P})}{R}\, dv', \tag{3–87}$$

where $\mathbf{a}'_n$ is the outward normal from the surface element ds' of the dielectric. Comparison of the two integrals on the right side of Eq. (3–87) with Eqs. (3–62) and (3–61), respectively, reveals that the electric potential (and therefore the electric field intensity also) due to a polarized dielectric may be calculated from the contributions of surface and volume charge distributions having, respectively, densities

$$\boxed{\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n} \tag{3–88}$$ [‡]

and

$$\boxed{\rho_p = -\nabla \cdot \mathbf{P}.} \tag{3–89}$$ [‡]

[†] We note here that V on the left side of Eq. (3–81) represents the *electric potential* at a field point, and V' on the right side is the *volume* of the polarized dielectric.

[‡] The prime sign on $\mathbf{a}_n$ and ∇ has been dropped for simplicity, since Eqs. (3–88) and (3–89) involve only source coordinates and no confusion will result.

These are referred to as ***polarization charge densities*** or ***bound-charge densities***. In other words, a *polarized dielectric may be replaced by an equivalent polarization surface charge density* ρ_{ps} *and an equivalent polarization volume charge density* ρ_p *for field calculations:*

$$V = \frac{1}{4\pi\epsilon_0}\oint_{S'} \frac{\rho_{ps}}{R}\, ds' + \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{\rho_p}{R}\, dv'. \tag{3-90}$$

Although Eqs. (3–88) and (3–89) were derived mathematically with the aid of a vector identity, a physical interpretation can be provided for the charge distributions. The sketch in Fig. 3–20 clearly indicates that charges from the ends of similarly oriented dipoles exist on surfaces not parallel to the direction of polarization. Consider an imaginary elemental surface Δs of a nonpolar dielectric. The application of an external electric field normal to Δs causes a separation d of the bound charges: positive charges $+q$ move a distance $d/2$ in the direction of the field, and negative charges $-q$ move an equal distance against the direction of the field. The net total charge ΔQ that crosses the surface Δs in the direction of the field is $nq\, d(\Delta s)$, where n is the number of molecules per unit volume. If the external field is not normal to Δs, the separation of the bound charges in the direction of $\mathbf{a}_n$ will be $\mathbf{d} \cdot \mathbf{a}_n$ and

$$\Delta Q = nq(\mathbf{d} \cdot \mathbf{a}_n)(\Delta s). \tag{3-91}$$

But $nq\mathbf{d}$, the dipole moment per unit volume, is by definition the polarization vector $\mathbf{P}$. We have

$$\Delta Q = \mathbf{P} \cdot \mathbf{a}_n(\Delta s) \tag{3-92}$$

and

$$\rho_{ps} = \frac{\Delta Q}{\Delta s} = \mathbf{P} \cdot \mathbf{a}_n,$$

as given in Eq. (3–88). Remember that $\mathbf{a}_n$ is always the *outward* normal. This relation correctly gives a positive surface charge on the right-hand surface in Fig. 3–20 and a negative surface charge on the left-hand surface.

For a surface S bounding a volume V, the net total charge flowing out of V as a result of polarization is obtained by integrating Eq. (3–92). The net charge *remaining* within the volume V is the *negative* of this integral:

$$\begin{aligned} Q &= -\oint_S \mathbf{P} \cdot \mathbf{a}_n\, ds \\ &= \int_V (-\nabla \cdot \mathbf{P})\, dv = \int_V \rho_p\, dv, \end{aligned} \tag{3-93}$$

which leads to the expression for the volume charge density in Eq. (3–89). Hence, when the divergence of $\mathbf{P}$ does not vanish, the bulk of the polarized dielectric appears to be charged. However, since we started with an electrically neutral dielectric body, the total charge of the body after polarization must remain zero. This can be readily verified by noting that

$$\begin{aligned}\text{Total charge} &= \oint_S \rho_{ps}\,ds + \int_V \rho_p\,dv \\ &= \oint_S \mathbf{P}\cdot\mathbf{a}_n\,ds - \int_V \nabla\cdot\mathbf{P}\,dv = 0,\end{aligned} \tag{3–94}$$

where the divergence theorem has again been applied.

3–8 Electric Flux Density and Dielectric Constant

Because a polarized dielectric gives rise to an equivalent volume charge density ρ_p, we expect the electric field intensity due to a given source distribution in a dielectric to be different from that in free space. In particular, the divergence postulated in Eq. (3–4) must be modified to include the effect of ρ_p; that is,

$$\nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}(\rho + \rho_p). \tag{3–95}$$

Using Eq. (3–89), we have

$$\nabla\cdot(\epsilon_0\mathbf{E} + \mathbf{P}) = \rho. \tag{3–96}$$

We now define a new fundamental field quantity, the ***electric flux density***, or ***electric displacement***, **D**, such that

$$\boxed{\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} \qquad (\text{C/m}^2).} \tag{3–97}$$

The use of the vector **D** enables us to write a divergence relation between the electric field and the distribution of *free charges* in any medium without the necessity of dealing explicitly with the polarization vector **P** or the polarization charge density ρ_p. Combining Eqs. (3–96) and (3–97), we obtain the new equation

$$\boxed{\nabla\cdot\mathbf{D} = \rho \qquad (\text{C/m}^3),} \tag{3–98}$$

where ρ is the volume density of *free charges*. Equations (3–98) and (3–5) are the two fundamental governing differential equations for electrostatics in any medium. Note that the permittivity of free space, ϵ_0, does not appear explicitly in these two equations.

The corresponding integral form of Eq. (3–98) is obtained by taking the volume integral of both sides. We have

$$\int_V \nabla\cdot\mathbf{D}\,dv = \int_V \rho\,dv \tag{3–99}$$

or

$$\boxed{\oint_S \mathbf{D}\cdot d\mathbf{s} = Q \qquad (\text{C}).} \tag{3–100}$$

Equation (3–100), another form of ***Gauss's law***, states that ***the total outward flux of the electric displacement (or, simply, the total outward electric flux) over any closed surface is equal to the total free charge enclosed in the surface.*** As was indicated in Section 3–4, Gauss's law is most useful in determining the electric field due to charge distributions under symmetry conditions.

When the dielectric properties of the medium are *linear* and *isotropic*, the polarization is directly proportional to the electric field intensity, and the proportionality constant is independent of the direction of the field. We write

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \tag{3–101}$$

where χ_e is a dimensionless quantity called ***electric susceptibility***. A dielectric medium is linear if χ_e is independent of E and homogeneous if χ_e is independent of space coordinates. Substitution of Eq. (3–101) in Eq. (3–97) yields

$$\boxed{\begin{aligned}\mathbf{D} &= \epsilon_0(1 + \chi_e)\mathbf{E} \\ &= \epsilon_0 \epsilon_r \mathbf{E} = \epsilon \mathbf{E} \qquad (\mathrm{C/m^2}),\end{aligned}} \tag{3–102}$$

where

$$\epsilon_r = 1 + \chi_e = \frac{\epsilon}{\epsilon_0} \tag{3–103}$$

is a dimensionless quantity known as the ***relative permittivity*** or the ***dielectric constant*** of the medium. The coefficient $\epsilon = \epsilon_0 \epsilon_r$ is the ***absolute permittivity*** (often called simply ***permittivity***) of the medium and is measured in farads per meter (F/m). Air has a dielectric constant of 1.00059; hence its permittivity is usually taken as that of free space. The dielectric constants of some common materials are included in Table 3–1 on p. 114 and Appendix B–3.

Note that ϵ_r can be a function of space coordinates. If ϵ_r is independent of position, the medium is said to be ***homogenous***. A linear, homogeneous, and isotropic medium is called a ***simple medium***. The relative permittivity of a simple medium is a constant.

Later in the book we will learn that the effects of a lossy medium can be represented by a complex dielectric constant, whose imaginary part provides a measure of power loss in the medium and is, in general, frequency-dependent. For ***anisotropic*** materials the dielectric constant is different for different directions of the electric field, and **D** and **E** vectors generally have different directions; permittivity is a tensor. In matrix form we may write

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}. \tag{3–104}$$

For crystals the reference coordinates can be chosen to be along the principal axes of the crystal so that the off-diagonal terms of the permittivity matrix in Eq. (3–104) are zero. We have

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}. \tag{3–105}$$

Media having the property represented by Eq. (3–105) are said to be ***biaxial***. We may write

$$D_x = \epsilon_1 E_x, \tag{3–106a}$$

$$D_y = \epsilon_2 E_y, \tag{3–106b}$$

$$D_z = \epsilon_3 E_z. \tag{3–106c}$$

If further, $\epsilon_1 = \epsilon_2$, then the medium is said to be ***uniaxial***. Of course, if $\epsilon_1 = \epsilon_2 = \epsilon_3$, we have an isotropic medium. We shall deal only with isotropic media in this book.

EXAMPLE 3–12 A positive point charge Q is at the center of a spherical dielectric shell of an inner radius R_i and an outer radius R_o. The dielectric constant of the shell is ϵ_r. Determine **E**, V, **D**, and **P** as functions of the radial distance R.

Solution The geometry of this problem is the same as that of Example 3–11. The conducting shell has now been replaced by a dielectric shell, but the procedure of solution is similar. Because of the spherical symmetry, we apply Gauss's law to find **E** and **D** in three regions: (a) $R > R_o$; (b) $R_i < R < R_o$; and (c) $R < R_i$. Potential V is found from the negative line integral of **E**, and polarization **P** is determined by the relation

$$\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E} = \epsilon_0(\epsilon_r - 1)\mathbf{E}. \tag{3–107}$$

The **E**, **D**, and **P** vectors have only radial components. Refer to Fig. 3–21(a), where the Gaussian surfaces are not shown in order to avoid cluttering up the figure.

a) $R > R_o$
The situation in this region is exactly the same as that in Example 3–11. We have, from Eqs. (3–73) and (3–74),

$$E_{R1} = \frac{Q}{4\pi\epsilon_0 R^2}$$

$$V_1 = \frac{Q}{4\pi\epsilon_0 R}.$$

From Eqs. (3–102) and (3–107) we obtain

$$D_{R1} = \epsilon_0 E_{R1} = \frac{Q}{4\pi R^2} \tag{3–108}$$

and

$$P_{R1} = 0. \tag{3–109}$$

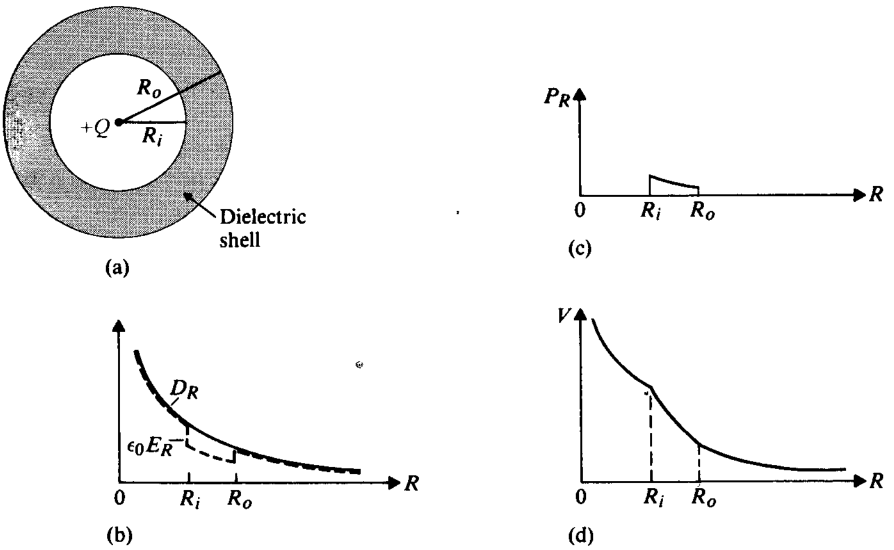


FIGURE 3–21
Field variations of a point charge $+Q$ at the center of a dielectric shell (Example 3–12).

b) $R_i < R < R_o$
The application of Gauss's law in this region gives us directly

$$E_{R2} = \frac{Q}{4\pi\epsilon_0\epsilon_r R^2} = \frac{Q}{4\pi\epsilon R^2}, \tag{3-110}$$

$$D_{R2} = \frac{Q}{4\pi R^2}, \tag{3-111}$$

$$P_{R2} = \left(1 - \frac{1}{\epsilon_r}\right)\frac{Q}{4\pi R^2}. \tag{3-112}$$

Note that D_{R2} has the same expression as D_{R1} and that both E_R and P_R have a discontinuity at $R = R_o$. In this region,

$$\begin{aligned} V_2 &= -\int_{\infty}^{R_o} E_{R1}\,dR - \int_{R_o}^{R} E_{R2}\,dR \\ &= V_1\Big|_{R=R_o} - \frac{Q}{4\pi\epsilon}\int_{R_o}^{R}\frac{1}{R^2}\,dR \\ &= \frac{Q}{4\pi\epsilon_0}\left[\left(1 - \frac{1}{\epsilon_r}\right)\frac{1}{R_o} + \frac{1}{\epsilon_r R}\right]. \end{aligned} \tag{3-113}$$

c) $R < R_i$
Since the medium in this region is the same as that in the region $R > R_o$, the application of Gauss's law yields the same expressions for E_R, D_R, and P_R in

both regions:

$$E_{R3} = \frac{Q}{4\pi\epsilon_0 R^2},$$

$$D_{R3} = \frac{Q}{4\pi R^2},$$

$$P_{R3} = 0.$$

To find V_3, we must add to V_2 at $R = R_i$ the negative line integral of E_{R3}:

$$\begin{aligned} V_3 &= V_2\Big|_{R=R_i} - \int_{R_i}^{R} E_{R3}\, dR \\ &= \frac{Q}{4\pi\epsilon_0}\left[\left(1 - \frac{1}{\epsilon_r}\right)\frac{1}{R_o} - \left(1 - \frac{1}{\epsilon_r}\right)\frac{1}{R_i} + \frac{1}{R}\right]. \end{aligned} \tag{3-114}$$

The variations of $\epsilon_0 E_R$ and D_R versus R are plotted in Fig. 3–21(b). The difference $(D_R - \epsilon_0 E_R)$ is P_R and is shown in Fig. 3–21(c). The plot for V in Fig. 3–21(d) is a composite graph for V_1, V_2, and V_3 in the three regions. We note that D_R is a continuous curve exhibiting no sudden changes in going from one medium to another and that P_R exists only in the dielectric region. ■

It is instructive to compare Figs. 3–21(b) and 3–21(d) with Figs. 3–19(b) and 3–19(c), respectively, of Example 3–11. From Eqs. (3–88) and (3–89) we find

$$\begin{aligned} \rho_{ps}\Big|_{R=R_i} &= \mathbf{P}\cdot(-\mathbf{a}_R)\Big|_{R=R_i} = -P_{R2}\Big|_{R=R_i} \\ &= -\left(1 - \frac{1}{\epsilon_r}\right)\frac{Q}{4\pi R_i^2} \end{aligned} \tag{3-115}$$

on the inner shell surface,

$$\begin{aligned} \rho_{ps}\Big|_{R=R_o} &= \mathbf{P}\cdot\mathbf{a}_R\Big|_{R=R_o} = P_{R2}\Big|_{R=R_o} \\ &= \left(1 - \frac{1}{\epsilon_r}\right)\frac{Q}{4\pi R_o^2} \end{aligned} \tag{3-116}$$

on the outer shell surface, and

$$\begin{aligned} \rho_p &= -\nabla\cdot\mathbf{P} \\ &= -\frac{1}{R^2}\frac{\partial}{\partial R}(R^2 P_{R2}) = 0. \end{aligned} \tag{3-117}$$

Equations (3–115), (3–116), and (3–117) indicate that there is no net polarization volume charge inside the dielectric shell. However, negative polarization surface charges exist on the inner surface and positive polarization surface charges on the outer surface. These surface charges produce an electric field intensity that is directed radially inward, thus reducing the **E** field in region 2 due to the point charge $+Q$ at the center.

TABLE 3–1
Dielectric Constants and Dielectric Strengths of Some Common Materials

Material	Dielectric Constant	Dielectric Strength (V/m)
Air (atmospheric pressure)	1.0	3×10^6
Mineral oil	2.3	15×10^6
Paper	2–4	15×10^6
Polystyrene	2.6	20×10^6
Rubber	2.3–4.0	25×10^6
Glass	4–10	30×10^6
Mica	6.0	200×10^6

3–8.1 DIELECTRIC STRENGTH

We have explained that an electric field causes small displacements of the bound charges in a dielectric material, resulting in polarization. If the electric field is very strong, it will pull electrons completely out of the molecules. The electrons will accelerate under the influence of the electric field, collide violently with the molecular lattice structure, and cause permanent dislocations and damage in the material. Avalanche effect of ionization due to collisions may occur. The material will become conducting, and large currents may result. This phenomenon is called a ***dielectric breakdown***. The maximum electric field intensity that a dielectric material can withstand without breakdown is the ***dielectric strength*** of the material. The approximate dielectric strengths of some common substances are given in Table 3–1. The dielectric strength of a material must not be confused with its dielectric constant.

A convenient number to remember is that the dielectric strength of air at the atmospheric pressure is 3 kV/mm. When the electric field intensity exceeds this value, air breaks down. Massive ionization takes place, and sparking (corona discharge) follows. Charge tends to concentrate at sharp points. In view of Eq. (3–72), the electric field intensity in the immediate vicinity of sharp points is much higher than that at points on a relatively flat surface with a small curvature. This is the principle upon which a lightning arrester with a sharp metal lightning rod on top of tall buildings works. When a cloud containing an abundance of electric charges approaches a tall building equipped with a lightning rod connected to the ground, charges of an opposite sign are attracted from the ground to the tip of the rod, where the electric field intensity is the strongest. As the electric field intensity exceeds the dielectric strength of the wet air, breakdown occurs, and the air near the tip is ionized and becomes conducting. The electric charges in the cloud are then discharged safely to the ground through the conducting path.

The fact that the electric field intensity tends to be higher at a point near the surface of a charged conductor with a larger curvature is illustrated quantitatively in the following example.

EXAMPLE 3–13 Consider two spherical conductors with radii b_1 and b_2 $(b_2 > b_1)$ that are connected by a conducting wire. The distance of separation between the conductors is assumed to be very large in comparison to b_2 so that the charges on the spherical conductors may be considered as uniformly distributed. A total charge Q is deposited on the spheres. Find (a) the charges on the two spheres, and (b) the electric field intensities at the sphere surfaces.

Solution

a) Refer to Fig. 3–22. Since the spherical conductors are at the same potential, we have

$$\frac{Q_1}{4\pi\epsilon_0 b_1} = \frac{Q_2}{4\pi\epsilon_0 b_2}$$

or

$$\frac{Q_1}{Q_2} = \frac{b_1}{b_2}.$$

Hence the charges on the spheres are directly proportional to their radii. But, since

$$Q_1 + Q_2 = Q,$$

we find that

$$Q_1 = \frac{b_1}{b_1 + b_2} Q \quad \text{and} \quad Q_2 = \frac{b_2}{b_1 + b_2} Q.$$

b) The electric field intensities at the surfaces of the two conducting spheres are

$$E_{1n} = \frac{Q_1}{4\pi\epsilon_0 b_1^2} \quad \text{and} \quad E_{2n} = \frac{Q_2}{4\pi\epsilon_0 b_2^2},$$

so

$$\frac{E_{1n}}{E_{2n}} = \left(\frac{b_2}{b_1}\right)^2 \frac{Q_1}{Q_2} = \frac{b_2}{b_1}.$$

The electric field intensities are therefore inversely proportional to the radii, being higher at the surface of the smaller sphere which has a larger curvature.

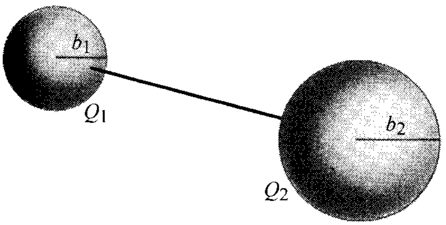


FIGURE 3–22
Two connected conducting spheres (Example 3–13).

3–9 Boundary Conditions for Electrostatic Fields

Electromagnetic problems often involve media with different physical properties and require the knowledge of the relations of the field quantities at an interface between two media. For instance, we may wish to determine how the **E** and **D** vectors change in crossing an interface. We already know the boundary conditions that must be satisfied at a conductor/free space interface. These conditions have been given in Eqs. (3–71) and (3–72). We now consider an interface between two general media shown in Fig. 3–23.

Let us construct a small path *abcda* with sides *ab* and *cd* in media 1 and 2, respectively, both being parallel to the interface and equal to Δw. Equation (3–8) is applied to this path. If we let sides $bc = da = \Delta h$ approach zero, their contributions to the line integral of **E** around the path can be neglected. We have

$$\oint_{abcda} \mathbf{E} \cdot d\boldsymbol{\ell} = \mathbf{E}_1 \cdot \Delta\mathbf{w} + \mathbf{E}_2 \cdot (-\Delta\mathbf{w}) = E_{1t}\Delta w - E_{2t}\Delta w = 0.$$

Therefore

$$\boxed{E_{1t} = E_{2t} \qquad (\text{V/m}),} \tag{3-118}$$

which states that ***the tangential component of an* E *field is continuous across an interface***. Eq. (3–118) simplifies to Eq. (3–71) if one of the media is a conductor. When media 1 and 2 are dielectrics with permittivities ϵ_1 and ϵ_2, respectively, we have

$$\frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2}. \tag{3-119}$$

In order to find a relation between the normal components of the fields at a boundary, we construct a small pillbox with its top face in medium 1 and bottom

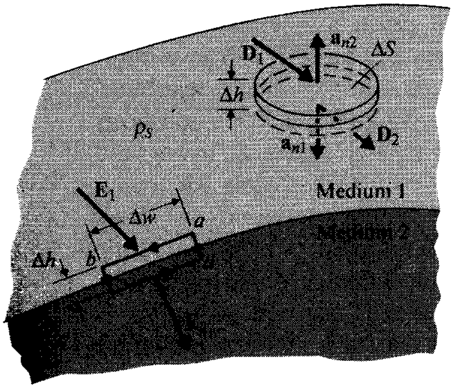


FIGURE 3–23
An interface between two media.

face in medium 2, as illustrated in Fig. 3–23. The faces have an area ΔS, and the height of the pillbox Δh is vanishingly small. Applying Gauss's law, Eq. (3–100), to the pillbox,† we have

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = (\mathbf{D}_1 \cdot \mathbf{a}_{n2} + \mathbf{D}_2 \cdot \mathbf{a}_{n1})\Delta S$$
$$= \mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2)\Delta S \qquad (3\text{–}120)$$
$$= \rho_s \Delta S,$$

where we have used the relation $\mathbf{a}_{n2} = -\mathbf{a}_{n1}$. Unit vectors $\mathbf{a}_{n1}$ and $\mathbf{a}_{n2}$ are, respectively, *outward* unit normals from media 1 and 2. From Eq. (3–120) we obtain

$$\boxed{\mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s} \qquad (3\text{–}121\text{a})$$

or

$$\boxed{D_{1n} - D_{2n} = \rho_s \qquad (\text{C/m}^2),} \qquad (3\text{–}121\text{b})$$

where the reference unit normal is outward from medium 2.

Eq. (3–121b) states that ***the normal component of* D *field is discontinuous across an interface where a surface charge exists—the amount of discontinuity being equal to the surface charge density***. If medium 2 is a conductor, $\mathbf{D}_2 = 0$ and Eq. (3–121b) becomes

$$D_{1n} = \epsilon_1 E_{1n} = \rho_s, \qquad (3\text{–}122)$$

which simplifies to Eq. (3–72) when medium 1 is free space.

When two dielectrics are in contact with *no free charges* at the interface, $\rho_s = 0$, we have

$$D_{1n} = D_{2n} \qquad (3\text{–}123)$$

or

$$\epsilon_1 E_{1n} = \epsilon_2 E_{2n}. \qquad (3\text{–}124)$$

Recapitulating, we find that the boundary conditions that must be satisfied for static electric fields are as follows:

Tangential components,	$E_{1t} = E_{2t}$;	(3–125)
Normal components,	$\mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$.	(3–126)

EXAMPLE 3–14 A lucite sheet ($\epsilon_r = 3.2$) is introduced perpendicularly in a uniform electric field $\mathbf{E}_o = \mathbf{a}_x E_o$ in free space. Determine $\mathbf{E}_i$, $\mathbf{D}_i$, and $\mathbf{P}_i$ inside the lucite.

† Equations (3–8) and (3–100) are assumed to hold for regions containing discontinuous media. See C. T. Tai, "On the presentation of Maxwell's theory," *Proceedings of the IEEE*, vol. 60, pp. 936–945, August 1972.

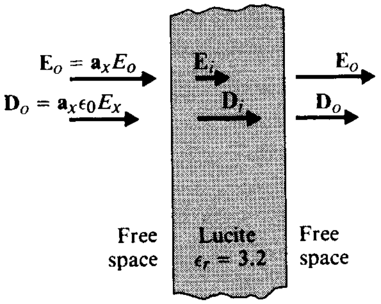


FIGURE 3-24
A lucite sheet in a uniform electric field (Example 3-14).

Solution We assume that the introduction of the lucite sheet does not disturb the original uniform electric field $\mathbf{E}_o$. The situation is depicted in Fig. 3-24. Since the interfaces are perpendicular to the electric field, only the normal field components need be considered. No free charges exist.

Boundary condition Eq. (3-123) at the left interface gives

$$\mathbf{D}_i = \mathbf{a}_x D_i = \mathbf{a}_x D_o$$

or

$$\mathbf{D}_i = \mathbf{a}_x \epsilon_0 E_o.$$

There is no change in electric flux density across the interface. The electric field intensity inside the lucite sheet is

$$\mathbf{E}_i = \frac{1}{\epsilon}\mathbf{D}_i = \frac{1}{\epsilon_o \epsilon_r}\mathbf{D}_i = \mathbf{a}_x \frac{E_o}{3.2}.$$

The polarization vector is zero outside the lucite sheet ($\mathbf{P}_o = 0$). Inside the sheet,

$$\mathbf{P}_i = \mathbf{D}_i - \epsilon_0 \mathbf{E}_i = \mathbf{a}_x\left(1 - \frac{1}{3.2}\right)\epsilon_0 E_o$$
$$= \mathbf{a}_x 0.6875 \epsilon_0 E_o \qquad (\text{C/m}^2).$$

Clearly, a similar application of the boundary condition Eq. (3-123) on the right interface will yield the original $\mathbf{E}_o$ and $\mathbf{D}_o$ in the free space on the right of the lucite sheet. Does the solution of this problem change if the original electric field is not uniform; that is, if $\mathbf{E}_o = \mathbf{a}_x E(y)$?

EXAMPLE 3-15 Two dielectric media with permittivities ϵ_1 and ϵ_2 are separated by a charge-free boundary as shown in Fig. 3-25. The electric field intensity in medium 1 at the point P_1 has a magnitude E_1 and makes an angle α_1 with the normal. Determine the magnitude and direction of the electric field intensity at point P_2 in medium 2.

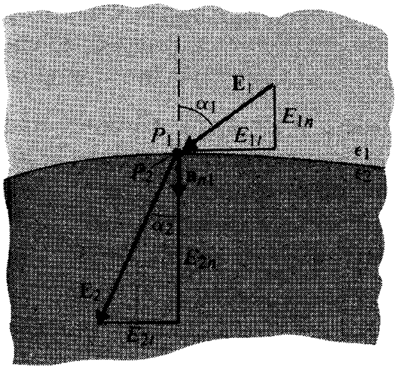


FIGURE 3–25
Boundary conditions at the interface between two dielectric media (Example 3–15).

Solution Two equations are needed to solve for two unknowns E_{2t} and E_{2n}. After E_{2t} and E_{2n} have been found, E_2 and α_2 will follow directly. Using Eqs. (3–118) and (3–123), we have

$$E_2 \sin \alpha_2 = E_1 \sin \alpha_1 \tag{3-127}$$

and

$$\epsilon_2 E_2 \cos \alpha_2 = \epsilon_1 E_1 \cos \alpha_1. \tag{3-128}$$

Division of Eq. (3–127) by Eq. (3–128) gives

$$\boxed{\frac{\tan \alpha_2}{\tan \alpha_1} = \frac{\epsilon_2}{\epsilon_1}.} \tag{3-129}$$

The magnitude of $\mathbf{E}_2$ is

$$\begin{aligned} E_2 &= \sqrt{E_{2t}^2 + E_{2n}^2} = \sqrt{(E_2 \sin \alpha_2)^2 + (E_2 \cos \alpha_2)^2} \\ &= \left[(E_1 \sin \alpha_1)^2 + \left(\frac{\epsilon_1}{\epsilon_2} E_1 \cos \alpha_1\right)^2\right]^{1/2} \end{aligned}$$

or

$$\boxed{E_2 = E_1\left[\sin^2 \alpha_1 + \left(\frac{\epsilon_1}{\epsilon_2} \cos \alpha_1\right)^2\right]^{1/2}.} \tag{3-130}$$

By examining Fig. 3–25, can you tell whether ϵ_1 is larger or smaller than ϵ_2? ▬

EXAMPLE 3–16 When a coaxial cable is used to carry electric power, the radius of the inner conductor is determined by the load current, and the overall size by the voltage and the type of insulating material used. Assume that the radius of the inner conductor is 0.4 (cm) and that concentric layers of rubber ($\epsilon_{rr} = 3.2$) and polystyrene ($\epsilon_{rp} = 2.6$) are used as insulating materials. Design a cable that is to work at a voltage rating of 20 (kV). In order to avoid breakdown due to voltage surges caused by lightning and other abnormal external conditions, the maximum electric field intensities in the insulating materials are not to exceed 25% of their dielectric strengths.

Solution From Table 3–1, p. 114, we find the dielectric strengths of rubber and polystyrene to be 25 × 10⁶ (V/m) and 20 × 10⁶ (V/m), respectively. Using Eq. (3–40) for specified 25% of dielectric strengths, we have the following.

In rubber: $$\text{Max } E_r = 0.25 \times 25 \times 10^6 = \frac{\rho_\ell}{2\pi\epsilon_0}\left(\frac{1}{3.2r_i}\right). \tag{3-131a}$$

In polystyrene: $$\text{Max } E_p = 0.25 \times 20 \times 10^6 = \frac{\rho_\ell}{2\pi\epsilon_0}\left(\frac{1}{2.6r_p}\right). \tag{3-131b}$$

Combination of Eqs. (3–131a) and (3–131b) yields

$$r_p = 1.54r_i = 0.616 \qquad \text{(cm)}. \tag{3-132}$$

Equation (3–132) indicates that the insulating layer of polystyrene should be placed outside of that of rubber, as shown in Fig. 3–26(a). (It would be interesting to determine what would happen if the polystyrene layer were placed inside the rubber layer.)

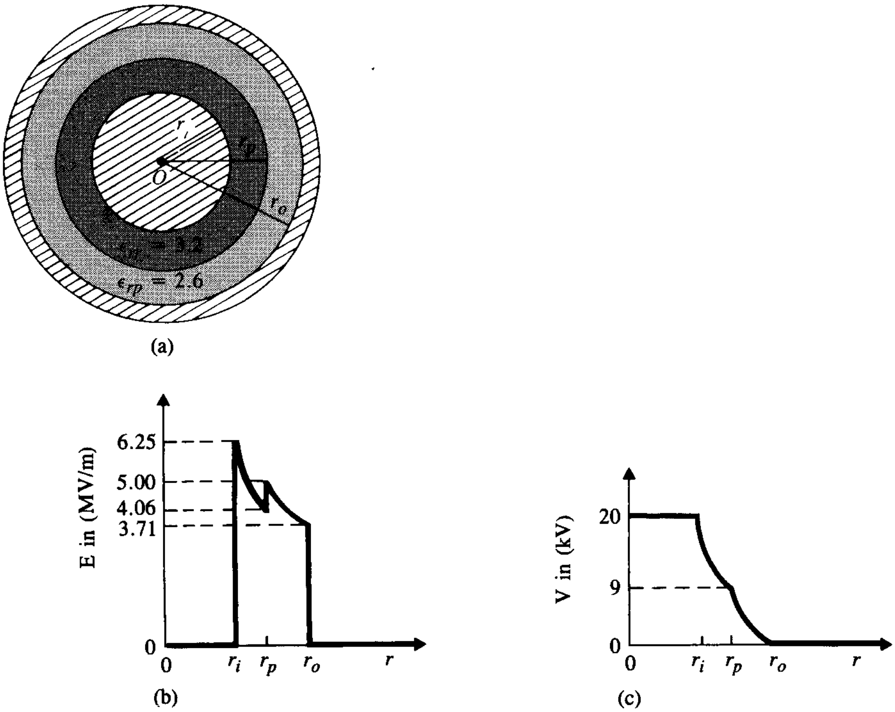


FIGURE 3–26
Cross section of coaxial cable with two different kinds of insulating material (Example 3–16).

The cable is to work at a potential difference of 20,000 (V) between the inner and outer conductors. We set

$$-\int_{r_o}^{r_p} E_p\,dr - \int_{r_p}^{r_i} E_r\,dr = 20{,}000,$$

where both E_p and E_r have the form given in Eq. (3–40). The above relation leads to

$$\frac{\rho_\ell}{2\pi\epsilon_0}\left(\frac{1}{\epsilon_{rp}}\ln\frac{r_o}{r_p} + \frac{1}{\epsilon_{rr}}\ln\frac{r_p}{r_i}\right) = 20{,}000$$

or

$$\frac{\rho_\ell}{2\pi\epsilon_0}\left(\frac{1}{2.6}\ln\frac{r_o}{1.54r_i} + \frac{1}{3.2}\ln 1.54\right) = 20{,}000. \tag{3–133}$$

Since $r_i = 0.4$ (cm) is given, r_o can be determined by finding the factor $\rho_\ell/2\pi\epsilon_0$ from Eq. (3–131a) and then using it in Eq. (3–133). We obtain $\rho_\ell/2\pi\epsilon_0 = 8 \times 10^4$, and $r_o = 2.08r_i = 0.832$ (cm).

In Figs. 3–26(b) and 3–26(c) are plotted the variations of the radial electric field intensity E and the potential V referred to that of the outer sheath. Note that E has discontinuous jumps, while the V curve is continuous. The reader should verify all the indicated numerical values. ■

3–10 Capacitance and Capacitors

From Section 3–6 we understand that a conductor in a static electric field is an equipotential body and that charges deposited on a conductor will distribute themselves on its surface in such a way that the electric field inside vanishes. Suppose the potential due to a charge Q is V. Obviously, increasing the total charge by some factor k would merely increase the surface charge density ρ_s everywhere by the same factor without affecting the charge distribution because the conductor remains an equipotential body in a static situation. We may conclude from Eq. (3–62) that the potential of an isolated conductor is directly proportional to the total charge on it. This may also be seen from the fact that increasing V by a factor of k increases $\mathbf{E} = -\nabla V$ by a factor of k. But from Eq. (3–72), $\mathbf{E} = \mathbf{a}_n\rho_s/\epsilon_0$; it follows that ρ_s, and consequently the total charge Q will also increase by a factor of k. The ratio Q/V therefore remains unchanged. We write

$$\boxed{Q = CV,} \tag{3–134}$$

where the constant of proportionality C is called the **capacitance** of the isolated conducting body. The capacitance is the electric charge that must be added to the body per unit increase in its electric potential. Its SI unit is coulomb per volt, or farad (F).

Of considerable importance in practice is the **capacitor**, which consists of two conductors separated by free space or a dielectric medium. The conductors may be of arbitrary shapes as in Fig. 3–27. When a d-c voltage source is connected between the conductors, a charge transfer occurs, resulting in a charge $+Q$ on one conductor

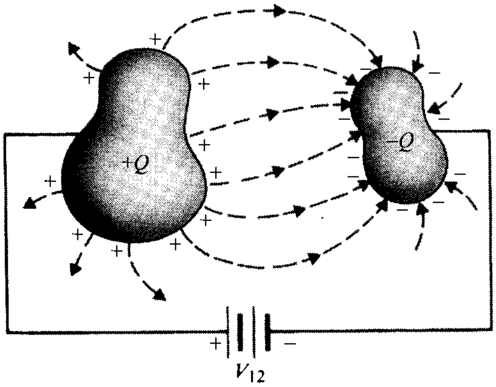


FIGURE 3–27
A two-conductor capacitor.

and $-Q$ on the other. Several electric field lines originating from positive charges and terminating on negative charges are shown in Fig. 3–27. Note that the field lines are perpendicular to the conductor surfaces, which are equipotential surfaces. Equation (3–134) applies here if V is taken to mean the potential difference between the two conductors, V_{12}. That is,

$$\boxed{C = \frac{Q}{V_{12}} \quad \text{(F).}} \tag{3–135}$$

The capacitance of a capacitor is a physical property of the two-conductor system. It depends on the geometry of the conductors and on the permittivity of the medium between them; it does *not* depend on either the charge Q or the potential difference V_{12}. A capacitor has a capacitance even when no voltage is applied to it and no free charges exist on its conductors. Capacitance C can be determined from Eq. (3–135) by either (1) assuming a V_{12} and determining Q in terms of V_{12}, or (2) assuming a Q and determining V_{12} in terms of Q. At this stage, since we have not yet studied the methods for solving boundary-value problems (which will be taken up in Chapter 4), we find C by the second method. The procedure is as follows:

1. Choose an appropriate coordinate system for the given geometry.
2. Assume charges $+Q$ and $-Q$ on the conductors.
3. Find $\mathbf{E}$ from Q by Eq. (3–122), Gauss's law, or other relations.
4. Find V_{12} by evaluating
$$V_{12} = -\int_2^1 \mathbf{E} \cdot d\boldsymbol{\ell}$$
from the conductor carrying $-Q$ to the other carrying $+Q$.
5. Find C by taking the ratio Q/V_{12}.

EXAMPLE 3–17 A parallel-plate capacitor consists of two parallel conducting plates of area S separated by a uniform distance d. The space between the plates is filled with a dielectric of a constant permittivity ϵ. Determine the capacitance.

Solution A cross section of the capacitor is shown in Fig. 3–28. It is obvious that the appropriate coordinate system to use is the Cartesian coordinate system. Following the procedure outlined above, we put charges $+Q$ and $-Q$ on the upper and lower conducting plates, respectively. The charges are assumed to be uniformly distributed over the conducting plates with surface densities $+\rho_s$ and $-\rho_s$, where

$$\rho_s = \frac{Q}{S}.$$

From Eq. (3–122) we have

$$\mathbf{E} = -\mathbf{a}_y \frac{\rho_s}{\epsilon} = -\mathbf{a}_y \frac{Q}{\epsilon S},$$

which is constant within the dielectric if the fringing of the electric field at the edges of the plates is neglected. Now

$$V_{12} = -\int_{y=0}^{y=d} \mathbf{E} \cdot d\ell = -\int_0^d \left(-\mathbf{a}_y \frac{Q}{\epsilon S}\right) \cdot (\mathbf{a}_y\, dy) = \frac{Q}{\epsilon S} d.$$

Therefore, *for a parallel-plate capacitor,*

$$\boxed{C = \frac{Q}{V_{12}} = \epsilon \frac{S}{d},} \qquad (3\text{–}136)$$

which is independent of Q or V_{12}.

For this problem we could have started by assuming a potential difference V_{12} between the upper and lower plates. The electric field intensity between the plates is uniform and equals

$$\mathbf{E} = -\mathbf{a}_y \frac{V_{12}}{d}.$$

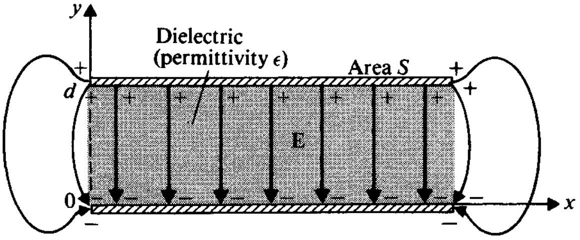


FIGURE 3–28
Cross section of a parallel-plate capacitor (Example 3–17).

The surface charge densities at the upper and lower conducting plates are $+\rho_s$ and $-\rho_s$, respectively, where, in view of Eq. (3–72),

$$\rho_s = \epsilon E_y = \epsilon \frac{V_{12}}{d}.$$

Therefore, $Q = \rho_s S = (\epsilon S/d)V_{12}$ and $C = Q/V_{12} = \epsilon S/d$, as before.

EXAMPLE 3–18 A cylindrical capacitor consists of an inner conductor of radius a and an outer conductor whose inner radius is b. The space between the conductors is filled with a dielectric of permittivity ϵ, and the length of the capacitor is L. Determine the capacitance of this capacitor.

Solution We use cylindrical coordinates for this problem. First we assume charges $+Q$ and $-Q$ on the surface of the inner conductor and the inner surface of the outer conductor, respectively. The **E** field in the dielectric can be obtained by applying Gauss's law to a cylindrical Gaussian surface within the dielectric $a < r < b$. (Note that Eq. (3–122) gives only the *normal component* of the **E** field *at* a conductor surface. Since the conductor surfaces are not planes here, the **E** field is not constant in the dielectric and Eq. (3–122) cannot be used to find **E** in the $a < r < b$ region.) Referring to Fig. 3–29 and applying Gauss's law, we have

$$\mathbf{E} = \mathbf{a}_r E_r = \mathbf{a}_r \frac{Q}{2\pi\epsilon L r}. \tag{3–137}$$

Again we neglect the fringing effect of the field near the edges of the conductors. The potential difference between the inner and outer conductors is

$$\begin{aligned} V_{ab} &= -\int_{r=b}^{r=a} \mathbf{E} \cdot d\boldsymbol{\ell} = -\int_b^a \left(\mathbf{a}_r \frac{Q}{2\pi\epsilon L r}\right) \cdot (\mathbf{a}_r\, dr) \\ &= \frac{Q}{2\pi\epsilon L} \ln\left(\frac{b}{a}\right). \end{aligned} \tag{3–138}$$

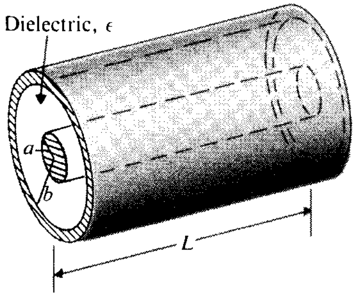


FIGURE 3–29
A cylindrical capacitor (Example 3–18).

Therefore, *for a cylindrical capacitor,*

$$C = \frac{Q}{V_{ab}} = \frac{2\pi\epsilon L}{\ln\left(\dfrac{b}{a}\right)}. \tag{3–139}$$

We could not solve this problem from an assumed V_{ab} because the electric field is not uniform between the inner and outer conductors. Thus we would not know how to express **E** and Q in terms of V_{ab} until we learned how to solve such a boundary-value problem. ■

EXAMPLE 3–19 A spherical capacitor consists of an inner conducting sphere of radius R_i and an outer conductor with a spherical inner wall of radius R_o. The space in between is filled with a dielectric of permittivity ϵ. Determine the capacitance.

Solution Assume charges $+Q$ and $-Q$ on the inner and outer conductors, respectively, of the spherical capacitor in Fig. 3–30. Applying Gauss's law to a spherical Gaussian surface with radius $R(R_i < R < R_o)$, we have

$$\mathbf{E} = \mathbf{a}_R E_R = \mathbf{a}_R \frac{Q}{4\pi\epsilon R^2}$$

$$V = -\int_{R_o}^{R_i} \mathbf{E} \cdot (\mathbf{a}_R\, dR) = -\int_{R_o}^{R_i} \frac{Q}{4\pi\epsilon R^2}\, dR = \frac{Q}{4\pi\epsilon}\left(\frac{1}{R_i} - \frac{1}{R_o}\right).$$

Therefore, *for a spherical capacitor,*

$$C = \frac{Q}{V} = \frac{4\pi\epsilon}{\dfrac{1}{R_i} - \dfrac{1}{R_o}}. \tag{3–140}$$

■

For an isolated conducting sphere of a radius R_i, $R_o \to \infty$, $C = 4\pi\epsilon R_i$.

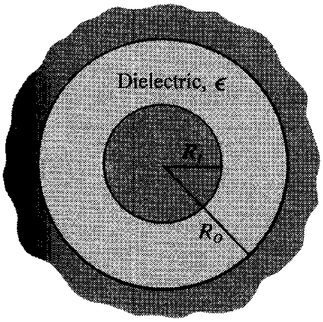


FIGURE 3–30
A spherical capacitor (Example 3–19).

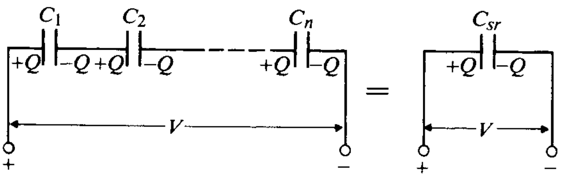


FIGURE 3–31
Series connection of capacitors.

3–10.1 SERIES AND PARALLEL CONNECTIONS OF CAPACITORS

Capacitors are often combined in various ways in electric circuits. The two basic ways are series and parallel connections. In the series, or head-to-tail, connection shown in Fig. 3–31,† the external terminals are from the first and last capacitors only. When a potential difference or electrostatic voltage V is applied, charge cumulations on the conductors connected to the external terminals are $+Q$ and $-Q$. Charges will be induced on the internally connected conductors such that $+Q$ and $-Q$ will appear on each capacitor independently of its capacitance. The potential differences across the individual capacitors are $Q/C_1, Q/C_2, \ldots, Q/C_n$, and

$$V = \frac{Q}{C_{sr}} = \frac{Q}{C_1} + \frac{Q}{C_2} + \cdots + \frac{Q}{C_n},$$

where C_{sr} is the equivalent capacitance of the series-connected capacitors. We have

$$\boxed{\frac{1}{C_{sr}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}.} \tag{3–141}$$

In the parallel connection of capacitors the external terminals are connected to the conductors of all the capacitors as in Fig. 3–32. When a potential difference V is applied to the terminals, the charge cumulated on a capacitor depends on its capacitance. The total charge is the sum of all the charges.

$$\begin{aligned} Q &= Q_1 + Q_2 + \cdots + Q_n \\ &= C_1 V + C_2 V + \cdots + C_n V = C_{||} V \end{aligned}$$

Therefore, the equivalent capacitance of the parallel-connected capacitors is

$$\boxed{C_{||} = C_1 + C_2 + \cdots + C_n.} \tag{3–142}$$

† Capacitors, whatever their actual shape, are conventionally represented in circuits by pairs of parallel bars.

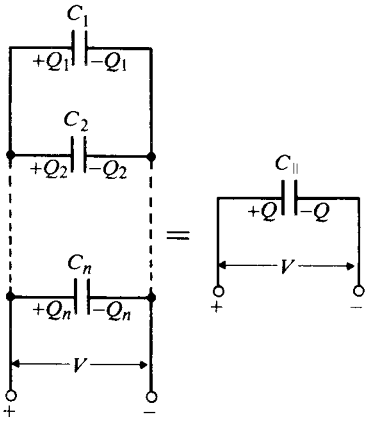


FIGURE 3-32
Parallel connection of capacitors.

We note that the formula for the equivalent capacitance of series-connected capacitors is similar to that for the equivalent resistance of parallel-connected resistors and that the formula for the equivalent capacitance of parallel-connected capacitors is similar to that for the equivalent resistance of series-connected resistors. Can you explain this?

EXAMPLE 3-20 Four capacitors $C_1 = 1\ (\mu F)$, $C_2 = 2\ (\mu F)$, $C_3 = 3\ (\mu F)$, and $C_4 = 4\ (\mu F)$ are connected as in Fig. 3-33. A d-c voltage of 100 (V) is applied to the external terminals a–b. Determine the following: (a) the total equivalent capacitance between terminals a–b, (b) the charge on each capacitor, and (c) the potential difference across each capacitor.

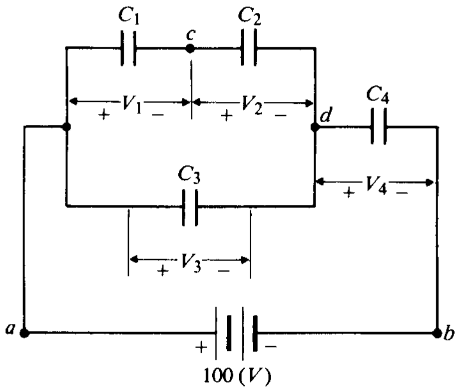


FIGURE 3-33
A combination of capacitors (Example 3-20).

Solution

a) The equivalent capacitance C_{12} of C_1 and C_2 in series is

$$C_{12} = \frac{1}{(1/C_1) + (1/C_2)} = \frac{C_1 C_2}{C_1 + C_2} = \frac{2}{3} \quad (\mu\text{F}).$$

The combination of C_{12} in parallel with C_3 gives

$$C_{123} = C_{12} + C_3 = \tfrac{11}{3} \quad (\mu\text{F}).$$

The total equivalent capacitance C_{ab} is then

$$C_{ab} = \frac{C_{123} C_4}{C_{123} + C_4} = \frac{44}{23} = 1.913 \quad (\mu\text{F}).$$

b) Since the capacitances are given, the voltages can be found as soon as the charges have been determined. We have four unknowns: Q_1, Q_2, Q_3, and Q_4. Four equations are needed for their determination.

Series connection of C_1 and C_2: $\quad Q_1 = Q_2.$

Kirchhoff's voltage law, $V_1 + V_2 = V_3$: $\quad \dfrac{Q_1}{C_1} + \dfrac{Q_2}{C_2} = \dfrac{Q_3}{C_3}.$

Kirchhoff's voltage law, $V_3 + V_4 = 100$: $\quad \dfrac{Q_3}{C_3} + \dfrac{Q_4}{C_4} = 100.$

Series connection at d: $\quad Q_2 + Q_3 = Q_4.$

Using the given values of C_1, C_2, C_3, and C_4 and solving the equations, we obtain

$$Q_1 = Q_2 = \frac{800}{23} = 34.8 \quad (\mu\text{C}),$$

$$Q_3 = \frac{3600}{23} = 156.5 \quad (\mu\text{C}),$$

$$Q_4 = \frac{4400}{23} = 191.3 \quad (\mu\text{C}).$$

c) Dividing the charges by the capacitances, we find

$$V_1 = \frac{Q_1}{C_1} = 34.8 \quad (\text{V}),$$

$$V_2 = \frac{Q_2}{C_2} = 17.4 \quad (\text{V}),$$

$$V_3 = \frac{Q_3}{C_3} = 52.2 \quad (\text{V}),$$

$$V_4 = \frac{Q_4}{C_4} = 47.8 \quad (\text{V}).$$

These results can be checked by verifying that $V_1 + V_2 = V_3$ and that $V_3 + V_4 = 100$ (V). ∎

3–10.2 CAPACITANCES IN MULTICONDUCTOR SYSTEMS

We now consider the situation of more than two conducting bodies in an isolated system, such as that shown in Fig. 3–34. The positions of the conductors are arbitrary, and one of the conductors may represent the ground. Obviously, the presence of a charge on any one of the conductors will affect the potential of all the others. Since the relation between potential and charge is linear, we may write the following set of N equations relating the potentials $V_1, V_2, \ldots, V_N$ of the N conductors to the charges $Q_1, Q_2, \ldots, Q_N$:

$$\begin{aligned} V_1 &= p_{11}Q_1 + p_{12}Q_2 + \cdots + p_{1N}Q_N, \\ V_2 &= p_{21}Q_1 + p_{22}Q_2 + \cdots + p_{2N}Q_N, \\ &\vdots \\ V_N &= p_{N1}Q_1 + p_{N2}Q_2 + \cdots + p_{NN}Q_N. \end{aligned} \tag{3–143}$$

In Eqs. (3–143) the p_{ij}'s are called the ***coefficients of potential***, which are constants whose values depend on the shape and position of the conductors as well as the permittivity of the surrounding medium. We note that in an isolated system,

$$Q_1 + Q_2 + Q_3 + \cdots + Q_N = 0. \tag{3–144}$$

The N linear equations in (3–143) can be inverted to express the charges as functions of potentials as follows:

$$\begin{aligned} Q_1 &= c_{11}V_1 + c_{12}V_2 + \cdots + c_{1N}V_N, \\ Q_2 &= c_{21}V_1 + c_{22}V_2 + \cdots + c_{2N}V_N, \\ &\vdots \\ Q_N &= c_{N1}V_1 + c_{N2}V_2 + \cdots + c_{NN}V_N, \end{aligned} \tag{3–145}$$

where the c_{ij}'s are constants whose values depend only on the p_{ij}'s in Eqs. (3–143). The coefficients c_{ii}'s are called the ***coefficients of capacitance***, which equal the ratios of the charge Q_i on and the potential V_i of the ith conductor ($i = 1, 2, \ldots, N$) with all other conductors grounded. The c_{ij}'s ($i \neq j$) are called the ***coefficients of induction.*** If a positive Q_i exists on the ith conductor, V_i will be positive, but the charge Q_j

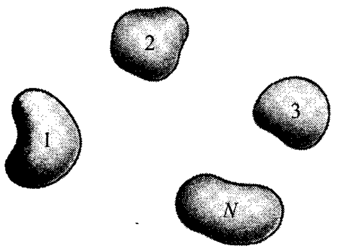


FIGURE 3–34
A multiconductor system.

induced on the jth ($j \neq i$) conductor will be negative. Hence the coefficients of capacitance c_{ii} are positive, and the coefficients of induction c_{ij} are negative. The condition of reciprocity guarantees that $p_{ij} = p_{ji}$ and $c_{ij} = c_{ji}$.

To establish a physical meaning to the coefficients of capacitance and the coefficients of induction, let us consider a four-conductor system as depicted in Fig. 3–34 with the stipulation that the conductor labeled N is now the conducting earth at zero potential and is designated by the number 0. A schematic diagram of the four-conductor system is shown in Fig. 3–35, in which the conductors 1, 2, and 3 have been drawn as simple dots (nodes). Coupling capacitances have been shown between pairs of nodes and between the three nodes and the ground. If Q_1, Q_2, Q_3 and V_1, V_2, V_3 denote the charges and the potentials, respectively, of conductors 1, 2, and 3, the first three equations in (3–145) become

$$Q_1 = c_{11}V_1 + c_{12}V_2 + c_{13}V_3, \tag{3–146a}$$
$$Q_2 = c_{12}V_1 + c_{22}V_2 + c_{23}V_3, \tag{3–146b}$$
$$Q_3 = c_{13}V_1 + c_{23}V_2 + c_{33}V_3, \tag{3–146c}$$

where we have used the symmetry relation $c_{ij} = c_{ji}$. On the other hand, we can write another set of three $Q \sim V$ relations based on the schematic diagram in Fig. 3–35:

$$Q_1 = C_{10}V_1 + C_{12}(V_1 - V_2) + C_{13}(V_1 - V_3), \tag{3–147a}$$
$$Q_2 = C_{20}V_2 + C_{12}(V_2 - V_1) + C_{23}(V_2 - V_3), \tag{3–147b}$$
$$Q_3 = C_{30}V_3 + C_{13}(V_3 - V_1) + C_{23}(V_3 - V_2), \tag{3–147c}$$

where C_{10}, C_{20}, and C_{30} are self-partial capacitances and $C_{ij}\,(i \neq j)$ are mutual partial capacitances.

Equations (3–147a), (3–147b), and (3–147c) can be rearranged as

$$Q_1 = (C_{10} + C_{12} + C_{13})V_1 - C_{12}V_2 - C_{13}V_3, \tag{3–148a}$$
$$Q_2 = -C_{12}V_1 + (C_{20} + C_{12} + C_{23})V_2 - C_{23}V_3, \tag{3–148b}$$
$$Q_3 = -C_{13}V_1 - C_{23}V_2 + (C_{30} + C_{13} + C_{23})V_3. \tag{3–148c}$$

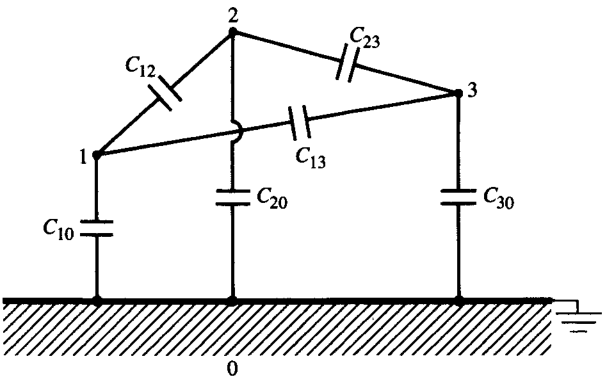


FIGURE 3–35
Schematic diagram of three conductors and the ground.

Comparing Eqs. (3–148) with Eqs. (3–146), we obtain

$$c_{11} = C_{10} + C_{12} + C_{13}, \quad (3\text{–}149a)$$

$$c_{22} = C_{20} + C_{12} + C_{23}, \quad (3\text{–}149b)$$

$$c_{33} = C_{30} + C_{13} + C_{23}, \quad (3\text{–}149c)$$

and

$$c_{12} = -C_{12}, \quad (3\text{–}150a)$$

$$c_{23} = -C_{23}, \quad (3\text{–}150b)$$

$$c_{13} = -C_{13}. \quad (3\text{–}150c)$$

On the basis of Eq. (3–149a) we can interpret the coefficient of capacitance c_{11} as the total capacitance between conductor 1 and all the other conductors connected together to ground; similarly for c_{22} and c_{33}. Equations (3–150) indicate that the coefficients of inductances are the negative of the mutual partial capacitances. Inverting Eqs. (3–149), we can express the conductor-to-ground capacitances in terms of the coefficients of capacitance and coefficients of induction:

$$C_{10} = c_{11} + c_{12} + c_{13}, \quad (3\text{–}151a)$$

$$C_{20} = c_{22} + c_{12} + c_{23}, \quad (3\text{–}151b)$$

$$C_{30} = c_{33} + c_{13} + c_{23}. \quad (3\text{–}151c)$$

EXAMPLE 3–21 Three horizontal parallel conducting wires, each of radius a and isolated from the ground, are separated from one another as shown in Fig. 3–36. Assuming $d \gg a$, determine the partial capacitances per unit length between the wires.

Solution We designate the wires as conductors 0, 1, and 2, as indicated in Fig. 3–36. Choosing conductor 0 as the reference and using Eq. (3–138), we can write two equations for the potential differences V_{10} and V_{20} due to the three wires as follows:

$$V_{10} = \frac{\rho_{\ell 0}}{2\pi\epsilon_0} \ln\frac{a}{d} + \frac{\rho_{\ell 1}}{2\pi\epsilon_0} \ln\frac{d}{a} + \frac{\rho_{\ell 2}}{2\pi\epsilon_0} \ln\frac{3d}{2d}$$

or

$$2\pi\epsilon_0 V_{10} = \rho_{\ell 0} \ln\frac{a}{d} + \rho_{\ell 1} \ln\frac{d}{a} + \rho_{\ell 2} \ln\frac{3}{2}, \quad (3\text{–}152a)$$

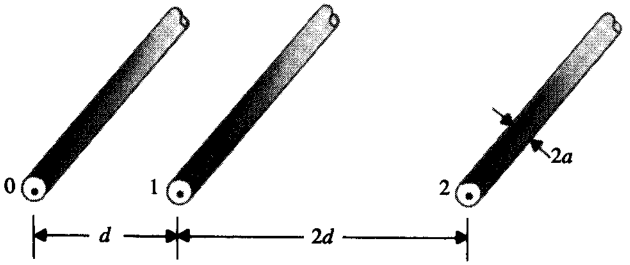


FIGURE 3–36
Three parallel wires (Example 3–21).

where $\rho_{\ell 0}$, $\rho_{\ell 1}$, and $\rho_{\ell 2}$ denote the charges per unit length on wires 0, 1, and 2 respectively. Similarly,

$$2\pi\epsilon_0 V_{20} = \rho_{\ell 0} \ln \frac{a}{3d} + \rho_{\ell 1} \ln \frac{d}{2d} + \rho_{\ell 2} \ln \frac{3d}{a}. \tag{3-152b}$$

For the isolated system of three conductors we have $\rho_{\ell 0} + \rho_{\ell 1} + \rho_{\ell 2} = 0$, or

$$\rho_{\ell 0} = -(\rho_{\ell 1} + \rho_{\ell 2}). \tag{3-153}$$

Combination of Eqs. (3–152a), (3–152b), and (3–153) yields

$$2\pi\epsilon_0 V_{10} = \rho_{\ell 1} 2 \ln \frac{d}{a} + \rho_{\ell 2} \ln \frac{3d}{2a}, \tag{3-154a}$$

$$2\pi\epsilon_0 V_{20} = \rho_{\ell 1} \ln \frac{3d}{2a} + \rho_{\ell 2} 2 \ln \frac{3d}{a}. \tag{3-154b}$$

Equations (3–154a) and (3–154b) can be used to solve for $\rho_{\ell 1}$ and $\rho_{\ell 2}$ as functions of V_{10} and V_{20}.

$$\rho_{\ell 1} = \Delta_0 \left(V_{10} 2 \ln \frac{3d}{a} - V_{20} \ln \frac{3d}{2a} \right), \tag{3-155a}$$

$$\rho_{\ell 2} = \Delta_0 \left(-V_{10} \ln \frac{3d}{2a} + V_{20} 2 \ln \frac{d}{a} \right), \tag{3-155b}$$

where

$$\Delta_0 = \frac{2\pi\epsilon_0}{4 \ln \frac{d}{a} \ln \frac{3d}{a} - \left(\ln \frac{3d}{2a} \right)^2}. \tag{3-156}$$

Comparing Eqs. (3–155) with Eqs. (3–146), (3–148), and (3–151), we obtain the following partial capacitances per unit length for the given three-wire system:

$$C_{12} = -c_{12} = \Delta_0 \ln \frac{3d}{2a}, \tag{3-157a}$$

$$C_{10} = c_{11} + c_{12} = \Delta_0 \left(2 \ln \frac{3d}{a} - \ln \frac{3d}{2a} \right), \tag{3-157b}$$

$$C_{20} = c_{22} + c_{12} = \Delta_0 \left(2 \ln \frac{d}{a} - \ln \frac{3d}{2a} \right). \tag{3-157c}$$

■

3–10.3 ELECTROSTATIC SHIELDING

Electrostatic shielding, a technique for reducing capacitive coupling between conducting bodies, is important in some practical applications. Let us consider the situation shown in Fig. 3–37, in which a grounded conducting shell 2 completely

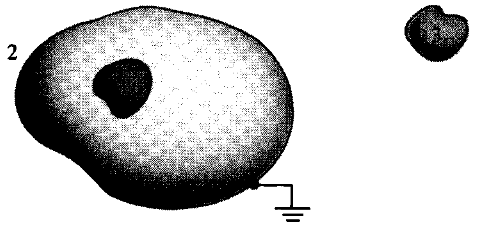


FIGURE 3–37
Illustrating electrostatic shielding.

encloses conducting body 1. Setting $V_2 = 0$ in Eq. (3–147a), we have

$$Q_1 = C_{10}V_1 + C_{12}V_1 + C_{13}(V_1 - V_3). \tag{3–158}$$

When $Q_1 = 0$, there is no field inside shell 2; hence body 1 and shell 2 have the same potential, $V_1 = V_2 = 0$. From Eq. (3–158) we see that the coupling capacitance C_{13} must vanish, since V_3 is arbitrary. This means that a change in V_3 will not affect Q_1, and vice versa. We then have electrostatic shielding between conducting bodies 1 and 3. Obviously, the same shielding effectiveness is obtained if the grounded conducting shell 2 encloses body 3 instead of body 1.

3–11 Electrostatic Energy and Forces

In Section 3–5 we indicated that electric potential at a point in an electric field is the work required to bring a unit positive charge from infinity (at reference zero-potential) to that point. To bring a charge Q_2 (slowly, so that kinetic energy and radiation effects may be neglected) from infinity *against* the field of a charge Q_1 in free space to a distance R_{12}, the amount of work required is

$$W_2 = Q_2V_2 = Q_2\frac{Q_1}{4\pi\epsilon_0 R_{12}}. \tag{3–159}$$

Because electrostatic fields are conservative, W_2 is independent of the path followed by Q_2. Another form of Eq. (3–159) is

$$W_2 = Q_1\frac{Q_2}{4\pi\epsilon_0 R_{12}} = Q_1V_1. \tag{3–160}$$

This work is stored in the assembly of the two charges as potential energy. Combining Eqs. (3–159) and (3–160), we can write

$$W_2 = \tfrac{1}{2}(Q_1V_1 + Q_2V_2). \tag{3–161}$$

Now suppose another charge Q_3 is brought from infinity to a point that is R_{13} from Q_1 and R_{23} from Q_2; an additional amount of work is required that equals

$$\Delta W = Q_3V_3 = Q_3\left(\frac{Q_1}{4\pi\epsilon_0 R_{13}} + \frac{Q_2}{4\pi\epsilon_0 R_{23}}\right). \tag{3–162}$$

The sum of ΔW in Eq. (3–162) and W_2 in Eq. (3–159) is the potential energy, W_3, stored in the assembly of the three charges Q_1, Q_2, and Q_3. That is,

$$W_3 = W_2 + \Delta W = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_1Q_2}{R_{12}} + \frac{Q_1Q_3}{R_{13}} + \frac{Q_2Q_3}{R_{23}}\right). \qquad (3\text{–}163)$$

We can rewrite W_3 in the following form:

$$\begin{aligned} W_3 &= \frac{1}{2}\left[Q_1\left(\frac{Q_2}{4\pi\epsilon_0 R_{12}} + \frac{Q_3}{4\pi\epsilon_0 R_{13}}\right) + Q_2\left(\frac{Q_1}{4\pi\epsilon_0 R_{12}} + \frac{Q_3}{4\pi\epsilon_0 R_{23}}\right)\right.\\ &\quad \left. + Q_3\left(\frac{Q_1}{4\pi\epsilon_0 R_{13}} + \frac{Q_2}{4\pi\epsilon_0 R_{23}}\right)\right] \\ &= \tfrac{1}{2}(Q_1V_1 + Q_2V_2 + Q_3V_3). \end{aligned} \qquad (3\text{–}164)$$

In Eq. (3–164), V_1, the potential at the position of Q_1, is caused by charges Q_2 and Q_3; it is *different* from the V_1 in Eq. (3–160) in the two-charge case. Similarly, V_2 and V_3 are the potentials at Q_2 and Q_3, respectively, in the three-charge assembly.

Extending this procedure of bringing in additional charges, we arrive at the following general expression for the potential energy of a group of N discrete point charges at rest. (The purpose of the subscript e on W_e is to denote that the energy is of an electric nature.) We have

$$\boxed{W_e = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k \qquad \text{(J)},} \qquad (3\text{–}165)$$

where V_k, the electric potential at Q_k, is caused by all the other charges and has the following expression:

$$V_k = \frac{1}{4\pi\epsilon_0}\sum_{\substack{j=1\\(j\neq k)}}^{N} \frac{Q_j}{R_{jk}}. \qquad (3\text{–}166)$$

Two remarks are in order here. First, W_e can be negative. For instance, W_2 in Eq. (3–159) will be negative if Q_1 and Q_2 are of opposite signs. In that case, work is done by the field (not against the field) established by Q_1 in moving Q_2 from infinity. Second, W_e in Eq. (3–165) represents only the interaction energy (mutual energy) and does not include the work required to assemble the individual point charges themselves (self-energy).

The SI unit for energy, ***joule*** (J), is too large a unit for work in physics of elementary particles, where energy is more conveniently measured in terms of a much smaller unit called ***electron-volt*** (eV). An electron-volt is the energy or work required to move an electron against a potential difference of one volt.

$$1 \quad \text{(eV)} = (1.60 \times 10^{-19}) \times 1 = 1.60 \times 10^{-19} \qquad \text{(J)}. \qquad (3\text{–}167)$$

Energy in (eV) is essentially that in (J) per unit electronic charge. The proton beams of the world's most powerful high-energy particle accelerator collide with a kinetic energy of two trillion electron-volts (2 TeV), or $(2 \times 10^{12}) \times (1.60 \times 10^{-19}) = 3.20 \times 10^{-7}$ (J). A binding energy of $W = 5 \times 10^{-19}$ (J) in an ionic crystal is equal to $W/e = 5 \times 10^{-19}/1.60 \times 10^{-19} = 3.125$ (eV), which is a more convenient number to use than the one in terms of joules.

EXAMPLE 3–22 Find the energy required to assemble a uniform sphere of charge of radius b and volume charge density ρ.

Solution Because of symmetry, it is simplest to assume that the sphere of charge is assembled by bringing up a succession of spherical layers of thickness dR. At a radius R shown in Fig. 3–38 the potential is

$$V_R = \frac{Q_R}{4\pi\epsilon_0 R},$$

where Q_R is the total charge contained in a sphere of radius R:

$$Q_R = \rho \tfrac{4}{3}\pi R^3.$$

The differential charge in a spherical layer of thickness dR is

$$dQ_R = \rho 4\pi R^2\, dR,$$

and the work or energy in bringing up dQ_R is

$$dW = V_R\, dQ_R = \frac{4\pi}{3\epsilon_0}\rho^2 R^4\, dR.$$

Hence the total work or energy required to assemble a uniform sphere of charge of radius b and charge density ρ is

$$W = \int dW = \frac{4\pi}{3\epsilon_0}\rho^2 \int_0^b R^4\, dR = \frac{4\pi\rho^2 b^5}{15\epsilon_0} \quad \text{(J)}. \qquad (3\text{–}168)$$

In terms of the total charge

$$Q = \rho \frac{4\pi}{3} b^3,$$

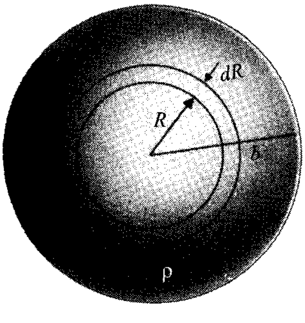


FIGURE 3–38
Assembling a uniform sphere of charge (Example 3–22).

we have

$$W = \frac{3Q^2}{20\pi\epsilon_0 b} \qquad \text{(J)}. \tag{3-169}$$

Equation (3–169) shows that the energy is directly proportional to the square of the total charge and inversely proportional to the radius. The sphere of charge in Fig. 3–38 could be a cloud of electrons, for instance.

For a continuous charge distribution of density ρ the formula for W_e in Eq. (3–165) for discrete charges must be modified. Without going through a separate proof we replace Q_k by $\rho\,dv$ and the summation by an integration and obtain

$$\boxed{W_e = \frac{1}{2}\int_{V'} \rho V\,dv \qquad \text{(J)}.} \tag{3-170}$$

In Eq. (3–170), V is the potential at the point where the volume charge density is ρ, and V' is the volume of the region where ρ exists.

EXAMPLE 3–23 Solve the problem in Example 3–22 by using Eq. (3–170).

Solution In Example 3–22 we solved the problem of assembling a sphere of charge by bringing up a succession of spherical layers of a differential thickness. Now we assume that the sphere of charge is already in place. Since ρ is a constant, it can be taken out of the integral sign. For a spherically symmetrical problem,

$$W_e = \frac{\rho}{2}\int_{V'} V\,dv = \frac{\rho}{2}\int_0^b V\,4\pi R^2\,dR, \tag{3-171}$$

where V is the potential at a point R from the center. To find V at R, we must find the negative of the line integral of $\mathbf{E}$ in two regions: (1) $\mathbf{E}_1 = \mathbf{a}_R E_{R1}$ from $R = \infty$ to $R = b$, and (2) $\mathbf{E}_2 = \mathbf{a}_R E_{R2}$ from $R = b$ to $R = R$. We have

$$\mathbf{E}_{R1} = \mathbf{a}_R \frac{Q}{4\pi\epsilon_0 R^2} = \mathbf{a}_R \frac{\rho b^3}{3\epsilon_0 R^2}, \qquad R \geq b,$$

and

$$\mathbf{E}_{R2} = \mathbf{a}_R \frac{Q_R}{4\pi\epsilon_0 R^2} = \mathbf{a}_R \frac{\rho R}{3\epsilon_0}, \qquad 0 < R \leq b.$$

Consequently, we obtain

$$\begin{aligned} V &= -\int_\infty^R \mathbf{E}\cdot d\mathbf{R} = -\left[\int_\infty^b E_{R1}\,dR + \int_b^R E_{R2}\,dR\right] \\ &= -\left[\int_\infty^b \frac{\rho b^3}{3\epsilon_0 R^2}\,dR + \int_b^R \frac{\rho R}{3\epsilon_0}\,dR\right] \\ &= \frac{\rho}{3\epsilon_0}\left(b^2 + \frac{b^2}{2} - \frac{R^2}{2}\right) = \frac{\rho}{3\epsilon_0}\left(\frac{3}{2}b^2 - \frac{R^2}{2}\right). \end{aligned} \tag{3-172}$$

Substituting Eq. (3–172) in Eq. (3–171), we get

$$W_e = \frac{\rho}{2}\int_0^b \frac{\rho}{3\epsilon_0}\left(\frac{3}{2}b^2 - \frac{R^2}{2}\right)4\pi R^2\,dR = \frac{4\pi\rho^2 b^5}{15\epsilon_0},$$

which is the same as the result in Eq. (3–168). ∎

Note that W_e in Eq. (3–170) includes the work (self-energy) required to assemble the distribution of macroscopic charges, because it is the energy of interaction of every infinitesimal charge element with all other infinitesimal charge elements. As a matter of fact, we have used Eq. (3–170) in Example 3–23 to find the self-energy of a uniform spherical charge. As the radius b approaches zero, the self-energy of a (mathematical) point charge of a given Q is infinite (see Eq. 3–169). The self-energies of point charges Q_k are not included in Eq. (3–165). Of course, there are, strictly, no point charges, inasmuch as the smallest charge unit, the electron, is itself a distribution of charge.

3–11.1 ELECTROSTATIC ENERGY IN TERMS OF FIELD QUANTITIES

In Eq. (3–170) the expression of electrostatic energy of a charge distribution contains the source charge density ρ and the potential function V. We frequently find it more convenient to have an expression of W_e in terms of field quantities $\mathbf{E}$ and/or $\mathbf{D}$, without knowing ρ explicitly. To this end, we substitute $\nabla \cdot \mathbf{D}$ for ρ in Eq. (3–170):

$$W_e = \tfrac{1}{2}\int_{V'} (\nabla \cdot \mathbf{D})V\,dv. \tag{3–173}$$

Now, using the vector identity (from Problem P.2–28),

$$\nabla \cdot (V\mathbf{D}) = V\nabla \cdot \mathbf{D} + \mathbf{D} \cdot \nabla V, \tag{3–174}$$

we can write Eq. (3–173) as

$$\begin{aligned} W_e &= \tfrac{1}{2}\int_{V'} \nabla \cdot (V\mathbf{D})\,dv - \tfrac{1}{2}\int_{V'} \mathbf{D} \cdot \nabla V\,dv \\ &= \tfrac{1}{2}\oint_{S'} V\mathbf{D} \cdot \mathbf{a}_n\,ds + \tfrac{1}{2}\int_{V'} \mathbf{D} \cdot \mathbf{E}\,dv, \end{aligned} \tag{3–175}$$

where the divergence theorem has been used to change the first volume integral into a closed surface integral and $\mathbf{E}$ has been substituted for $-\nabla V$ in the second volume integral. Since V' can be any volume that includes all the charges, we may choose it to be a very large sphere with radius R. As we let $R \to \infty$, electric potential V and the magnitude of electric displacement D fall off at least as fast as $1/R$ and $1/R^2$, respectively.† The area of the bounding surface S' increases as R^2. Hence the surface integral in Eq. (3–175) decreases at least as fast as $1/R$ and will vanish as $R \to \infty$. We are then left with only the second integral on the right side of Eq. (3–175).

† For point charges $V \propto 1/R$ and $D \propto 1/R^2$; for dipoles $V \propto 1/R^2$ and $D \propto 1/R^3$.

$$\boxed{W_e = \frac{1}{2}\int_{V'} \mathbf{D}\cdot\mathbf{E}\,dv \qquad \text{(J).}} \tag{3–176a}$$

Using the relation $\mathbf{D} = \epsilon\mathbf{E}$ for a linear medium, Eq. (3–176a) can be written in two other forms:

$$\boxed{W_e = \frac{1}{2}\int_{V'} \epsilon E^2\,dv \qquad \text{(J)}} \tag{3–176b}$$

and

$$\boxed{W_e = \frac{1}{2}\int_{V'} \frac{D^2}{\epsilon}\,dv \qquad \text{(J).}} \tag{3–176c}$$

We can always define an ***electrostatic energy density*** w_e mathematically, such that its volume integral equals the total electrostatic energy:

$$W_e = \int_{V'} w_e\,dv. \tag{3–177}$$

We can therefore write

$$w_e = \tfrac{1}{2}\mathbf{D}\cdot\mathbf{E} \qquad (\text{J/m}^3) \tag{3–178a}$$

or

$$w_e = \tfrac{1}{2}\epsilon E^2 \qquad (\text{J/m}^3) \tag{3–178b}$$

or

$$w_e = \frac{D^2}{2\epsilon} \qquad (\text{J/m}^3). \tag{3–178c}$$

However, this definition of energy density is artificial because a physical justification has not been found to localize energy with an electric field; all we know is that the volume integrals in Eqs. (3–176a, b, c) give the correct total electrostatic energy.

EXAMPLE 3–24 In Fig. 3–39 a parallel-plate capacitor of area S and separation d is charged to a voltage V. The permittivity of the dielectric is ϵ. Find the stored electrostatic energy.

Solution With the d-c source (batteries) connected as shown, the upper and lower plates are charged positive and negative, respectively. If the fringing of the field at

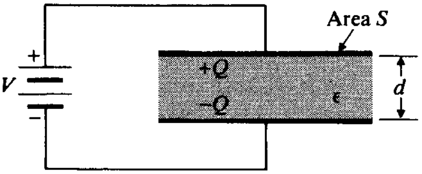


FIGURE 3–39
A charged parallel-plate capacitor (Example 3–24).

the edges is neglected, the electric field in the dielectric is uniform (over the plate) and constant (across the dielectric) and has a magnitude

$$E = \frac{V}{d}.$$

Using Eq. (3–176b), we have

$$W_e = \frac{1}{2}\int_{V'} \epsilon\left(\frac{V}{d}\right)^2 dv = \frac{1}{2}\epsilon\left(\frac{V}{d}\right)^2 (Sd) = \frac{1}{2}\left(\epsilon\frac{S}{d}\right)V^2. \tag{3–179}$$

The quantity in the parentheses of the last expression, $\epsilon S/d$, is the capacitance of the parallel-plate capacitor (see Eq. 3–136). So,

$$\boxed{W_e = \tfrac{1}{2}CV^2 \qquad \text{(J)}.} \tag{3–180a}$$

Since $Q = CV$, Eq. (3–180a) can be put in two other forms:

$$\boxed{W_e = \tfrac{1}{2}QV \qquad \text{(J)}} \tag{3–180b}$$

and

$$\boxed{W_e = \frac{Q^2}{2C} \qquad \text{(J)}.} \tag{3–180c}$$

It so happens that Eqs. (3–180a, b, c) hold true for any two-conductor capacitor (see Problem P.3–43).

EXAMPLE 3–25 Use energy formulas (3–176) and (3–180) to find the capacitance of a cylindrical capacitor having a length L, an inner conductor of radius a, an outer conductor of inner radius b, and a dielectric of permittivity ϵ, as shown in Fig. 3–29.

Solution By applying Gauss's law, we know that

$$\mathbf{E} = \mathbf{a}_r E_r = \mathbf{a}_r \frac{Q}{2\pi\epsilon L r}, \qquad a < r < b.$$

The electrostatic energy stored in the dielectric region is, from Eq. (3–176b),

$$\begin{aligned} W_e &= \frac{1}{2}\int_a^b \epsilon\left(\frac{Q}{2\pi\epsilon L r}\right)^2 (L 2\pi r\, dr) \\ &= \frac{Q^2}{4\pi\epsilon L}\int_a^b \frac{dr}{r} = \frac{Q^2}{4\pi\epsilon L}\ln\frac{b}{a}. \end{aligned} \tag{3–181}$$

On the other hand, W_e can also be expressed in the form of Eq. (3–180c). Equating (3–180c) and (3–181), we obtain

$$\frac{Q^2}{2C} = \frac{Q^2}{4\pi\epsilon L} \ln \frac{b}{a}$$

or

$$C = \frac{2\pi\epsilon L}{\ln \dfrac{b}{a}},$$

which is the same as that given in Eq. (3–139). ▬

3–11.2 ELECTROSTATIC FORCES

Coulomb's law governs the force between two point charges. In a more complex system of charged bodies, using Coulomb's law to determine the force on one of the bodies that is caused by the charges on other bodies would be very tedious. This would be so even in the simple case of finding the force between the plates of a charged parallel-plate capacitor. We will now discuss a method for calculating the force on an object in a charged system from the electrostatic energy of the system. This method is based on the ***principle of virtual displacement***. We will consider two cases: (1) that of an isolated system of bodies with fixed charges, and (2) that of a system of conducting bodies with fixed potentials.

System of Bodies with Fixed Charges We consider an isolated system of charged conducting, as well as dielectric, bodies separated from one another with no connection to the outside world. The charges on the bodies are constant. Imagine that the electric forces have displaced one of the bodies by a differential distance $d\ell$ (a virtual displacement). The mechanical work done *by the system* would be

$$dW = \mathbf{F}_Q \cdot d\ell, \tag{3–182}$$

where $\mathbf{F}_Q$ is the total electric force acting on the body under the condition of constant charges. Since we have an isolated system with no external supply of energy, this mechanical work must be done at the expense of the stored electrostatic energy; that is,

$$dW = -dW_e = \mathbf{F}_Q \cdot d\ell. \tag{3–183}$$

Noting from Eq. (2–88) in Section 2–6 that the differential change of a scalar resulting from a position change $d\ell$ is the dot product of the gradient of the scalar, and $d\ell$, we write

$$dW_e = (\nabla W_e) \cdot d\ell. \tag{3–184}$$

Since $d\ell$ is arbitrary, comparison of Eqs. (3–183) and (3–184) leads to

$$\boxed{\mathbf{F}_Q = -\nabla W_e \qquad \text{(N)}.} \tag{3–185}$$

Equation (3–185) is a very simple formula for the calculation of $\mathbf{F}_Q$ from the electrostatic energy of the system. In Cartesian coordinates the component forces are

$$(F_Q)_x = -\frac{\partial W_e}{\partial x}, \tag{3–186a}$$

$$(F_Q)_y = -\frac{\partial W_e}{\partial y}, \tag{3–186b}$$

$$(F_Q)_z = -\frac{\partial W_e}{\partial z}. \tag{3–186c}$$

If the body under consideration is constrained to rotate about an axis, say the z-axis, the mechanical work done by the system for a virtual angular displacement $d\phi$ would be

$$dW = (T_Q)_z\, d\phi, \tag{3–187}$$

where $(T_Q)_z$ is the z-component of the torque acting on the body under the condition of constant charges. The foregoing procedure will lead to

$$\boxed{(T_Q)_z = -\frac{\partial W_e}{\partial \phi} \qquad (\text{N}\cdot\text{m}).} \tag{3–188}$$

System of Conducting Bodies with Fixed Potentials Now consider a system in which conducting bodies are held at fixed potentials through connections to such external sources as batteries. Uncharged dielectric bodies may also be present. A displacement $d\ell$ by a conducting body would result in a change in total electrostatic energy and would require the sources to transfer charges to the conductors in order to keep them at their fixed potentials. If a charge dQ_k (which may be positive or negative) is added to the kth conductor that is maintained at potential V_k, the work done or energy supplied by the sources is $V_k\, dQ_k$. The total energy supplied by the sources to the system is

$$dW_s = \sum_k V_k\, dQ_k. \tag{3–189}$$

The mechanical work done by the system as a consequence of the virtual displacement is

$$dW = \mathbf{F}_V \cdot d\boldsymbol{\ell}, \tag{3–190}$$

where $\mathbf{F}_V$ is the electric force on the conducting body under the condition of constant potentials. The charge transfers also change the electrostatic energy of the system by an amount dW_e, which, in view of Eq. (3–165), is

$$dW_e = \frac{1}{2}\sum_k V_k\, dQ_k = \frac{1}{2}\, dW_s. \tag{3–191}$$

Conservation of energy demands that

$$dW + dW_e = dW_s. \tag{3-192}$$

Substitution of Eqs. (3–189), (3–190), and (3–191) in Eq. (3–192) gives

$$\begin{aligned} \mathbf{F}_V \cdot d\boldsymbol{\ell} &= dW_e \\ &= (\nabla W_e) \cdot d\boldsymbol{\ell} \end{aligned}$$

or

$$\boxed{\mathbf{F}_V = \nabla W_e \qquad \text{(N)}.} \tag{3-193}$$

Comparison of Eqs. (3–193) and (3–185) reveals that the only difference between the formulas for the electric forces in the two cases is in the sign. It is clear that if the conducting body is constrained to rotate about the z-axis, the z-component of the electric torque will be

$$\boxed{(T_V)_z = \frac{\partial W_e}{\partial \phi} \qquad \text{(N·m)},} \tag{3-194}$$

which differs from Eq. (3–188) also only by a sign change.

EXAMPLE 3–26 Determine the force on the conducting plates of a charged parallel-plate capacitor. The plates have an area S and are separated in air by a distance x.

Soiution We solve the problem in two ways: (a) by assuming fixed charges, and then (b) by assuming fixed potentials. The fringing of field around the edges of the plates will be neglected.

a) *Fixed charges.* With fixed charges $\pm Q$ on the plates, an electric field intensity $E_x = Q/(\epsilon_0 S) = V/x$ exists in the air between the plates regardless of their separation (unchanged by a virtual displacement). From Eq. (3–180b),

$$W_e = \tfrac{1}{2}QV = \tfrac{1}{2}QE_x x,$$

where Q and E_x are constants. Using Eq. (3–186a), we obtain

$$(F_Q)_x = -\frac{\partial}{\partial x}\left(\frac{1}{2}QE_x x\right) = -\frac{1}{2}QE_x = -\frac{Q^2}{2\epsilon_0 S}, \tag{3-195}$$

where the negative signs indicate that the force is opposite to the direction of increasing x. It is an attractive force.

b) *Fixed potentials.* With fixed potentials it is more convenient to use the expression in Eq. (3–180a) for W_e. Capacitance C for the parallel-plate air capacitor is $\epsilon_0 S/x$. We have, from Eq. (3–193),

$$(F_V)_x = \frac{\partial W_e}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{2}CV^2\right) = \frac{V^2}{2}\frac{\partial}{\partial x}\left(\frac{\epsilon_0 S}{x}\right) = -\frac{\epsilon_0 S V^2}{2x^2}. \tag{3-196}$$

How different are $(F_Q)_x$ in Eq. (3–195) and $(F_V)_x$ in Eq. (3–196)? Recalling the relation

$$Q = CV = \frac{\epsilon_0 SV}{x},$$

we find

$$(F_Q)_x = (F_V)_x. \tag{3-197}$$

The force is the same in both cases in spite of the apparent sign difference in the formulas as expressed by Eqs. (3–185) and (3–193). A little reflection on the physical problem will convince us that this must be true. Since the charged capacitor has fixed dimensions, a given Q will result in a fixed V, and vice versa. Therefore there is a unique force between the plates regardless of whether Q or V is given, and the force certainly does not depend on virtual displacements. A change in the conceptual constraint (fixed Q or fixed V) cannot change the unique force between the plates. ∎

The preceding discussion holds true for a general charged two-conductor capacitor with capacitance C. The electrostatic force F_ℓ in the direction of a virtual displacement $d\ell$ for fixed charges is

$$(F_Q)_\ell = -\frac{\partial W_e}{\partial \ell} = -\frac{\partial}{\partial \ell}\left(\frac{Q^2}{2C}\right) = \frac{Q^2}{2C^2}\frac{\partial C}{\partial \ell}. \tag{3-198}$$

For fixed potentials,

$$(F_V)_\ell = \frac{\partial W_e}{\partial \ell} = \frac{\partial}{\partial \ell}\left(\frac{1}{2}CV^2\right) = \frac{V^2}{2}\frac{\partial C}{\partial \ell} = \frac{Q^2}{2C^2}\frac{\partial C}{\partial \ell}. \tag{3-199}$$

It is clear that the forces calculated from the two procedures, which assumed different constraints imposed on the same charged capacitor, are equal.

Review Questions

R.3–1 Write the differential form of the fundamental postulates of electrostatics in free space.

R.3–2 Under what conditions will the electric field intensity be both solenoidal and irrotational?

R.3–3 Write the integral form of the fundamental postulates of electrostatics in free space, and state their meaning in words.

R.3–4 When the formula for the electric field intensity of a point charge, Eq. (3–12), was derived,

a) why was it necessary to stipulate that q is in a boundless free space?
b) why did we *not* construct a cubic or a cylindrical surface around q?

R.3–5 In what ways does the electric field intensity vary with distance for

a) a point charge? **b)** an electric dipole?

R.3–6 State *Coulomb's law.*

R.3–7 Explain the principle of operation of ink-jet printers.

R.3–8 State *Gauss's law.* Under what conditions is Gauss's law especially useful in determining the electric field intensity of a charge distribution?

R.3–9 Describe the ways in which the electric field intensity of an infinitely long, straight line charge of uniform density varies with distance.

R.3–10 Is Gauss's law useful in finding the **E** field of a finite line charge? Explain.

R.3–11 See Example 3–6, Fig. 3–9. Could a cylindrical pillbox with circular top and bottom faces be chosen as a Gaussian surface? Explain.

R.3–12 Make a two-dimensional sketch of the electric field lines and the equipotential lines of a point charge.

R.3–13 At what value of θ is the **E** field of a z-directed electric dipole pointed in the negative z-direction?

R.3–14 Refer to Eq. (3–64). Explain why the absolute sign around z is required.

R.3–15 If the electric potential at a point is zero, does it follow that the electrical field intensity is also zero at that point? Explain.

R.3–16 If the electric field intensity at a point is zero, does it follow that the electric potential is also zero at that point? Explain.

R.3–17 If an uncharged spherical conducting shell of a finite thickness is placed in an external electric field $\mathbf{E}_o$, what is the electric field intensity at the center of the shell? Describe the charge distributions on both the outer and the inner surfaces of the shell.

R.3–18 What are *electrets*? How can they be made?

R.3–19 Can $\nabla'(1/R)$ in Eq. (3–84) be replaced by $\nabla(1/R)$? Explain.

R.3–20 Define *polarization vector.* What is its SI unit?

R.3–21 What are *polarization charge densities*? What are the SI units for $\mathbf{P} \cdot \mathbf{a}_n$ and $\nabla \cdot \mathbf{P}$?

R.3–22 What do we mean by *simple medium*?

R.3–23 What properties do *anisotropic materials* have?

R.3–24 What characterizes a *uniaxial medium*?

R.3–25 Define *electric displacement vector.* What is its SI unit?

R.3–26 Define *electric susceptibility.* What is its unit?

R.3–27 What is the difference between the *permittivity* and the *dielectric constant* of a medium?

R.3–28 Does the electric flux density due to a given charge distribution depend on the properties of the medium? Does the electric field intensity? Explain.

R.3–29 What is the difference between the *dielectric constant* and the *dielectric strength* of a dielectric material?

R.3–30 Explain the principle of operation of lightning arresters.

R.3–31 What are the general boundary conditions for electrostatic fields at an interface between two different dielectric media?

R.3–32 What are the boundary conditions for electrostatic fields at an interface between a conductor and a dielectric with permittivity ϵ?

R.3–33 What is the boundary condition for electrostatic potential at an interface between two different dielectric media?

R.3–34 Does a force exist between a point charge and a dielectric body? Explain.

R.3–35 Define *capacitance* and *capacitor.*

R.3–36 Assume that the permittivity of the dielectric in a parallel-plate capacitor is not constant. Will Eq. (3–136) hold if the average value of permittivity is used for ϵ in the formula? Explain.

R.3–37 Given three 1-μF capacitors, explain how they should be connected in order to obtain a total capacitance of

a) $\frac{1}{3}$ (μF), **b)** $\frac{2}{3}$ (μF), **c)** $\frac{3}{2}$ (μF), **d)** 3 (μF).

R.3–38 What are *coefficients of potential, coefficients of capacitance,* and *coefficients of induction*?

R.3–39 What are *partial capacitances*? How are they different from coefficients of capacitance?

R.3–40 Explain the principle of electrostatic shielding.

R.3–41 What is the definition of an *electron-volt*? How does it compare with a joule?

R.3–42 What is the expression for the electrostatic energy of an assembly of four discrete point charges?

R.3–43 What is the expression for the electrostatic energy of a continuous distribution of charge in a volume? on a surface? along a line?

R.3–44 Provide a mathematical expression for electrostatic energy in terms of **E** and/or **D**.

R.3–45 Discuss the meaning and use of the *principle of virtual displacement.*

R.3–46 What is the relation between the force and the stored energy in a system of stationary charged objects under the condition of constant charges? Under the condition of fixed potentials?

Problems

P.3–1 Refer to Fig. 3–4.

a) Find the relation between the angle of arrival, α, of the electron beam at the screen and the deflecting electric field intensity E_d.

b) Find the relation between w and L such that $d_1 = d_0/20$.

P.3–2 The cathode-ray oscilloscope (CRO) shown in Fig. 3–4 is used to measure the voltage applied to the parallel deflection plates.

a) Assuming no breakdown in insulation, what is the maximum voltage that can be measured if the distance of separation between the plates is h?

b) What is the restriction on L if the diameter of the screen is D?

c) What can be done with a fixed geometry to double the CRO's maximum measurable voltage?

P.3–3 The deflection system of a cathode-ray oscilloscope usually consists of two pairs of parallel plates producing orthogonal electric fields. Assume the presence of another set of plates in Fig. 3–4 that establishes a uniform electric field $\mathbf{E}_x = \mathbf{a}_x E_x$ in the deflection region. Deflection voltages $v_x(t)$ and $v_y(t)$ are applied to produce $\mathbf{E}_x$ and $\mathbf{E}_y$, respectively. Determine the types of waveforms that $v_x(t)$ and $v_y(t)$ should have if the electrons are to trace the following graphs on the fluorescent screen:

a) a horizontal line,
b) a straight line having a negative unity slope,
c) a circle,
d) two cycles of a sine wave.

P.3–4 Write a short article explaining the principle of operation of xerography. (Use library resources if needed.)

P.3–5 Two point charges, Q_1 and Q_2, are located at (1, 2, 0) and (2, 0, 0), respectively. Find the relation between Q_1 and Q_2 such that the total force on a test charge at the point $P(-1, 1, 0)$ will have

a) no x-component, **b)** no y-component.

P.3–6 Two very small conducting spheres, each of a mass 1.0×10^{-4} (kg), are suspended at a common point by very thin nonconducting threads of a length 0.2 (m). A charge Q is placed on each sphere. The electric force of repulsion separates the spheres, and an equilibrium is reached when the suspending threads make an angle of 10°. Assuming a gravitational force of 9.80 (N/kg) and a negligible mass for the threads, find Q.

P.3–7 Find the force between a charged circular loop of radius b and uniform charge density ρ_ℓ and a point charge Q located on the loop axis at a distance h from the plane of the loop. What is the force when $h \gg b$, and when $h = 0$? Plot the force as a function of h.

P.3–8 A line charge of uniform density ρ_ℓ in free space forms a semicircle of radius b. Determine the magnitude and direction of the electric field intensity at the center of the semicircle.

P.3–9 Three uniform line charges—$\rho_{\ell 1}$, $\rho_{\ell 2}$, and $\rho_{\ell 3}$, each of length L—form an equilateral triangle. Assuming that $\rho_{\ell 1} = 2\rho_{\ell 2} = 2\rho_{\ell 3}$, determine the electric field intensity at the center of the triangle.

P.3–10 Assuming that the electric field intensity is $\mathbf{E} = \mathbf{a}_x 100x$ (V/m), find the total electric charge contained inside

a) a cubical volume 100 (mm) on a side centered symmetrically at the origin,
b) a cylindrical volume around the z-axis having a radius 50 (mm) and a height 100 (mm) centered at the origin.

P.3–11 A spherical distribution of charge $\rho = \rho_0[1 - (R^2/b^2)]$ exists in the region $0 \leq R \leq b$. This charge distribution is concentrically surrounded by a conducting shell with inner radius R_i $(>b)$ and outer radius R_o. Determine $\mathbf{E}$ everywhere.

P.3–12 Two infinitely long coaxial cylindrical surfaces, $r = a$ and $r = b$ $(b > a)$, carry surface charge densities ρ_{sa} and ρ_{sb}, respectively.

a) Determine $\mathbf{E}$ everywhere.
b) What must be the relation between a and b in order that $\mathbf{E}$ vanishes for $r > b$?

P.3–13 Determine the work done in carrying a -2 (μC) charge from $P_1(2, 1, -1)$ to $P_2(8, 2, -1)$ in the field $\mathbf{E} = \mathbf{a}_x y + \mathbf{a}_y x$

a) along the parabola $x = 2y^2$,
b) along the straight line joining P_1 and P_2.

P.3–14 At what values of θ does the electric field intensity of a z-directed dipole have no z-component?

P.3–15 Three charges $(+q$, $-2q$, and $+q)$ are arranged along the z-axis at $z = d/2$, $z = 0$, and $z = -d/2$, respectively.

a) Determine V and $\mathbf{E}$ at a distant point $P(R, \theta, \phi)$.
b) Find the equations for equipotential surfaces and streamlines.
c) Sketch a family of equipotential lines and streamlines.

(Such an arrangement of three charges is called a ***linear electrostatic quadrupole***.)

P.3–16 A finite line charge of length L carrying uniform line charge density ρ_ℓ is coincident with the x-axis.

a) Determine V in the plane bisecting the line charge.
b) Determine $\mathbf{E}$ from ρ_ℓ directly by applying Coulomb's law.
c) Check the answer in part (b) with $-\nabla V$.

P.3–17 In Example 3–5 we obtained the electric field intensity around an infinitely long line charge of a uniform charge density in a very simple manner by applying Gauss's law. Since $|\mathbf{E}|$ is a function of r only, any coaxial cylinder around the infinite line charge is an equipotential surface. In practice, all conductors are of finite length. A finite line charge carrying a constant charge density ρ_ℓ along the axis, however, does not produce a constant potential on a concentric cylindrical surface. Given the finite line charge ρ_ℓ of length L in Fig. 3–40, find the potential on the cylindrical surface of radius b as a function of x and plot it.

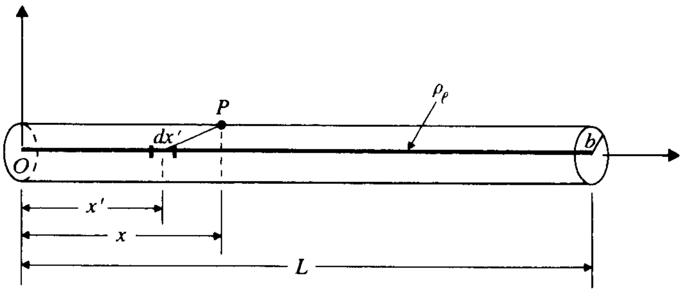


FIGURE 3–40
A finite line charge (Problem P.3–17).

(*Hint:* Find dV at P due to charge $\rho_\ell dx'$ and integrate.)

P.3–18 A charge Q is distributed uniformly over an $L \times L$ square plate. Determine V and $\mathbf{E}$ at a point on the axis perpendicular to the plate and through its center.

P.3–19 A charge Q is distributed uniformly over the wall of a circular tube of radius b and height h. Determine V and $\mathbf{E}$ on its axis

a) at a point outside the tube, then
b) at a point inside the tube.

P.3–20 An early model of the atomic structure of a chemical element was that the atom was a spherical cloud of uniformly distributed positive charge Ne, where N is the atomic number and e is the magnitude of electronic charge. Electrons, each carrying a negative charge $-e$, were considered to be imbedded in the cloud. Assuming the spherical charge cloud to have a radius R_0 and neglecting collision effects,

a) find the force experienced by an imbedded electron at a distance r from the center;
b) describe the motion of the electron;
c) explain why this atomic model is unsatisfactory.

P.3–21 A simple classical model of an atom consists of a nucleus of a positive charge Ne surrounded by a spherical electron cloud of the same total negative charge. (N is the atomic number and e is the magnitude of electronic charge.) An external electric field $\mathbf{E}_o$ will cause the nucleus to be displaced a distance r_o from the center of the electron cloud, thus polarizing the atom. Assuming a uniform charge distribution within the electron cloud of radius b, find r_o.

P.3–22 The polarization in a dielectric cube of side L centered at the origin is given by $\mathbf{P} = P_o(\mathbf{a}_x x + \mathbf{a}_y y + \mathbf{a}_z z)$.

a) Determine the surface and volume bound-charge densities.
b) Show that the total bound charge is zero.

P.3–23 Determine the electric field intensity at the center of a small spherical cavity cut out of a large block of dielectric in which a polarization **P** exists.

P.3–24 Solve the following problems:

a) Find the breakdown voltage of a parallel-plate capacitor, assuming that conducting plates are 50 (mm) apart and the medium between them is air.
b) Find the breakdown voltage if the entire space between the conducting plates is filled with plexiglass, which has a dielectric constant 3 and a dielectric strength 20 (kV/mm).
c) If a 10-(mm) thick plexiglass is inserted between the plates, what is the maximum voltage that can be applied to the plates without a breakdown?

P.3–25 Assume that the $z = 0$ plane separates two lossless dielectric regions with $\epsilon_{r1} = 2$ and $\epsilon_{r2} = 3$. If we know that $\mathbf{E}_1$ in region 1 is $\mathbf{a}_x 2y - \mathbf{a}_y 3x + \mathbf{a}_z(5 + z)$, what do we also know about $\mathbf{E}_2$ and $\mathbf{D}_2$ in region 2? Can we determine $\mathbf{E}_2$ and $\mathbf{D}_2$ at any point in region 2? Explain.

P.3–26 Determine the boundary conditions for the tangential and the normal components of **P** at an interface between two perfect dielectric media with dielectric constants ϵ_{r1} and ϵ_{r2}.

P.3–27 What are the boundary conditions that must be satisfied by the electric potential at an interface between two perfect dielectrics with dielectric constants ϵ_{r1} and ϵ_{r2}?

P.3–28 Dielectric lenses can be used to collimate electromagnetic fields. In Fig. 3–41 the left surface of the lens is that of a circular cylinder, and the right surface is a plane. If $\mathbf{E}_1$ at point $P(r_o, 45°, z)$ in region 1 is $\mathbf{a}_r 5 - \mathbf{a}_\phi 3$, what must be the dielectric constant of the lens in order that $\mathbf{E}_3$ in region 3 is parallel to the x-axis?

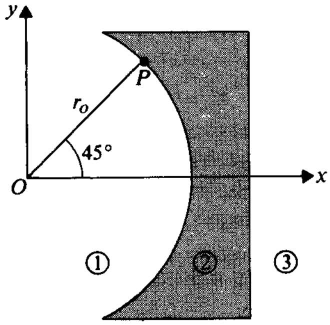


FIGURE 3–41
A dielectric lens (Problem P.3–28).

P.3–29 Refer to Example 3–16. Assuming the same r_i and r_o and requiring the maximum electric field intensities in the insulating materials not to exceed 25% of their dielectric strengths, determine the voltage rating of the coaxial cable

a) if $r_p = 1.75r_i$;
b) if $r_p = 1.35r_i$.
c) Plot the variations of E_r and V versus r for both part (a) and part (b).

P.3–30 The space between a parallel-plate capacitor of area S is filled with a dielectric whose permittivity varies linearly from ϵ_1 at one plate ($y = 0$) to ϵ_2 at the other plate ($y = d$). Neglecting fringing effect, find the capacitance.

P.3–31 Assume that the outer conductor of the cylindrical capacitor in Example 3–18 is grounded and that the inner conductor is maintained at a potential V_0.

a) Find the electric field intensity, $\mathbf{E}(a)$, at the surface of the inner conductor.
b) With the inner radius, b, of the outer conductor fixed, find a so that $E(a)$ is minimized.
c) Find this minimum $E(a)$.
d) Determine the capacitance under the conditions of part (b).

P.3–32 The radius of the core and the inner radius of the outer conductor of a very long coaxial transmission line are r_i and r_o, respectively. The space between the conductors is filled with two coaxial layers of dielectrics. The dielectric constants of the dielectrics are ϵ_{r1} for $r_i < r < b$ and ϵ_{r2} for $b < r < r_o$. Determine its capacitance per unit length.

P.3–33 A cylindrical capacitor of length L consists of coaxial conducting surfaces of radii r_i and r_o. Two dielectric media of different dielectric constants ϵ_{r1} and ϵ_{r2} fill the space between the conducting surfaces as shown in Fig. 3–42. Determine its capacitance.

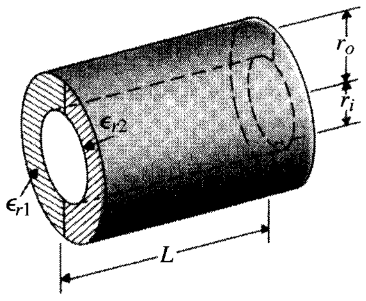


FIGURE 3–42
A cylindrical capacitor with two dielectric media (Problem P.3–33).

P.3–34 A capacitor consists of two coaxial metallic cylindrical surfaces of a length 30 (mm) and radii 5 (mm) and 7 (mm). The dielectric material between the surfaces has a relative permittivity $\epsilon_r = 2 + (4/r)$, where r is measured in mm. Determine the capacitance of the capacitor.

P.3–35 Assuming the earth to be a large conducting sphere (radius = 6.37×10^3 km) surrounded by air, find

a) the capacitance of the earth;
b) the maximum charge that can exist on the earth before the air breaks down.

P.3–36 Determine the capacitance of an isolated conducting sphere of radius b that is coated with a dielectric layer of uniform thickness d. The dielectric has an electric susceptibility χ_e.

P.3–37 A capacitor consists of two concentric spherical shells of radii R_i and R_o. The space between them is filled with a dielectric of relative permittivity ϵ_r from R_i to $b(R_i < b < R_o)$ and another dielectric of relative permittivity $2\epsilon_r$ from b to R_o.

a) Determine **E** and **D** everywhere in terms of an applied voltage V.

b) Determine the capacitance.

P.3–38 The two parallel conducting wires of a power transmission line have a radius a and are spaced at a distance d apart. The wires are at a height h above the ground. Assuming the ground to be perfectly conducting and both d and h to be much larger than a, find the expressions for the mutual and self-partial capacitances per unit length.

P.3–39 An isolated system consists of three very long parallel conducting wires. The axes of all three wires lie in a plane. The two outside wires are of a radius b and both are at a distance $d = 500b$ from a center wire of a radius $2b$. Determine the partial capacitances per unit length.

P.3–40 Calculate the amount of electrostatic energy of a uniform sphere of charge with radius b and volume charge density ρ stored in the following regions:

a) inside the sphere,

b) outside the sphere.

Check your results with those in Example 3–22.

P.3–41 Einstein's theory of relativity stipulates that the work required to assemble a charge is stored as energy in the mass and is equal to mc^2, where m is the mass and $c \cong 3 \times 10^8$ (m/s) is the velocity of light. Assuming the electron to be a perfect sphere, find its radius from its charge and mass (9.1×10^{-31} kg).

P.3–42 Find the electrostatic energy stored in the region of space $R > b$ around an electric dipole of moment **p**.

P.3–43 Prove that Eqs. (3–180) for stored electrostatic energy hold true for any two-conductor capacitor.

P.3–44 A parallel-plate capacitor of width w, length L, and separation d is partially filled with a dielectric medium of dielectric constant ϵ_r, as shown in Fig. 3–43. A battery of V_0 volts is connected between the plates.

a) Find **D**, **E**, and ρ_s in each region.

b) Find distance x such that the electrostatic energy stored in each region is the same.

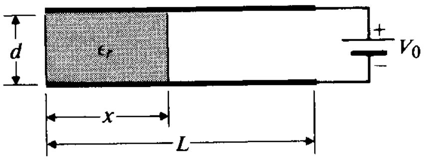


FIGURE 3–43
A parallel-plate capacitor (Problem P.3–44).

P.3–45 Using the principle of virtual displacement, derive an expression for the force between two point charges $+Q$ and $-Q$ separated by a distance x in free space.

P.3–46 A constant voltage V_0 is applied to a partially filled parallel-plate capacitor shown in Fig. 3–44. The permittivity of the dielectric is ϵ, and the area of the plates is S. Find the force on the upper plate.

P.3–47 The conductors of an isolated two-wire transmission line, each of radius b, are spaced at a distance D apart. Assuming $D \gg b$ and a voltage V_0 between the lines, find the force per unit length on the lines.

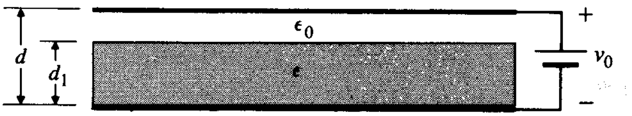


FIGURE 3–44
A parallel-plate capacitor (Problem P.3–46).

P.3–48 A parallel-plate capacitor of width w, length L, and separation d has a solid dielectric slab of permittivity ϵ in the space between the plates. The capacitor is charged to a voltage V_0 by a battery, as indicated in Fig. 3–45. Assuming that the dielectric slab is withdrawn to the position shown, determine the force acting on the slab

a) with the switch closed,

b) after the switch is first opened.

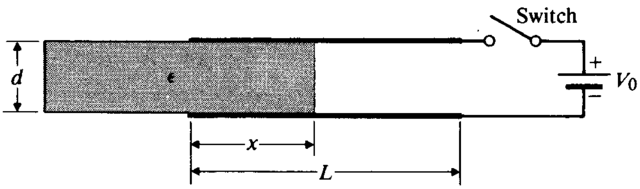


FIGURE 3–45
A partially filled parallel-plate capacitor (Problem P.3–48).

4

Solution of Electrostatic Problems

4–1 Introduction

Electrostatic problems are those which deal with the effects of electric charges at rest. These problems can present themselves in several different ways according to what is initially known. The solution usually calls for the determination of electric potential, electric field intensity, and/or electric charge distribution. If the charge distribution is given, both the electric potential and the electric field intensity can be found by the formulas developed in Chapter 3. In many practical problems, however, the exact charge distribution is not known everywhere, and the formulas in Chapter 3 cannot be applied directly for finding the potential and field intensity. For instance, if the charges at certain discrete points in space and the potentials of some conducting bodies are given, it is rather difficult to find the distribution of surface charges on the conducting bodies and/or the electric field intensity in space. When the conducting bodies have boundaries of a simple geometry, the ***method of images*** may be used to great advantage. This method will be discussed in Section 4–4.

In another type of problem the potentials of all conducting bodies may be known, and we wish to find the potential and field intensity in the surrounding space as well as the distribution of surface charges on the conducting boundaries. Differential equations must be solved subject to the appropriate boundary conditions. These are ***boundary-value problems***. The techniques for solving boundary-value problems in the various coordinate systems will be discussed in Sections 4–5 through 4–7.

4–2 Poisson's and Laplace's Equations

In Section 3–8 we pointed out that Eqs. (3–98) and (3–5) are the two fundamental governing differential equations for electrostatics in any medium. These equations are repeated below for convenience.

$$\text{Eq. (3–98):} \qquad \nabla \cdot \mathbf{D} = \rho. \tag{4–1}$$

$$\text{Eq. (3–5):} \qquad \nabla \times \mathbf{E} = 0. \tag{4–2}$$

The irrotational nature of **E** indicated by Eq. (4–2) enables us to define a scalar electric potential V, as in Eq. (3–43).

$$\text{Eq. (3–43):} \qquad \mathbf{E} = -\nabla V. \tag{4–3}$$

In a linear and isotropic medium $\mathbf{D} = \epsilon\mathbf{E}$, and Eq. (4–1) becomes

$$\nabla \cdot \epsilon\mathbf{E} = \rho. \tag{4–4}$$

Substitution of Eq. (4–3) in Eq. (4–4) yields

$$\nabla \cdot (\epsilon \nabla V) = -\rho, \tag{4–5}$$

where ϵ can be a function of position. For a simple medium; that is, for a medium that is also homogeneous, ϵ is a constant and can then be taken out of the divergence operation. We have

$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon}.} \tag{4–6}$$

In Eq. (4–6) we have introduced a new operator, ∇^2 (del square), the ***Laplacian operator***, which stands for "the divergence of the gradient of," or $\nabla \cdot \nabla$. Equation (4–6) is known as ***Poisson's equation***; it states that the Laplacian (the divergence of the gradient) of V equals $-\rho/\epsilon$ *for a simple medium*, where ϵ is the permittivity of the medium (which is a constant) and ρ is the volume density of free charges (which may be a function of space coordinates).

Since both divergence and gradient operations involve first-order spatial derivatives, Poisson's equation is a second-order partial differential equation that holds at every point in space where the second-order derivatives exist. In Cartesian coordinates,

$$\nabla^2 V = \nabla \cdot \nabla V = \left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}\right) \cdot \left(\mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}\right);$$

and Eq. (4–6) becomes

$$\boxed{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\epsilon} \qquad (\text{V/m}^2).} \tag{4–7}$$

Similarly, by using Eqs. (2–93) and (2–110) we can easily verify the following expressions for $\nabla^2 V$ in cylindrical and spherical coordinates.

Cylindrical coordinates:

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}. \tag{4–8}$$

Spherical coordinates:

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}. \tag{4-9}$$

The solution of Poisson's equation in three dimensions subject to prescribed boundary conditions is, in general, not an easy task.

At points in a simple medium where there is no free charge, $\rho = 0$ and Eq. (4–6) reduces to

$$\boxed{\nabla^2 V = 0,} \tag{4-10}$$

which is known as ***Laplace's equation.*** Laplace's equation occupies a very important position in electromagnetics. It is the governing equation for problems involving a set of conductors, such as capacitors, maintained at different potentials. Once V is found from Eq. (4–10), $\mathbf{E}$ can be determined from $-\nabla V$, and the charge distribution on the conductor surfaces can be determined from $\rho_s = \epsilon E_n$ (Eq. 3–72).

EXAMPLE 4–1 The two plates of a parallel-plate capacitor are separated by a distance d and maintained at potentials 0 and V_0, as shown in Fig. 4–1. Assuming negligible fringing effect at the edges, determine (a) the potential at any point between the plates, and (b) the surface charge densities on the plates.

Solution

a) Laplace's equation is the governing equation for the potential between the plates, since $\rho = 0$ there. Ignoring the fringing effect of the electric field is tantamount to assuming that the field distribution between the plates is the same as though the plates were infinitely large and that there is no variation of V in the x and z directions. Equation (4–7) then simplifies to

$$\frac{d^2V}{dy^2} = 0, \tag{4-11}$$

where d^2/dy^2 is used instead of $\partial^2/\partial y^2$, since y is the only space variable here.

Integration of Eq. (4–11) with respect to y gives

$$\frac{dV}{dy} = C_1,$$

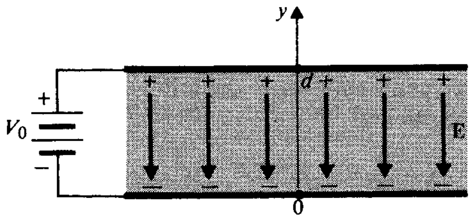


FIGURE 4–1
A parallel-plate capacitor (Example 4–1).

where the constant of integration C_1 is yet to be determined. Integrating again, we obtain

$$V = C_1 y + C_2. \tag{4-12}$$

Two boundary conditions are required for the determination of the two constants of integration, C_1 and C_2:

$$\text{At } y = 0, \qquad V = 0. \tag{4-13a}$$

$$\text{At } y = d, \qquad V = V_0. \tag{4-13b}$$

Substitution of Eqs. (4–13a) and (4–13b) in Eq. (4–12) yields immediately $C_1 = V_0/d$ and $C_2 = 0$. Hence the potential at any point y between the plates is, from Eq. (4–12),

$$V = \frac{V_0}{d} y. \tag{4-14}$$

The potential increases linearly from $y = 0$ to $y = d$.

b) In order to find the surface charge densities, we must first find **E** at the conducting plates at $y = 0$ and $y = d$. From Eqs. (4–3) and (4–14) we have

$$\mathbf{E} = -\mathbf{a}_y \frac{dV}{dy} = -\mathbf{a}_y \frac{V_0}{d}, \tag{4-15}$$

which is a constant and is independent of y. Note that the direction of **E** is opposite to the direction of increasing V. The surface charge densities at the conducting plates are obtained by using Eq. (3–72),

$$E_n = \mathbf{a}_n \cdot \mathbf{E} = \frac{\rho_s}{\epsilon}.$$

At the lower plate,

$$\mathbf{a}_n = \mathbf{a}_y, \qquad E_{n\ell} = -\frac{V_0}{d}, \qquad \rho_{s\ell} = -\frac{\epsilon V_0}{d}.$$

At the upper plate,

$$\mathbf{a}_n = -\mathbf{a}_y, \qquad E_{nu} = \frac{V_0}{d}, \qquad \rho_{su} = \frac{\epsilon V_0}{d}.$$

Electric field lines in an electrostatic field begin from positive charges and end in negative charges. ▬

EXAMPLE 4–2 Determine the **E** field both inside and outside a spherical cloud of electrons with a uniform volume charge density $\rho = -\rho_0$ (where ρ_0 is a positive quantity) for $0 \le R \le b$ and $\rho = 0$ for $R > b$ by solving Poisson's and Laplace's equations for V.

Solution We recall that this problem was solved in Chapter 3 (Example 3–7) by applying Gauss's law. We now use the same problem to illustrate the solution of one-dimensional Poisson's and Laplace's equations. Since there are no variations in θ and ϕ directions, we are dealing only with functions of R in spherical coordinates.

a) Inside the cloud,

$$0 \leq R \leq b, \qquad \rho = -\rho_0.$$

In this region, Poisson's equation ($\nabla^2 V_i = -\rho/\epsilon_0$) holds. Dropping $\partial/\partial\theta$ and $\partial/\partial\phi$ terms from Eq. (4–9), we have

$$\frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{dV_i}{dR}\right) = \frac{\rho_0}{\epsilon_0},$$

which reduces to

$$\frac{d}{dR}\left(R^2\frac{dV_i}{dR}\right) = \frac{\rho_0}{\epsilon_0}R^2. \tag{4–16}$$

Integration of Eq. (4–16) gives

$$\frac{dV_i}{dR} = \frac{\rho_0}{3\epsilon_0}R + \frac{C_1}{R^2}. \tag{4–17}$$

The electric field intensity inside the electron cloud is

$$\mathbf{E}_i = -\nabla V_i = -\mathbf{a}_R\left(\frac{dV_i}{dR}\right).$$

Since $\mathbf{E}_i$ cannot be infinite at $R = 0$, the integration constant C_1 in Eq. (4–17) must vanish. We obtain

$$\mathbf{E}_i = -\mathbf{a}_R\frac{\rho_0}{3\epsilon_0}R, \qquad 0 \leq R \leq b. \tag{4–18}$$

b) Outside the cloud,

$$R \geq b, \qquad \rho = 0.$$

Laplace's equation holds in this region. We have $\nabla^2 V_o = 0$ or

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{dV_o}{dR}\right) = 0. \tag{4–19}$$

Integrating Eq. (4–19), we obtain

$$\frac{dV_o}{dR} = \frac{C_2}{R^2} \tag{4–20}$$

or

$$\mathbf{E}_o = -\nabla V_o = -\mathbf{a}_R\frac{dV_o}{dR} = -\mathbf{a}_R\frac{C_2}{R^2}. \tag{4–21}$$

The integration constant C_2 can be found by equating $\mathbf{E}_o$ and $\mathbf{E}_i$ at $R = b$, where there is no discontinuity in medium characteristics.

$$\frac{C_2}{b^2} = \frac{\rho_0}{3\epsilon_0}b,$$

from which we find

$$C_2 = \frac{\rho_0 b^3}{3\epsilon_0} \tag{4–22}$$

and

$$\mathbf{E}_o = -\mathbf{a}_R\frac{\rho_0 b^3}{3\epsilon_0 R^2}, \qquad R \geq b. \tag{4–23}$$

Since the total charge contained in the electron cloud is

$$Q = -\rho_0 \frac{4\pi}{3} b^3,$$

Eq. (4–23) can be written as

$$\mathbf{E}_o = \mathbf{a}_R \frac{Q}{4\pi\epsilon_0 R^2}, \qquad (4\text{–}24)$$

which is the familiar expression for the electric field intensity at a point R from a point charge Q. ∎

Further insight to this problem can be gained by examining the potential as a function of R. Integrating Eq. (4–17), remembering that $C_1 = 0$, we have

$$V_i = \frac{\rho_0 R^2}{6\epsilon_0} + C_1'. \qquad (4\text{–}25)$$

It is important to note that C_1' is a new integration constant and is not the same as C_1. Substituting Eq. (4–22) in Eq. (4–20) and integrating, we obtain

$$V_o = -\frac{\rho_0 b^3}{3\epsilon_0 R} + C_2'. \qquad (4\text{–}26)$$

However, C_2' in Eq. (4–26) must vanish, since V_o is zero at infinity ($R \to \infty$). As electrostatic potential is continuous at a boundary, we determine C_1' by equating V_i and V_o at $R = b$:

$$\frac{\rho_0 b^2}{6\epsilon_0} + C_1' = -\frac{\rho_0 b^2}{3\epsilon_0}$$

or

$$C_1' = -\frac{\rho_0 b^2}{2\epsilon_0}; \qquad (4\text{–}27)$$

and, from Eq. (4–25),

$$V_i = -\frac{\rho_0}{3\epsilon_0}\left(\frac{3b^2}{2} - \frac{R^2}{2}\right). \qquad (4\text{–}28)$$

We see that V_i in Eq. (4–28) is the same as V in Eq. (3–172), with $\rho = -\rho_0$.

4–3 Uniqueness of Electrostatic Solutions

In the two relatively simple examples in the last section we obtained the solutions by direct integration. In more complicated situations, other methods of solution must be used. Before these methods are discussed, it is important to know that ***a solution of Poisson's equation*** (of which Laplace's equation is a special case) ***that satisfies the given boundary conditions is a unique solution.*** This statement is called the ***uniqueness theorem.*** The implication of the uniqueness theorem is that a solution of an electrostatic problem satisfying its boundary conditions is *the only possible*

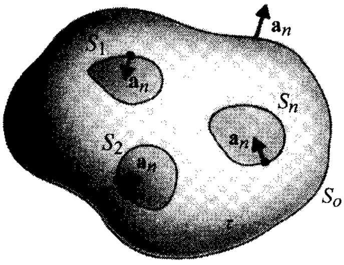


FIGURE 4-2
Surface S_o enclosing volume τ with conducting bodies.

solution, irrespective of the method by which the solution is obtained. A solution obtained even by intelligent guessing is the only correct solution. The importance of this theorem will be appreciated when we discuss the method of images in Section 4–4.

To prove the uniqueness theorem, suppose a volume τ is bounded outside by a surface S_o, which may be a surface at infinity. Inside the closed surface S_o there are a number of charged conducting bodies with surfaces $S_1, S_2, \ldots, S_n$ at specified potentials, as depicted in Fig. 4–2. Now assume that, contrary to the uniqueness theorem, there are two solutions, V_1 and V_2, to Poisson's equation in τ:

$$\nabla^2 V_1 = -\frac{\rho}{\epsilon}, \qquad (4\text{–}29\text{a})$$

$$\nabla^2 V_2 = -\frac{\rho}{\epsilon}. \qquad (4\text{–}29\text{b})$$

Also assume that both V_1 and V_2 satisfy the *same* boundary conditions on S_1, $S_2, \ldots, S_n$ and S_o. Let us try to define a new difference potential:

$$V_d = V_1 - V_2. \qquad (4\text{–}30)$$

From Eqs. (4–29a) and (4–29b) we see that V_d satisfies Laplace's equation in τ:

$$\nabla^2 V_d = 0. \qquad (4\text{–}31)$$

On conducting boundaries the potentials are specified and $V_d = 0$.

Recalling the vector identity (Problem P.2–28),

$$\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f; \qquad (4\text{–}32)$$

and letting $f = V_d$ and $\mathbf{A} = \nabla V_d$; we have

$$\nabla \cdot (V_d \nabla V_d) = V_d \nabla^2 V_d + |\nabla V_d|^2, \qquad (4\text{–}33)$$

where, because of Eq. (4–31), the first term on the right side vanishes. Integration of Eq. (4–33) over the volume τ yields

$$\oint_S (V_d \nabla V_d) \cdot \mathbf{a}_n \, ds = \int_\tau |\nabla V_d|^2 \, dv, \qquad (4\text{–}34)$$

where $\mathbf{a}_n$ denotes the unit normal outward from τ. Surface S consists of S_o as well as $S_1, S_2, \ldots,$ and S_n. Over the conducting boundaries, $V_d = 0$. Over the large surface

S_o, which encloses the whole system, the surface integral on the left side of Eq. (4–34) can be evaluated by considering S_o as the surface of a very large sphere with radius R. As R increases, both V_1 and V_2 (and therefore also V_d) fall off as $1/R$; consequently, ∇V_d falls off as $1/R^2$, making the integrand $(V_d \nabla V_d)$ fall off as $1/R^3$. The surface area S_o, however, increases as R^2. Hence the surface integral on the left side of Eq. (4–34) decreases as $1/R$ and approaches zero at infinity. So must also the volume integral on the right side. We have

$$\int_\tau |\nabla V_d|^2 \, dv = 0. \tag{4–35}$$

Since the integrand $|\nabla V_d|^2$ is nonnegative everywhere, Eq. (4–35) can be satisfied only if $|\nabla V_d|$ is identically zero. A vanishing gradient everywhere means that V_d has the same value at all points in τ as it has on the bounding surfaces, $S_1, S_2, \ldots, S_n$, where $V_d = 0$. It follows that $V_d = 0$ throughout the volume τ. Therefore $V_1 = V_2$, and there is only one possible solution.

It is easy to see that the uniqueness theorem holds if the surface charge distributions ($\rho_s = \epsilon E_n = -\epsilon\, \partial V/\partial n$), rather than the potentials, of the conducting bodies are specified. In such a case, ∇V_d will be zero, which in turn, makes the left side of Eq. (4–34) vanish and leads to the same conclusion. In fact, the uniqueness theorem applies even if an inhomogeneous dielectric (one whose permittivity varies with position) is present. The proof, however, is more involved and will be omitted here.

4–4 Method of Images

There is a class of electrostatic problems with boundary conditions that appear to be difficult to satisfy if the governing Poisson's or Laplace's equation is to be solved directly, but the conditions on the bounding surfaces in these problems can be set up by appropriate ***image*** (equivalent) ***charges***, and the potential distributions can then be determined in a straightforward manner. This method of replacing bounding surfaces by appropriate image charges in lieu of a formal solution of Poisson's or Laplace's equation is called the ***method of images.***

Consider the case of a positive point charge, Q, located at a distance d above a large grounded (zero-potential) conducting plane, as shown in Fig. 4–3(a). The problem is to find the potential at every point above the conducting plane ($y > 0$). The formal procedure for doing so would be to solve Laplace's equation in Cartesian coordinates:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0, \tag{4–36}$$

which must hold for $y > 0$ except at the point charge. The solution $V(x, y, z)$ should satisfy the following conditions:

1. At all points on the grounded conducting plane, the potential is zero; that is,

$$V(x, 0, z) = 0.$$

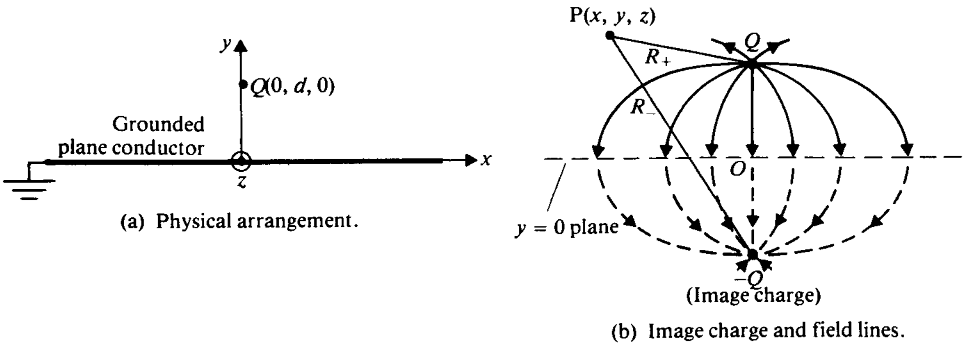


FIGURE 4–3
Point charge and grounded plane conductor.

2. At points very close to Q the potential approaches that of the point charge alone; that is

$$V \to \frac{Q}{4\pi\epsilon_0 R}, \text{ as } R \to 0,$$

where R is the distance to Q.

3. At points very far from $Q(x \to \pm\infty, y \to +\infty$, or $z \to \pm\infty)$ the potential approaches zero.
4. The potential function is even with respect to the x and z coordinates; that is,

$$V(x, y, z) = V(-x, y, z)$$

and

$$V(x, y, z) = V(x, y, -z).$$

It does appear difficult to construct a solution for V that will satisfy all of these conditions.

From another point of view, we may reason that the presence of a positive charge Q at $y = d$ would induce negative charges on the surface of the conducting plane, resulting in a surface charge density ρ_s. Hence the potential at points above the conducting plane would be

$$V(x, y, z) = \frac{Q}{4\pi\epsilon_0\sqrt{x^2 + (y-d)^2 + z^2}} + \frac{1}{4\pi\epsilon_0}\int_S \frac{\rho_s}{R_1}\, ds,$$

where R_1 is the distance from ds to the point under consideration and S is the surface of the entire conducting plane. The trouble here is that ρ_s must first be determined from the boundary condition $V(x, 0, z) = 0$. Moreover, the indicated surface integral is difficult to evaluate even after ρ_s has been determined at every point on the conducting plane. In the following subsections we demonstrate how the method of images greatly simplifies these problems.

4-4.1 POINT CHARGE AND CONDUCTING PLANES

The problem in Fig. 4–3(a) is that of a positive point charge, Q, located at a distance d above a large plane conductor that is at zero potential. If we remove the conductor and replace it by an image point charge $-Q$ at $y = -d$, then the potential at a point $P(x, y, z)$ *in the* $y > 0$ *region* is

$$V(x, y, z) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right), \tag{4-37}$$

where R_+ and R_- are the distances from Q and $-Q$, respectively, to the point P.

$$R_+ = [x^2 + (y - d)^2 + z^2]^{1/2},$$
$$R_- = [x^2 + (y + d)^2 + z^2]^{1/2}.$$

It is easy to prove by direct substitution (Problem P.4–5a) that $V(x, y, z)$ in Eq. (4–37) satisfies the Laplace's equation in Eq. (4–36), and it is obvious that all four conditions listed after Eq. (4–36) are satisfied. Therefore Eq. (4–37) is a solution of this problem; and, in view of the uniqueness theorem, it is the only solution.

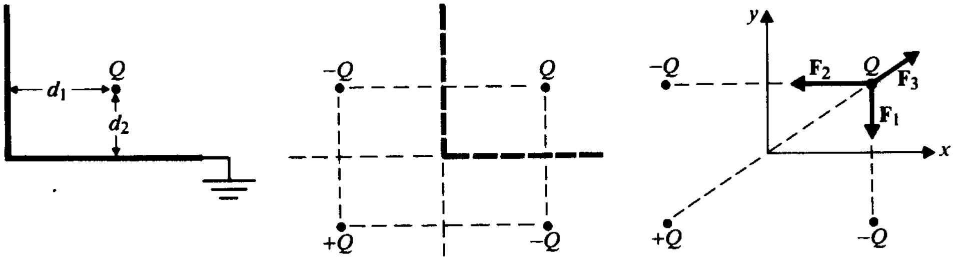
Electric field intensity **E** in the $y > 0$ region can be found easily from $-\nabla V$ with Eq. (4–37). It is exactly the same as that between two point charges, $+Q$ and $-Q$, spaced a distance $2d$ apart. A few of the field lines are shown in Fig. 4–3(b). The solution of this electrostatic problem by the method of images is extremely simple; but it must be emphasized that the image charge is located *outside* the region in which the field is to be determined. In this problem the point charges $+Q$ and $-Q$ *cannot* be used to calculate the V or **E** in the $y < 0$ region. As a matter of fact, both V and **E** are zero in the $y < 0$ region. Can you explain that?

It is readily seen that the electric field of a line charge ρ_ℓ above an infinite conducting plane can be found from ρ_ℓ and its image $-\rho_\ell$ (with the conducting plane removed).

EXAMPLE 4–3 A positive point charge Q is located at distances d_1 and d_2, respectively, from two grounded perpendicular conducting half-planes, as shown in Fig. 4–4(a). Determine the force on Q caused by the charges induced on the planes.

Solution A formal solution of Poisson's equation, subject to the zero-potential boundary condition at the conducting half-planes, would be quite difficult. Now an image charge $-Q$ in the fourth quadrant would make the potential of the horizontal half-plane (but not that of the vertical half-plane) zero. Similarly, an image charge $-Q$ in the second quadrant would make the potential of the vertical half-plane (but not that of the horizontal plane) zero. But if a third image charge $+Q$ is added in the third quadrant, we see from symmetry that the image-charge arrangement in Fig. 4–4(b) satisfies the zero-potential boundary condition on both half-planes and is electrically equivalent to the physical arrangement in Fig. 4–4(*a*).

Negative surface charges will be induced on the half-planes, but their effect on Q can be determined from that of the three image charges. Referring to Fig. 4–4(c),



(a) Physical arrangement. (b) Equivalent image-charge arrangement. (c) Forces on charge Q.

FIGURE 4–4
Point charge and perpendicular conducting planes.

we have, for the net force on Q,

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3,$$

where

$$\mathbf{F}_1 = -\mathbf{a}_y \frac{Q^2}{4\pi\epsilon_0(2d_2)^2},$$

$$\mathbf{F}_2 = -\mathbf{a}_x \frac{Q^2}{4\pi\epsilon_0(2d_1)^2},$$

$$\mathbf{F}_3 = \frac{Q^2}{4\pi\epsilon_0[(2d_1)^2 + (2d_2)^2]^{3/2}} (\mathbf{a}_x 2d_1 + \mathbf{a}_y 2d_2).$$

Therefore,

$$\mathbf{F} = \frac{Q^2}{16\pi\epsilon_0} \left\{ \mathbf{a}_x \left[\frac{d_1}{(d_1^2 + d_2^2)^{3/2}} - \frac{1}{d_1^2} \right] + \mathbf{a}_y \left[\frac{d_2}{(d_1^2 + d_2^2)^{3/2}} - \frac{1}{d_2^2} \right] \right\}.$$

The electric potential and electric field intensity at points *in the first quadrant* and the surface charge density induced on the two half-planes can also be found from the system of four charges (Problem P.4–8).

4–4.2 LINE CHARGE AND PARALLEL CONDUCTING CYLINDER

We now consider the problem of a line charge ρ_ℓ (C/m) located at a distance d from the axis of a parallel, conducting, circular cylinder of radius a. Both the line charge and the conducting cylinder are assumed to be infinitely long. Figure 4–5(a) shows a cross section of this arrangement. Preparatory to the solution of this problem by the method of images, we note the following: (1) The image must be a parallel line charge inside the cylinder in order to make the cylindrical surface at $r = a$ an equipotential surface. Let us call this image line charge ρ_i. (2) Because of symmetry with respect to the line OP, the image line charge must lie somewhere along OP, say at point P_i, which is at a distance d_i from the axis (Fig. 4–5b). We need to determine the two unknowns, ρ_i and d_i.

As a first approach, let us assume that

$$\boxed{\rho_i = -\rho_\ell.} \tag{4-38}$$

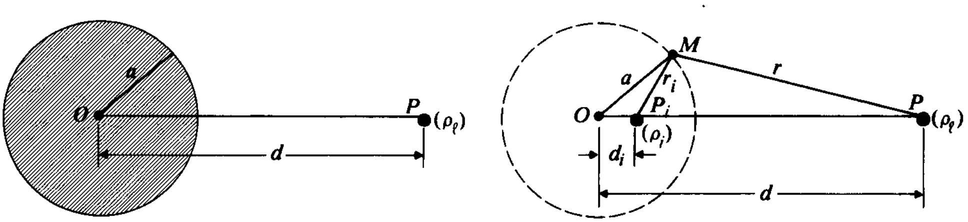
At this stage, Eq. (4–38) is just a trial solution (an intelligent guess), and we are not sure that it will hold true. We will, on one hand, proceed with this trial solution until we find that it fails to satisfy the boundary conditions. On the other hand, if Eq. (4–38) leads to a solution that does satisfy all boundary conditions, then by the uniqueness theorem it is the only solution. Our next job will be to see whether we can determine d_i.

The electric potential at a distance r from a line charge of density ρ_ℓ can be obtained by integrating the electric field intensity $\mathbf{E}$ given in Eq. (3–40):

$$\begin{aligned} V &= -\int_{r_0}^{r} E_r\,dr = -\frac{\rho_\ell}{2\pi\epsilon_0}\int_{r_0}^{r}\frac{1}{r}\,dr \\ &= \frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{r_0}{r}. \end{aligned} \tag{4-39}$$

Note that the reference point for zero potential, r_0, cannot be at infinity because setting $r_0 = \infty$ in Eq. (4–39) would make V infinite everywhere else. Let us leave r_0 unspecified for the time being. The potential at a point on or outside the cylindrical surface is obtained by adding the contributions of ρ_ℓ and ρ_i. In particular, at a point M on the cylindrical surface shown in Fig. 4–5(b) we have

$$\begin{aligned} V_M &= \frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{r_0}{r} - \frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{r_0}{r_i} \\ &= \frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{r_i}{r}. \end{aligned} \tag{4-40}$$



(a) Line charge and parallel conducting cylinder.

(b) Line charge and its image.

FIGURE 4–5
Cross section of line charge and its image in a parallel, conducting, circular cylinder.

In Eq. (4–40) we have chosen, for simplicity, a point equidistant from ρ_ℓ and ρ_i as the reference point for zero potential so that the $\ln r_0$ terms cancel. Otherwise, a constant term should be included in the right side of Eq. (4–40), but it would not affect what follows. Equipotential surfaces are specified by

$$\frac{r_i}{r} = \text{Constant}. \tag{4-41}$$

If an equipotential surface is to coincide with the cylindrical surface ($\overline{OM} = a$), the point P_i must be located in such a way as to make triangles OMP_i and OPM similar. Note that these two triangles already have one common angle, $\angle MOP_i$. Point P_i should be chosen to make $\angle OMP_i = \angle OPM$. We have

$$\frac{\overline{P_iM}}{\overline{PM}} = \frac{\overline{OP_i}}{\overline{OM}} = \frac{\overline{OM}}{\overline{OP}}$$

or

$$\frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d} = \text{Constant}. \tag{4-42}$$

From Eq. (4–42) we see that if

$$\boxed{d_i = \frac{a^2}{d}} \tag{4-43}$$

the image line charge $-\rho_\ell$, together with ρ_ℓ, will make the dashed cylindrical surface in Fig. 4–5(b) equipotential. As the point M changes its location on the dashed circle, both r_i and r will change; but their ratio remains a constant that equals a/d. Point P_i is called the ***inverse point*** of P with respect to a circle of radius a.

The image line charge $-\rho_\ell$ can then replace the cylindrical conducting surface, and V and $\mathbf{E}$ at any point outside the surface can be determined from the line charges ρ_ℓ and $-\rho_\ell$. By symmetry we find that the parallel cylindrical surface surrounding the original line charge ρ_ℓ with radius a and its axis at a distance d_i to the right of P is also an equipotential surface. This observation enables us to calculate the capacitance per unit length of an open-wire transmission line consisting of two parallel conductors of circular cross section.

EXAMPLE 4–4 Determine the capacitance per unit length between two long, parallel, circular conducting wires of radius a. The axes of the wires are separated by a distance D.

Solution Refer to the cross section of the two-wire transmission line shown in Fig. 4–6. The equipotential surfaces of the two wires can be considered to have been generated by a pair of line charges $+\rho_\ell$ and $-\rho_\ell$ separated by a distance $(D - 2d_i) = d - d_i$. The potential difference between the two wires is that between any two points

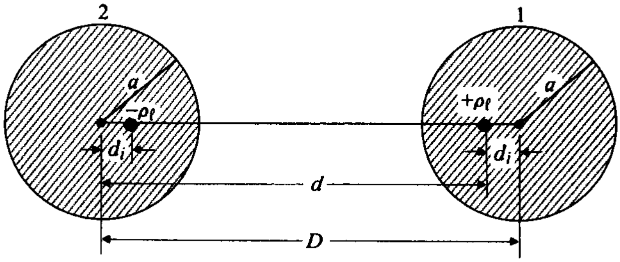


FIGURE 4–6
Cross section of two-wire transmission line and equivalent line charges (Example 4–4).

on the respective wires. Let subscripts 1 and 2 denote the wires surrounding the equivalent line charges $+\rho_\ell$ and $-\rho_\ell$, respectively. We have, from Eqs. (4–40) and (4–42).

$$V_2 = \frac{\rho_\ell}{2\pi\epsilon_0} \ln \frac{a}{d}$$

and, similarly,

$$V_1 = -\frac{\rho_\ell}{2\pi\epsilon_0} \ln \frac{a}{d}.$$

We note that V_1 is a positive quantity, whereas V_2 is negative because $a < d$. The capacitance per unit length is

$$C = \frac{\rho_\ell}{V_1 - V_2} = \frac{\pi\epsilon_0}{\ln (d/a)}, \tag{4–44}$$

where

$$d = D - d_i = D - \frac{a^2}{d},$$

from which we obtain†

$$d = \tfrac{1}{2}(D + \sqrt{D^2 - 4a^2}). \tag{4–45}$$

Using Eq. (4–45) in Eq. (4–44), we have

$$\boxed{C = \frac{\pi\epsilon_0}{\ln [(D/2a) + \sqrt{(D/2a)^2 - 1}]} \quad \text{(F/m).}} \tag{4–46}$$

Since

$$\ln [x + \sqrt{x^2 - 1}] = \cosh^{-1} x$$

† The other solution, $d = \frac{1}{2}(D - \sqrt{D^2 - 4a^2})$, is discarded because both D and d are usually much larger than a.

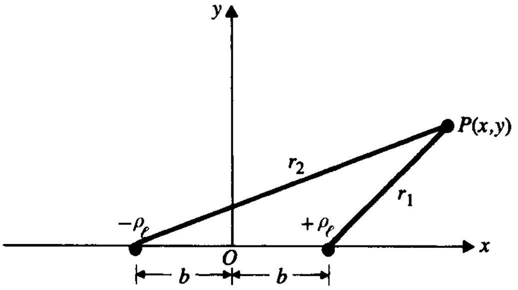


FIGURE 4–7
Cross section of a pair of line charges.

for $x > 1$, Eq. (4–46) can be written alternatively as

$$\boxed{C = \frac{\pi\epsilon_0}{\cosh^{-1}(D/2a)} \qquad \text{(F/m)}.} \tag{4–47}$$

The potential distribution and electric field intensity around the two-wire line in Fig. 4–6 can also be determined easily from the equivalent line charges.

We now consider the more general case of a two-wire line of different radii. We know that our problem would be solved if we could find the location of the equivalent line charges that make the wire surfaces equipotential. Let us then first study the potential distribution around a pair of positive and negative line charges, a cross section of which is given in Fig. 4–7. The potential at any point $P(x, y)$ due to $+\rho_\ell$ and $-\rho_\ell$ is, from Eq. (4–40),

$$V_P = \frac{\rho_\ell}{2\pi\epsilon_0} \ln \frac{r_2}{r_1}. \tag{4–48}$$

In the xy-plane the equipotential lines are defined by $r_2/r_1 = k$ (constant). We have

$$\frac{r_2}{r_1} = \frac{\sqrt{(x+b)^2 + y^2}}{\sqrt{(x-b)^2 + y^2}} = k, \tag{4–49}$$

which reduces to

$$\left(x - \frac{k^2+1}{k^2-1} b\right)^2 + y^2 = \left(\frac{2k}{k^2-1} b\right)^2 \tag{4–50}$$

Equation (4–49) represents a family of circles in the xy-plane with radii

$$a = \left|\frac{2kb}{k^2-1}\right|, \tag{4–51}$$

where the absolute-value sign is necessary because k in Eq. (4–49) can be less than unity and a must be positive. The centers of the circles are displaced from the origin by a distance

$$c = \frac{k^2 + 1}{k^2 - 1} b. \tag{4-52}$$

A particularly simple relation exists among a, b, and c:

$$c^2 = a^2 + b^2, \tag{4-53}$$

or

$$b = \sqrt{c^2 - a^2}. \tag{4-54}$$

Two families of the displaced circular equipotential lines are shown in Fig. 4–8: one family around $+\rho_\ell$ for $k > 1$ and another around $-\rho_\ell$ for $k < 1$. The y-axis is the zero-potential line (a circle of infinite radius) corresponding to $k = 1$. The dashed lines in Fig. 4–8 are circles representing electric field lines, which are everywhere perpendicular to the equipotential lines (Problem P.4–12). Thus the electrostatic

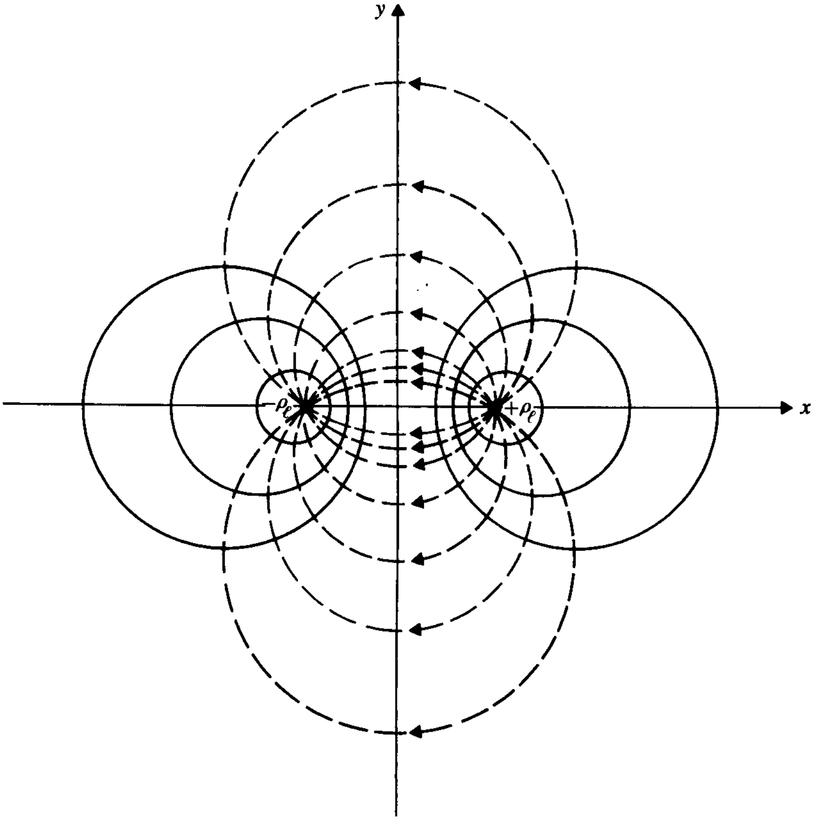


FIGURE 4–8
Equipotential (solid) and electric field (dashed) lines around a pair of line charges.

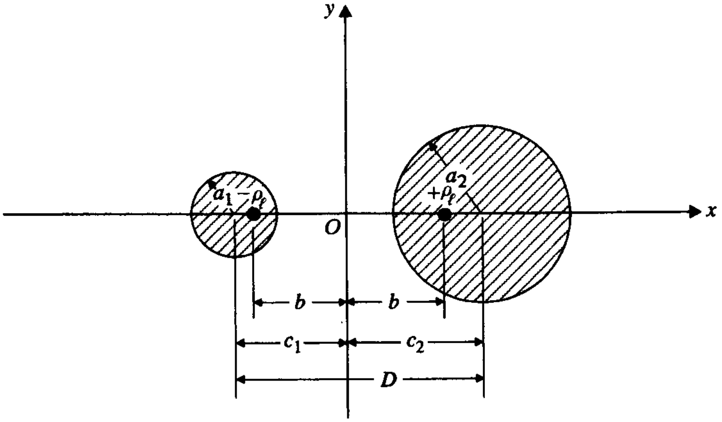


FIGURE 4–9
Cross section of two parallel wires with different radii.

problem of a two-wire line with different radii is that of two equipotential circles of unequal radii, one on each side of the y-axis in Fig. 4–8; it can be solved by determining the locations of the equivalent line charges.

Assume that the radii of the wires are a_1 and a_2 and that their axes are separated by a distance D, as shown in Fig. 4–9. The distance b of the line charges to the origin is to be determined. This can be done by first expressing c_1 and c_2 in terms of a_1, a_2, and D. From Eq. (4–54) we have

$$b^2 = c_1^2 - a_1^2 \tag{4-55}$$

and

$$b^2 = c_2^2 - a_2^2. \tag{4-56}$$

But

$$c_1 + c_2 = D. \tag{4-57}$$

Solution of Eqs. (4–55), (4–56), and (4–57) gives

$$c_1 = \frac{1}{2D}(D^2 + a_1^2 - a_2^2) \tag{4-58}$$

and

$$c_2 = \frac{1}{2D}(D^2 + a_2^2 - a_1^2). \tag{4-59}$$

The distance b can then be found from Eq. (4–55) or Eq. (4–56).

An interesting variation of the two-wire problem is that of an off-center conductor inside a conducting cylindrical tunnel shown in Fig. 4–10(a). Here the two equipotential surfaces are on the same side of a pair of equal and opposite line charges. This is depicted in Fig. 4–10(b). We have, in addition to Eqs. (4–55) and (4–56),

$$c_2 - c_1 = D. \tag{4-60}$$

Combination of, Eqs. (4–55), (4–56), and (4–60) yields

$$c_1 = \frac{1}{2D}(a_2^2 - a_1^2 - D^2) \tag{4-61}$$

and

$$c_2 = \frac{1}{2D}(a_2^2 - a_1^2 + D^2). \tag{4-62}$$

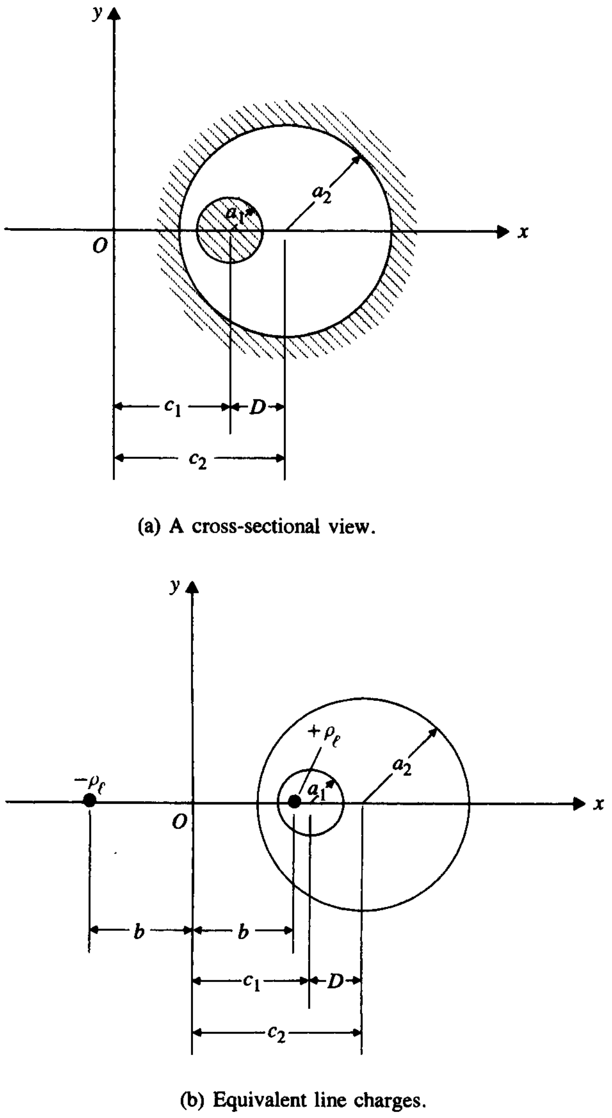


FIGURE 4–10
An off-center conductor inside a cylindrical tunnel.

The distance b can be found from Eq. (4–55) or Eq. (4–56). With the locations of the equivalent line charges known, the determination of the potential and electric field distributions and of the capacitance between the conductors per unit length becomes straightforward (Problems P.4–13 and P.4–14).

4–4.3 POINT CHARGE AND CONDUCTING SPHERE

The method of images can also be applied to solve the electrostatic problem of a point charge in the presence of a spherical conductor. Referring to Fig. 4–11(a), in which a positive point charge Q is located at a distance d from the center of a grounded conducting sphere of radius a ($a < d$), we now proceed to find the V and $\mathbf{E}$ at points external to the sphere. By reason of symmetry we expect the image charge Q_i to be a negative point charge situated inside the sphere and on the line joining O and Q. Let it be at a distance d_i from O. It is obvious that Q_i cannot be equal to $-Q$, since $-Q$ and the original Q do not make the spherical surface $R = a$ a zero-potential surface as required. (What would the zero-potential surface be if $Q_i = -Q$?) We must therefore treat both d_i and Q_i as unknowns.

In Fig. 4–11(b) the conducting sphere has been replaced by the image point charge Q_i, which makes the potential at all points on the spherical surface $R = a$ zero. At a typical point M, the potential caused by Q and Q_i is

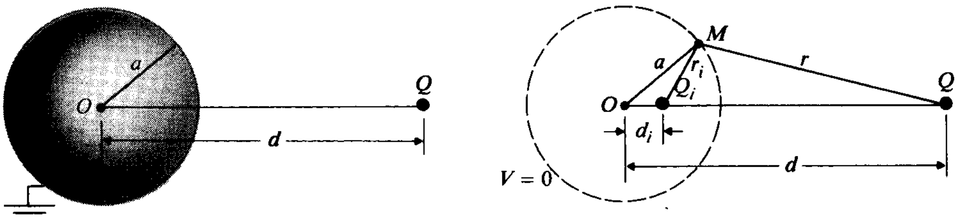
$$V_M = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{Q_i}{r_i}\right) = 0, \tag{4–63}$$

which requires

$$\frac{r_i}{r} = -\frac{Q_i}{Q} = \text{Constant}. \tag{4–64}$$

Noting that the requirement on the ratio r_i/r is the same as that in Eq. (4–41), we conclude from Eqs. (4–42), (4–43), and (4–64) that

$$-\frac{Q_i}{Q} = \frac{a}{d}$$



(a) Point charge and grounded conducting sphere. (b) Point charge and its image.

FIGURE 4–11
Point charge and its image in a grounded sphere.

or

$$\boxed{Q_i = -\frac{a}{d}Q} \tag{4–65}$$

and

$$\boxed{d_i = \frac{a^2}{d}.} \tag{4–66}$$

The point Q_i is thus the ***inverse point*** of Q with respect to a sphere of radius a. The V and $\mathbf{E}$ of all points external to the grounded sphere can now be calculated from the V and $\mathbf{E}$ caused by the two point charges Q and $-aQ/d$.

EXAMPLE 4–5 A point charge Q is at a distance d from the center of a grounded conducting sphere of radius a ($a < d$). Determine (a) the charge distribution induced on the surface of the sphere, and (b) the total charge induced on the sphere.

Solution The physical problem is that shown in Fig. 4–11(a). We solve the problem by the method of images and replace the grounded sphere by the image charge $Q_i = -aQ/d$ at a distance $d_i = a^2/d$ from the center of the sphere, as shown in Fig. 4–12. The electric potential V at an arbitrary point $P(R, \theta)$ is

$$V(R, \theta) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_Q} - \frac{a}{dR_{Q_i}}\right), \tag{4–67}$$

where, by the law of cosines,

$$R_Q = [R^2 + d^2 - 2Rd\cos\theta]^{1/2} \tag{4–68}$$

and

$$R_{Q_i} = \left[R^2 + \left(\frac{a^2}{d}\right)^2 - 2R\left(\frac{a^2}{d}\right)\cos\theta\right]^{1/2}. \tag{4–69}$$

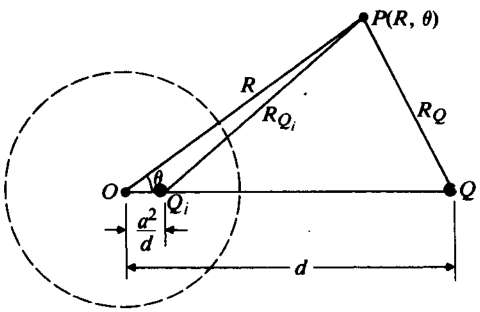


FIGURE 4–12
Diagram for computing induced charge distribution (Example 4–5).

Note that θ is measured from the line OQ. The R-component of the electric field intensity, E_R, is

$$E_R(R, \theta) = -\frac{\partial V(R, \theta)}{\partial R}. \tag{4-70}$$

Using Eq. (4–67) in Eq. (4–70) and noting Eqs. (4–68) and (4–69), we have

$$E_R(R, \theta) = \frac{Q}{4\pi\epsilon_0}\left\{\frac{R - d\cos\theta}{(R^2 + d^2 - 2Rd\cos\theta)^{3/2}} - \frac{a[R - (a^2/d)\cos\theta]}{d[R^2 + (a^2/d)^2 - 2R(a^2/d)\cos\theta]^{3/2}}\right\}. \tag{4-71}$$

a) To find the induced surface charge on the sphere, we set $R = a$ in Eq. (4–71) and evaluate

$$\rho_s = \epsilon_0 E_R(a, \theta), \tag{4-72}$$

which yields the following after simplification:

$$\rho_s = -\frac{Q(d^2 - a^2)}{4\pi a(a^2 + d^2 - 2ad\cos\theta)^{3/2}}. \tag{4-73}$$

Eq. (4–73) tells us that the induced surface charge is negative and that its magnitude is maximum at $\theta = 0$ and minimum at $\theta = \pi$, as expected.

b) The total charge induced on the sphere is obtained by integrating ρ_s over the surface of the sphere. We have

$$\begin{aligned}\text{Total induced charge} &= \oint \rho_s\, ds = \int_0^{2\pi}\int_0^{\pi} \rho_s a^2 \sin\theta\, d\theta\, d\phi \\ &= -\frac{a}{d}Q = Q_i.\end{aligned} \tag{4-74}$$

We note that the total induced charge is exactly equal to the image charge Q_i that replaced the sphere. Can you explain this? ▬

If the conducting sphere is electrically neutral and is not grounded, the image of a point charge Q at a distance d from the center of the sphere would still be Q_i at d_i given by Eqs. (4–65) and (4–66), respectively, in order to make the spherical surface $R = a$ equipotential. However, an additional point charge

$$Q' = -Q_i = \frac{a}{d}Q \tag{4-75}$$

at the center would be needed to make the net charge on the replaced sphere zero. The electrostatic problem of a point charge Q in the presence of an electrically neutral sphere can then be solved as a problem with three point charges: Q' at $R = 0$, Q_i at $R = a^2/d$, and Q at $R = d$.

4–4.4 CHARGED SPHERE AND GROUNDED PLANE

When a charged conducting sphere is near a large, grounded, conducting plane, as in Fig. 4–13(a), the charge distribution on and the electric field between the conducting bodies are obviously nonuniform. Since the geometry contains a mixture of spherical and Cartesian coordinates, field determination and capacitance calculation through a solution of Laplace's equation is a rather difficult problem. We shall now show how the repeated application of the method of images can be used to solve this problem.

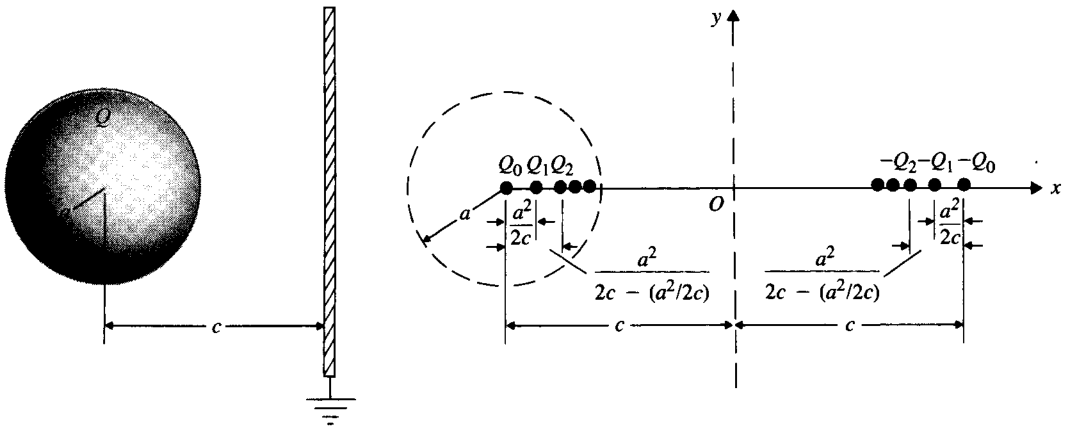
Assume that a charge Q_0 is put at the center of the sphere. We wish to find a system of image charges that, together with Q_0, will make both the sphere and the plane equipotential surfaces. The problem of a charged sphere near a grounded plane can then be replaced by that of the much simpler system of point charges. A cross section in the xy-plane is shown in Fig. 4–13(b). The presence of Q_0 at $(-c, 0)$ requires an image charge $-Q_0$ at $(c, 0)$ to make the yz-plane equipotential; but the pair of charges Q_0 and $-Q_0$ destroy the equipotential property of the sphere unless, according to Eqs. (4–65) and (4–66), an image charge $Q_1 = (a/2c)Q_0$ is placed at $(-c + a^2/2c, 0)$ inside the dashed circle. This, in turn, requires an image charge $-Q_1$ to make the yz-plane equipotential. This process of successive application of the method of images is continued, and we obtain two groups of image point charges: one group $(-Q_0, -Q_1, -Q_2, \ldots)$ on the right side of the y-axis, and another group $(Q_1, Q_2, \ldots)$ inside the sphere. We have

$$Q_1 = \left(\frac{a}{2c}\right)Q_0 = \alpha Q_0, \tag{4–76a}$$

$$Q_2 = \frac{a}{\left(2c - \dfrac{a^2}{2c}\right)} Q_1 = \frac{\alpha^2}{1 - \alpha^2} Q_0, \tag{4–76b}$$

$$Q_3 = \frac{a}{2c - \dfrac{a^2}{\left(2c - \dfrac{a^2}{2c}\right)}} Q_2 = \frac{\alpha^3}{(1 - \alpha^2)\left(1 - \dfrac{\alpha^3}{1 - \alpha^3}\right)} Q_0, \tag{4–76c}$$

$$\vdots$$



(a) Physical arrangement.

(b) Two groups of image point charges.

FIGURE 4–13
Charged sphere and grounded conducting plane.

where

$$\alpha = \frac{a}{2c}. \tag{4-77}$$

The total charge on the sphere is

$$\begin{aligned} Q &= Q_0 + Q_1 + Q_2 + \cdots \\ &= Q_0\left(1 + \alpha + \frac{\alpha^2}{1-\alpha^2} + \cdots\right). \end{aligned} \tag{4-78}$$

The series in Eq. (4–78) usually converges rapidly ($\alpha < 1/2$). Now since the charge pairs $(-Q_0, Q_1), (-Q_1, Q_2), \ldots$ yield a zero potential on the sphere, only the original Q_0 contributes to the potential of the sphere, which is

$$V_0 = \frac{Q_0}{4\pi\epsilon_0 a}. \tag{4-79}$$

Hence the capacitance between the sphere and the conducting plane is, from Eqs. (4–78) and (4–79),

$$C = \frac{Q}{V_0} = 4\pi\epsilon_0 a\left(1 + \alpha + \frac{\alpha^2}{1-\alpha^2} + \cdots\right), \tag{4-80}$$

which is larger than the capacitance of an isolated sphere of radius a, as expected. The potential and electric field distributions between the sphere and the conducting plane can also be obtained from the image point charges.

4–5 Boundary-Value Problems in Cartesian Coordinates

We saw in the preceding section that the method of images is very useful in solving certain types of electrostatic problems involving free charges near conducting boundaries that are geometrically simple. However, if the problem consists of a system of conductors maintained at specified potentials and with no isolated free charges, it cannot be solved by the method of images. This type of problem requires the solution of Laplace's equation. Example 4–1 (p. 154) was such a problem where the electric potential was a function of only one coordinate. Of course, Laplace's equation applied to three dimensions is a partial differential equation, where the potential is, in general, a function of all three coordinates. We will now develop a method for solving three-dimensional problems where the boundaries, over which the potential or its normal derivative is specified, coincide with the coordinate surfaces of an orthogonal, curvilinear coordinate system. In such cases, the solution can be expressed as a product of three one-dimensional functions, each depending separately on one coordinate variable only. The procedure is called the ***method of separation of variables.***

Problems (electromagnetic or otherwise) governed by partial differential equations with prescribed boundary conditions are called ***boundary-value problems.***

Boundary-value problems for potential functions can be classified into three types: (1) ***Dirichlet problems***, in which the value of the potential is specified everywhere on the boundaries; (2) ***Neumann problems***, in which the normal derivative of the potential is specified everywhere on the boundaries; (3) ***Mixed boundary-value problems***, in which the potential is specified over some boundaries and the normal derivative of the potential is specified over the remaining ones. Different specified boundary conditions will require the choice of different potential functions, but the procedure of solving these types of problems—that is, by the method of separation of variables—for the three types of problems is the same. The solutions of Laplace's equation are often called ***harmonic functions***.

Laplace's equation for scalar electric potential V in Cartesian coordinates is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{4–81}$$

To apply the method of separation of variables, we assume that the solution $V(x, y, z)$ can be expressed as a product in the following form:

$$V(x, y, z) = X(x)Y(y)Z(z), \tag{4–82}$$

where $X(x)$, $Y(y)$, and $Z(z)$ are functions of only x, y, and z, respectively. Substituting Eq. (4–82) in Eq. (4–81), we have

$$Y(y)Z(z)\frac{d^2X(x)}{dx^2} + X(x)Z(z)\frac{d^2Y(y)}{dy^2} + X(x)Y(y)\frac{d^2Z(z)}{dz^2} = 0,$$

which, when divided through by the product $X(x)Y(y)Z(z)$, yields

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = 0. \tag{4–83}$$

Note that each of the three terms on the left side of Eq. (4–83) is a function of only one coordinate variable and that only ordinary derivatives are involved. In order for Eq. (4–83) to be satisfied for *all values* of x, y, z, each of the three terms must be a constant. For instance, if we differentiate Eq. (4–83) with respect to x, we have

$$\frac{d}{dx}\left[\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}\right] = 0, \tag{4–84}$$

since the other two terms are independent of x. Equation (4–84) requires that

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -k_x^2, \tag{4–85}$$

where k_x^2 is a constant of integration to be determined from the boundary conditions of the problem. The negative sign on the right side of Eq. (4–85) is arbitrary, just as the square sign on k_x is arbitrary. The ***separation constant*** k_x can be a real or an imaginary number. If k_x is imaginary, k_x^2 is a negative real number, making $-k_x^2$ a

TABLE 4-1
Possible Solutions of $X''(x) + k_x^2 X(x) = 0$

k_x^2	k_x	$X(x)$	Exponential forms† of $X(x)$
0	0	$A_0 x + B_0$	
+	k	$A_1 \sin kx + B_1 \cos kx$	$C_1 e^{jkx} + D_1 e^{-jkx}$
−	jk	$A_2 \sinh kx + B_2 \cosh kx$	$C_2 e^{kx} + D_2 e^{-kx}$

† The exponential forms of $X(x)$ are related to the trigonometric and hyperbolic forms listed in the third column by the following formulas:

$$e^{\pm jkx} = \cos kx \pm j \sin kx, \quad \cos kx = \tfrac{1}{2}(e^{jkx} + e^{-jkx}), \quad \sin kx = \frac{1}{2j}(e^{jkx} - e^{-jkx});$$

$$e^{\pm kx} = \cosh kx \pm \sinh kx, \quad \cosh kx = \tfrac{1}{2}(e^{kx} + e^{-kx}), \quad \sinh kx = \tfrac{1}{2}(e^{kx} - e^{-kx}).$$

positive real number. It is convenient to rewrite Eq. (4–85) as

$$\frac{d^2X(x)}{dx^2} + k_x^2 X(x) = 0. \tag{4–86}$$

In a similar manner, we have

$$\frac{d^2Y(y)}{dy^2} + k_y^2 Y(y) = 0 \tag{4–87}$$

and

$$\frac{d^2Z(z)}{dz^2} + k_z^2 Z(z) = 0, \tag{4–88}$$

where the separation constants k_y and k_z will, in general, be different from k_x; but, because of Eq. (4–83), the following condition must be satisfied:

$$k_x^2 + k_y^2 + k_z^2 = 0. \tag{4–89}$$

Our problem has now been reduced to finding the appropriate solutions—$X(x)$, $Y(y)$, and $Z(z)$—from the second-order *ordinary* differential equations Eqs. (4–86), (4–87), and (4–88), respectively. The possible solutions of Eq. (4–86) are known from our study of ordinary differential equations with constant coefficients. They are listed in Table 4–1. That the listed solutions satisfy Eq. (4–86) is easily verified by direct substitution.

Of the listed solutions in Table 4–1, the first one, $A_0 x + B_0$ for $k_x = 0$, is a straight line with a slope A_0 and an intercept B_0 at $x = 0$. When $A_0 = 0$, $X(x) = B_0$, which means that V, the solution of Laplace's equation, is independent of the dimension x.

We are, of course, familiar with the sine and cosine functions, both of which are periodic with a period 2π. If plotted versus x, $\sin kx$ and $\cos kx$ have a period $2\pi/k$. Frequently, a careful examination of a given problem enables us to decide whether a sine or a cosine function is the proper choice. For example, if the solution is to vanish at $x = 0$, $\sin kx$ must be chosen; on the other hand, if the solution is expected to be symmetrical with respect to $x = 0$, then $\cos kx$ is the right choice. In

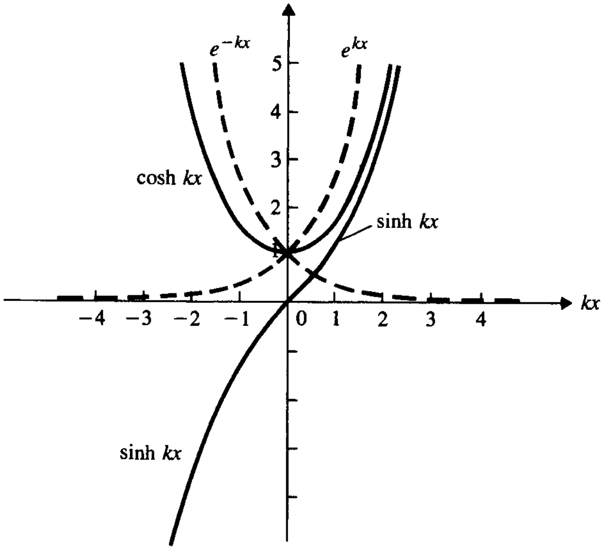


FIGURE 4–14
Hyperbolic and exponential functions.

the general case, both terms are required. Sometimes it may be desirable to write $A_1 \sin kx + B_1 \cos kx$ as $A_s \sin(kx + \psi_s)$ or $A_c \cos(kx + \psi_c)$.†

For $k_x = jk$ the solution converts to hyperbolic functions:

$$\sin jkx = -j \sinh kx$$

and

$$\cos jkx = \cosh kx.$$

Hyperbolic functions are combinations of exponential functions with real exponents, and are nonperiodic. They are plotted in Fig. 4–14 for easy reference. The important characteristics of sinh kx are that it is an odd function of x and that its value approaches $\pm\infty$ as x goes to $\pm\infty$. The function cosh kx is an even function of x, equals unity at $x = 0$, and approaches $+\infty$ as x goes to $+\infty$ or $-\infty$.

The specified boundary conditions will determine the choice of the proper form of the solution and of the constants A and B or C and D. The solutions of Eqs. (4–87) and (4–88) for $Y(y)$ and $Z(z)$ are entirely similar.

EXAMPLE 4–6 Two grounded, semi-infinite, parallel-plane electrodes are separated by a distance b. A third electrode perpendicular to and insulated from both is maintained at a constant potential V_0 (see Fig. 4–15). Determine the potential distribution in the region enclosed by the electrodes.

† $A_s \sin(kx + \psi_s) = (A_s \cos\psi_s) \sin kx + (A_s \sin\psi_s) \cos kx$; $A_1 = A_s \cos\psi_s$, $B_1 = A_s \sin\psi_s$; $A_s = (A_1^2 + B_1^2)^{1/2}$, $\psi_s = \tan^{-1}(B_1/A_1)$. $A_c \cos(kx + \psi_c) = (-A_c \sin\psi_c) \sin kx + (A_c \cos\psi_c) \cos kx$; $A_1 = -A_c \sin\psi_c$, $B_1 = A_c \cos\psi_c$; $A_c = (A_1^2 + B_1^2)^{1/2}$, $\psi_c = \tan^{-1}(-A_1/B_1)$.

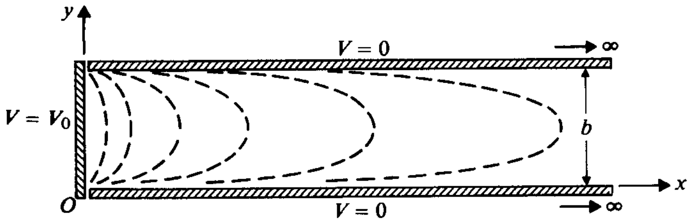


FIGURE 4-15
Cross-sectional figure for Example 4-6. The plane electrodes are infinite in z-direction.

Solution Referring to the coordinates in Fig. 4-15, we write down the boundary conditions for the potential function $V(x, y, z)$ as follows.

With V independent of z:

$$V(x, y, z) = V(x, y). \tag{4-90a}$$

In the x-direction:

$$V(0, y) = V_0 \tag{4-90b}$$

$$V(\infty, y) = 0. \tag{4-90c}$$

In the y-direction:

$$V(x, 0) = 0 \tag{4-90d}$$

$$V(x, b) = 0. \tag{4-90e}$$

Condition (4-90a) implies $k_z = 0$, and from Table 4-1,

$$Z(z) = B_0. \tag{4-91}$$

The constant A_0 vanishes because Z is independent of z. From Eq. (4-89) we have

$$k_y^2 = -k_x^2 = k^2, \tag{4-92}$$

where k is a real number. This choice of k implies that k_x is imaginary and that k_y is real. The use of $k_x = jk$, together with the condition of Eq. (4-90c), requires us to choose the exponentially decreasing form for $X(x)$, which is

$$X(x) = D_2 e^{-kx}. \tag{4-93}$$

In the y-direction, $k_y = k$. Condition (4-90d) indicates that the proper choice for $Y(y)$ from Table 4-1 is

$$Y(y) = A_1 \sin ky. \tag{4-94}$$

Combining the solutions given by Eqs. (4-91), (4-93), and (4-94) in Eq. (4-82), we obtain an appropriate solution of the following form:

$$\begin{aligned} V_n(x, y) &= (B_0 D_2 A_1) e^{-kx} \sin ky \\ &= C_n e^{-kx} \sin ky, \end{aligned} \tag{4-95}$$

where the arbitrary constant C_n has been written for the product $B_0 D_2 A_1$.

Now, of the five boundary conditions listed in Eqs. (4-90a) through (4-90e) we have used conditions (4-90a), (4-90c), and (4-90d). To meet condition (4-90e), we require

$$V_n(x, b) = C_n e^{-kx} \sin kb = 0, \tag{4–96}$$

which can be satisfied, for all values of x, only if

$$\sin kb = 0$$

or

$$kb = n\pi$$

or

$$k = \frac{n\pi}{b}, \qquad n = 1, 2, 3, \ldots. \tag{4–97}$$

Therefore, Eq. (4–95) becomes

$$V_n(x\ y) = C_n e^{-n\pi x/b} \sin \frac{n\pi}{b} y. \tag{4–98}$$

Question: Why are 0 and negative integral values of n not included in Eq. (4–97)?

We can readily verify by direct substitution that $V_n(x, y)$ in Eq. (4–98) satisfies the Laplace's equation (4–81). However, $V_n(x, y)$ alone cannot satisfy the remaining boundary condition (4–90b) at $x = 0$ for all values of y from 0 to b. Since Laplace's equation is a *linear* partial differential equation, a sum (superposition) of $V_n(x, y)$ of the form in Eq. (4–98) with different values of n is also a solution. At $x = 0$, we write

$$\begin{aligned} V(0, y) &= \sum_{n=1}^{\infty} V_n(0, y) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{b} y \\ &= V_0, \qquad 0 < y < b. \end{aligned} \tag{4–99}$$

Equation (4–99) is essentially a Fourier-series expansion of the periodic rectangular wave at $x = 0$ shown in Fig. 4–16, which has a constant value V_0 in the interval $0 < y < b$.

In order to evaluate the coefficients C_n, we multiply both sides of Eq. (4–99) by $\sin \frac{m\pi}{b} y$ and integrate the products from $y = 0$ to $y = b$:

$$\sum_{n=1}^{\infty} \int_0^b C_n \sin \frac{n\pi}{b} y \sin \frac{m\pi}{b} y\, dy = \int_0^b V_0 \sin \frac{m\pi}{b} y\, dy. \tag{4–100}$$

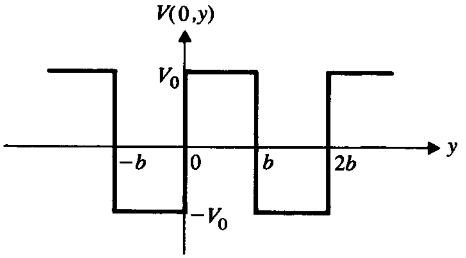


FIGURE 4–16
For Fourier-series expansion of boundary condition at $x = 0$ (Example 4–6).

The integral on the right side of Eq. (4–100) is easily evaluated:

$$\int_0^b V_0 \sin\frac{m\pi}{b}y\,dy = \begin{cases} \dfrac{2bV_0}{m\pi} & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even.} \end{cases} \tag{4–101}$$

Each integral on the left side of Eq. (4–100) is

$$\int_0^b C_n \sin\frac{n\pi}{b}y \sin\frac{m\pi}{b}y\,dy = \frac{C_n}{2}\int_0^b \left[\cos\frac{(n-m)\pi}{b}y - \cos\frac{(n+m)\pi}{b}y\right]dy$$

$$= \begin{cases} \dfrac{C_n}{2}b & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases} \tag{4–102}$$

Substituting Eqs. (4–101) and (4–102) in Eq. (4–100), we obtain

$$C_n = \begin{cases} \dfrac{4V_0}{n\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \tag{4–103}$$

The desired potential distribution is, then, a superposition of $V_n(x, y)$ in Eq. (4–98).

$$V(x, y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/b} \sin\frac{n\pi}{b}y$$

$$= \frac{4V_0}{\pi}\sum_{n=\text{odd}}^{\infty} \frac{1}{n} e^{-n\pi x/b} \sin\frac{n\pi}{b}y, \tag{4–104}$$

$$n = 1, 3, 5, \ldots,$$

$$x > 0 \quad \text{and} \quad 0 < y < b.$$

Equation (4–104) is a rather complicated expression to plot in two dimensions; but since the amplitude of the sine terms in the series decreases very rapidly as n increases, only the first few terms are needed to obtain a good approximation. Several equipotential lines are sketched in Fig. 4–15. ▬

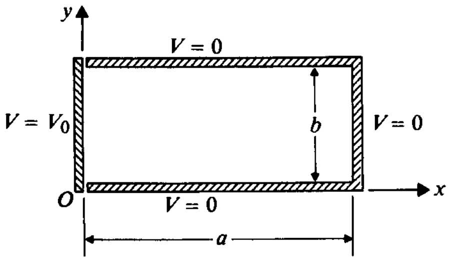


FIGURE 4–17
Cross-sectional figure for Example 4–7.

EXAMPLE 4–7 Consider the region enclosed on three sides by grounded conducting planes shown in Fig. 4–17. The end plate on the left is insulated from the grounded sides and has a constant potential V_0. All planes are assumed to be infinite in extent in the z-direction. Determine the potential distribution within this region.

Solution The boundary conditions for the potential function $V(x, y, z)$ are as follows.

With V independent of z:

$$V(x, y, z) = V(x, y). \tag{4–105a}$$

In the x-direction:

$$V(0, y) = V_0, \tag{4–105b}$$

$$V(a, y) = 0. \tag{4–105c}$$

In the y-direction:

$$V(x, 0) = 0, \tag{4–105d}$$

$$V(x, b) = 0. \tag{4–105e}$$

Condition (4–105a) implies that $k_z = 0$, and, from Table 4–1,

$$Z(z) = B_0. \tag{4–106}$$

As a consequence, Eq. (4–89) reduces to

$$k_y^2 = -k_x^2 = k^2, \tag{4–107}$$

which is the same as Eq. (4–92) in Example 4–6.

The boundary conditions in the y-direction, Eqs. (4–105d) and Eq. (4–105e), are the same as those specified by Eqs. (4–90d) and (4–90e). To make $V(x, 0) = 0$ for all values of x between 0 and a, $Y(0)$ must be zero, and we have

$$Y(y) = A_1 \sin ky, \tag{4–108}$$

as in Eq. (4–94). However, $X(x)$ given by Eq. (4–93) is obviously not a solution here because it does not satisfy the boundary condition (4–105c). In this case it is convenient to use the general form for $k_x = jk$ given in the third column of Table 4–1. (The exponential solution form given in the last column could be used as well, but it would not be as convenient because it is not as easy to see the condition under which the sum of two exponential terms vanishes at $x = a$ as it is to make a sinh term zero. This will be clear presently.) We have

$$X(x) = A_2 \sinh kx + B_2 \cosh kx. \tag{4–109}$$

A relation exists between the arbitrary constants A_2 and B_2 because of the boundary condition in Eq. (4–105c), which demands that $X(a) = 0$; that is,

$$0 = A_2 \sinh ka + B_2 \cosh ka$$

or

$$B_2 = -A_2 \frac{\sinh ka}{\cosh ka}.$$

From Eq. (4–109) we have

$$X(x) = A_2\left[\sinh kx - \frac{\sinh ka}{\cosh ka}\cosh kx\right]$$
$$= \frac{A_2}{\cosh ka}[\cosh ka \sinh kx - \sinh ka \cosh kx] \tag{4–110}$$
$$= A_3 \sinh k(x-a),$$

where A_3 has been written for $A_2/\cosh ka$. It is evident that Eq. (4–110) satisfies the condition $X(a) = 0$. With experience we should be able to write the solution given in Eq. (4–110) directly, without the steps leading to it, as only a shift in the argument of the sinh function is needed to make it vanish at $x = a$.

Collecting Eqs. (4–106), (4–108) and (4–110), we obtain the appropriate product solution

$$V_n(x, y) = B_0A_1A_3 \sinh k(x-a) \sin ky$$
$$= C_n' \sinh\frac{n\pi}{b}(x-a)\sin\frac{n\pi}{b}y, \qquad n = 1, 2, 3, \ldots, \tag{4–111}$$

where $C_n' = B_0A_1A_3$, and k has been set to equal $n\pi/b$ in order to satisfy boundary condition (4–105e).

We have now used all of the boundary conditions except Eq. (4–105b), which may be satisfied by a Fourier-series expansion of $V(0, y) = V_0$ over the interval from $y = 0$ to $y = b$. We have

$$V_0 = \sum_{n=1}^{\infty} V_n(0, y) = -\sum_{n=1}^{\infty} C_n' \sinh\frac{n\pi}{b}a \sin\frac{n\pi}{b}y, \qquad 0 < y < b. \tag{4–112}$$

We note that Eq. (4–112) is of the same form as Eq. (4–99), except that C_n is replaced by $-C_n' \sinh(n\pi a/b)$. The values for the coefficient C_n' can then be written down from Eq. (4–103):

$$C_n' = \begin{cases} -\dfrac{4V_0}{n\pi \sinh(n\pi a/b)} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \tag{4–113}$$

The desired potential distribution within the enclosed region in Fig. 4–17 is a summation of $V_n(x, y)$ in Eq. (4–111):

$$V(x, y) = \sum_{n=1}^{\infty} C_n' \sinh\frac{n\pi}{b}(x-a)\sin\frac{n\pi}{b}y$$
$$= \frac{4V_0}{\pi}\sum_{n=\text{odd}}^{\infty} \frac{\sinh[n\pi(a-x)/b]}{n\sinh(n\pi a/b)}\sin\frac{n\pi}{b}y, \tag{4–114}$$
$$n = 1, 3, 5, \ldots,$$
$$0 < x < a \quad \text{and} \quad 0 < y < b.$$

The electric field distribution within the enclosure is obtained by the relation

$$\mathbf{E}(x, y) = -\nabla V(x, y).$$

■

4–6 Boundary-Value Problems in Cylindrical Coordinates

For problems with circular cylindrical boundaries we write the governing equations in the cyclindrical coordinate system. Laplace's equation for scalar electric potential V in cylindrical coordinates is, from Eq. (4–8),

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\partial^2 V}{\partial z^2}=0. \tag{4–115}$$

A general solution of Eq. (4–115) requires the knowlege of ***Bessel functions***, the discussion of which will be deferred until Chapter 10. In situations in which the lengthwise dimension of the cylindrical geometry is large in comparison to its radius, the associated field quantities may be considered to be approximately independent of z. In such cases, $\partial^2 V/\partial z^2 = 0$ and Eq. (4–115) becomes the governing equation of a two-dimensional problem:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}=0. \tag{4–116}$$

Applying the method of separation of variables, we assume a product solution

$$V(r, \phi) = R(r)\Phi(\phi), \tag{4–117}$$

where $R(r)$ and $\Phi(\phi)$ are, respectively, functions of r and ϕ only. Substituting solution (4–117) in Eq. (4–116) and dividing by $R(r)\Phi(\phi)$, we have

$$\frac{r}{R(r)}\frac{d}{dr}\left[r\frac{dR(r)}{dr}\right]+\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2}=0. \tag{4–118}$$

In Eq. (4–118) the first term on the left side is a function of r only, and the second term is a function of ϕ only. (Note that ordinary derivatives have replaced partial derivatives.) For Eq. (4–118) to hold for all values of r and ϕ, each term must be a constant and be the negative of the other. We have

$$\frac{r}{R(r)}\frac{d}{dr}\left[r\frac{dR(r)}{dr}\right]=k^2 \tag{4–119}$$

and

$$\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2}=-k^2, \tag{4–120}$$

where k is a separation constant.

Equation (4–120) can be rewritten as

$$\frac{d^2\Phi(\phi)}{d\phi^2}+k^2\Phi(\phi)=0. \tag{4–121}$$

This is of the same form as Eq. (4–86), and its solution can be any one of those listed in Table 4–1. For circular cylindrical configurations, potential functions and therefore $\Phi(\phi)$ are periodic in ϕ, and the hyperbolic functions do not apply. In fact, if the range of ϕ is unrestricted, k must be an integer. Let k equal n. The appropriate solution is

$$\Phi(\phi) = A_\phi \sin n\phi + B_\phi \cos n\phi, \tag{4-122}$$

where A_ϕ and B_ϕ are arbitrary constants.

We now turn our attention to Eq. (4–119), which can be rearranged as

$$r^2 \frac{d^2R(r)}{dr^2} + r\frac{dR(r)}{dr} - n^2R(r) = 0, \tag{4-123}$$

where integer n has been written for k, implying a 2π range for ϕ. The solution of Eq. (4–123) is

$$R(r) = A_r r^n + B_r r^{-n}. \tag{4-124}$$

This can be verified by direct substitution. Taking the product of the solutions in (4–122) and (4–124), we obtain a general solution of z-independent Laplace's equation (4–116) for circular cylindrical regions with an unrestricted range for ϕ:

$$V_n(r, \phi) = r^n(A_n \sin n\phi + B_n \cos n\phi) + r^{-n}(A_n' \sin n\phi + B_n' \cos n\phi), \qquad n \neq 0. \tag{4-125}$$

Depending on the boundary conditions the complete solution of a problem may be a summation of the terms in Eq. (4–125). It is useful to note that, when the region of interest includes the cylindrical axis where $r = 0$, the terms containing the r^{-n} factor cannot exist. On the other hand, if the region of interest includes the point at infinity, the terms containing the r^n factor cannot exist, since the potential must be zero as $r \to \infty$.

Eq. (4–121) has the simplest form when $k = 0$. We have

$$\frac{d^2\Phi(\phi)}{d\phi^2} = 0. \tag{4-126}$$

The general solution of Eq. (4–126) is $\Phi(\phi) = A_0\phi + B_0$. If there is no circumferential variation, A_0 vanishes,† and we have

$$\Phi(\phi) = B_0, \qquad k = 0. \tag{4-127}$$

The equation for $R(r)$ also becomes simpler when $k = 0$. We obtain from Eq. (4–119)

$$\frac{d}{dr}\left[r\frac{dR(r)}{dr}\right] = 0, \tag{4-128}$$

which has a solution

$$R(r) = C_0 \ln r + D_0, \qquad k = 0. \tag{4-129}$$

The product of Eqs. (4–127) and (4–129) gives a solution that is independent of either z or ϕ:

$$V(r) = C_1 \ln r + C_2, \tag{4-130}$$

where the arbitrary constants C_1 and C_2 are determined from boundary conditions.

---

† The term $A_0\phi$ should be retained if there is circumferential variation, such as in problems involving a wedge. (See Problem P.4–23.)

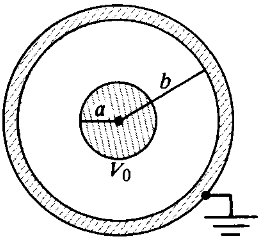


FIGURE 4–18
Cross section of a coaxial cable (Example 4–8).

We shall now illustrate the above procedures with two examples. One (Example 4–8) deals with a situation that is circularly symmetrical, and the other (Example 4–9) solves a problem with circumferential variation.

EXAMPLE 4–8 Consider a very long coaxial cable. The inner conductor has a radius a and is maintained at a potential V_0. The outer conductor has an inner radius b and is grounded. Determine the potential distribution in the space between the conductors.

Solution Figure 4–18 shows a cross section of the coaxial cable. We assume no z-dependence and, by symmetry, also no ϕ-dependence ($k = 0$). Therefore, the electric potential is a function of r only and is given by Eq. (4–130).

The boundary conditions are

$$V(b) = 0, \tag{4-131a}$$

$$V(a) = V_0. \tag{4-131b}$$

Substitution of Eqs. (4–131a) and (4–131b) in Eq. (4–130) leads to two relations:

$$C_1 \ln b + C_2 = 0, \tag{4-132a}$$

$$C_1 \ln a + C_2 = V_0. \tag{4-132b}$$

From Eqs. (4–132a) and (4–132b), C_1 and C_2 are readily determined:

$$C_1 = -\frac{V_0}{\ln(b/a)}, \qquad C_2 = \frac{V_0 \ln b}{\ln(b/a)}.$$

Therefore, the potential distribution in the space $a \le r \le b$ is

$$V(r) = \frac{V_0}{\ln(b/a)} \ln\left(\frac{b}{r}\right). \tag{4-133}$$

Obviously, equipotential surfaces are coaxial cylindrical surfaces. ■

EXAMPLE 4–9 An infinitely long, thin, conducting circular tube of radius b is split in two halves. The upper half is kept at a potential $V = V_0$ and the lower half at $V = -V_0$. Determine the potential distribution both inside and outside the tube.

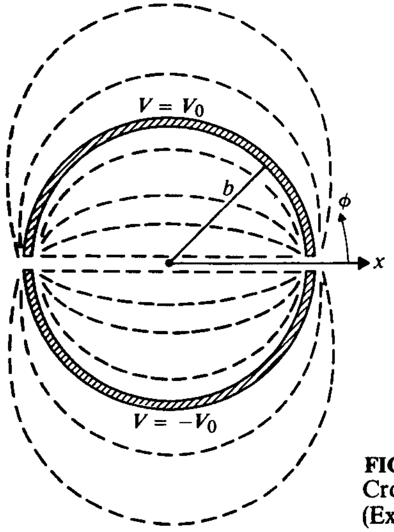


FIGURE 4–19
Cross section of split circular cylinder and equipotential lines (Example 4–9).

Solution A cross section of the split circular tube is shown in Fig. 4–19. Since the tube is assumed to be infinitely long, the potential is independent of z and the two-dimensional Laplace's equation (4–116) applies. The boundary conditions are

$$V(b, \phi) = \begin{cases} V_0 & \text{for } 0 < \phi < \pi, \\ -V_0 & \text{for } \pi < \phi < 2\pi. \end{cases} \tag{4–134}$$

These conditions are plotted in Fig. 4–20. Obviously, $V(r, \phi)$ is an odd function of ϕ. We shall determine $V(r, \phi)$ inside and outside the tube separately.

a) Inside the tube,

$$r < b.$$

Because this region includes $r = 0$, terms containing the r^{-n} factor cannot exist. Moreover, since $V(r, \phi)$ is an odd function of ϕ, the appropriate form of solution

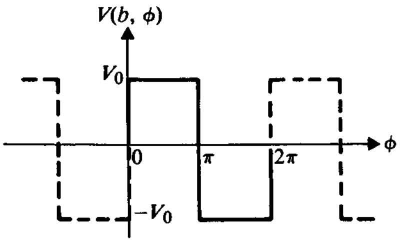


FIGURE 4–20
Boundary condition for Example 4–9.

is, from Eq. (4–125),

$$V_n(r, \phi) = A_n r^n \sin n\phi. \tag{4–135}$$

However, a single such term does not satisfy the boundary conditions specified in Eq. (4–134). We form a series solution

$$\begin{aligned} V(r, \phi) &= \sum_{n=1}^{\infty} V_n(r, \phi) \\ &= \sum_{n=1}^{\infty} A_n r^n \sin n\phi, \end{aligned} \tag{4–136}$$

and require that Eq. (4–134) be satisfied at $r = b$. This amounts to expanding the rectangular wave (period $= 2\pi$) shown in Fig. 4–20 into a Fourier sine series.

$$\sum_{n=1}^{\infty} A_n b^n \sin n\phi = \begin{cases} V_0 & \text{for } 0 < \phi < \pi, \\ -V_0 & \text{for } \pi < \phi < 2\pi. \end{cases} \tag{4–137}$$

The coefficients A_n can be found by the method illustrated in Example 4–6. As a matter of fact, because we already have the result in Eq. (4–103), we can directly write

$$A_n = \begin{cases} \dfrac{4V_0}{n\pi b^n} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \tag{4–138}$$

The potential distribution inside the tube is obtained by substituting Eq. (4–138) in Eq. (4–136):

$$V(r, \phi) = \frac{4V_0}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{1}{n} \left(\frac{r}{b}\right)^n \sin n\phi, \qquad r < b. \tag{4–139}$$

b) Outside the tube,

$$r > b.$$

In this region the potential must decrease to zero as $r \to \infty$. Terms containing the factor r^n cannot exist, and the appropriate form of solution is, from Eq. (4–125),

$$\begin{aligned} V(r, \phi) &= \sum_{n=1}^{\infty} V_n(r, \phi) \\ &= \sum_{n=1}^{\infty} B'_n r^{-n} \sin n\phi. \end{aligned} \tag{4–140}$$

At $r = b$,

$$\begin{aligned} V(b, \phi) &= \sum_{n=1}^{\infty} B'_n b^{-n} \sin n\phi \\ &= \begin{cases} V_0 & \text{for } 0 < \phi < \pi, \\ -V_0 & \text{for } \pi < \phi < 2\pi. \end{cases} \end{aligned} \tag{4–141}$$

The coefficients B'_n in Eq. (4–141) are analogous to A_n in Eq. (4–137). From Eq. (4–138) we obtain

$$B'_n = \begin{cases} \dfrac{4V_0 b^n}{n\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases} \tag{4–142}$$

Therefore, the potential distribution outside the tube is, from Eq. (4–140),

$$V(r, \phi) = \frac{4V_0}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{1}{n}\left(\frac{b}{r}\right)^n \sin n\phi, \qquad r > b. \tag{4–143}$$

Several equipotential lines both inside and outside the tube have been sketched in Fig. 4–19. ▬

4–7 Boundary-Value Problems in Spherical Coordinates

The general solution of Laplace's equation in spherical coordinates is a very involved procedure, so we will limit our discussion to cases in which the electric potential is independent of the azimuthal angle ϕ. Even with this limitation we will need to introduce some new functions. From Eq. (4–9) we have

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) = 0. \tag{4–144}$$

Applying the method of separation of variables, we assume a product solution

$$V(R, \theta) = \Gamma(R)\Theta(\theta). \tag{4–145}$$

Substitution of this solution in Eq. (4–144) yields, after rearrangement,

$$\frac{1}{\Gamma(R)}\frac{d}{dR}\left[R^2 \frac{d\Gamma(R)}{dR}\right] + \frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left[\sin\theta \frac{d\Theta(\theta)}{d\theta}\right] = 0. \tag{4–146}$$

In Eq. (4–146) the first term on the left side is a function of R only, and the second term is a function of θ only. If the equation is to hold for all values of R and θ, each term must be a constant and be the negative of the other. We write

$$\frac{1}{\Gamma(R)}\frac{d}{dR}\left[R^2 \frac{d\Gamma(R)}{dR}\right] = k^2 \tag{4–147}$$

and

$$\frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left[\sin\theta \frac{d\Theta(\theta)}{d\theta}\right] = -k^2, \tag{4–148}$$

where k is a separation constant. We must now solve the two second-order, ordinary differential equations, Eqs. (4–147) and (4–148).

TABLE 4–2
Several Legendre Polynomials

n	$P_n(\cos\theta)$
0	1
1	$\cos\theta$
2	$\frac{1}{2}(3\cos^2\theta - 1)$
3	$\frac{1}{2}(5\cos^3\theta - 3\cos\theta)$

Equation (4–147) can be rewritten as

$$R^2\frac{d^2\Gamma(R)}{dR^2} + 2R\frac{d\Gamma(R)}{dR} - k^2\Gamma(R) = 0, \qquad (4\text{–}149)$$

which has a solution of the form

$$\Gamma_n(R) = A_nR^n + B_nR^{-(n+1)}. \qquad (4\text{–}150)$$

In Eq. (4–150), A_n and B_n are arbitrary constants, and the following relation between n and k can be verified by substitution:

$$n(n+1) = k^2, \qquad (4\text{–}151)$$

where $n = 0, 1, 2, \ldots$ is a positive integer.

With the value of k^2 given in Eq. (4–151), we have, from Eq. (4–148),

$$\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta(\theta)}{d\theta}\right] + n(n+1)\Theta(\theta)\sin\theta = 0, \qquad (4\text{–}152)$$

which is a form of ***Legendre's equation.*** For problems involving the full range of θ, from 0 to π, the solutions to Legendre's equation (4–152) are called ***Legendre functions***, usually denoted by $P(\cos\theta)$. Since Legendre functions for integral values of n are polynomials in $\cos\theta$, they are also called ***Legendre polynomials.*** We write

$$\Theta_n(\theta) = P_n(\cos\theta). \qquad (4\text{–}153)$$

Table 4–2 lists the expressions for Legendre polynomials† for several values of n.

Combining solutions (4–150) and (4–153) in Eq. (4–145), we have, for spherical boundary-value problems with no azimuthal variation,

$$V_n(R, \theta) = [A_nR^n + B_nR^{-(n+1)}]P_n(\cos\theta). \qquad (4\text{–}154)$$

Depending on the boundary conditions of the given problem, the complete solution may be a summation of the terms in Eq. (4–154). We illustrate the application of

† Actually, Legendre polynomials are Legendre functions of the first kind. There is another set of solutions to Legendre's equation, called Legendre functions of the second kind; but they have singularities at $\theta = 0$ and π and must therefore be excluded if the polar axis is a region of interest.

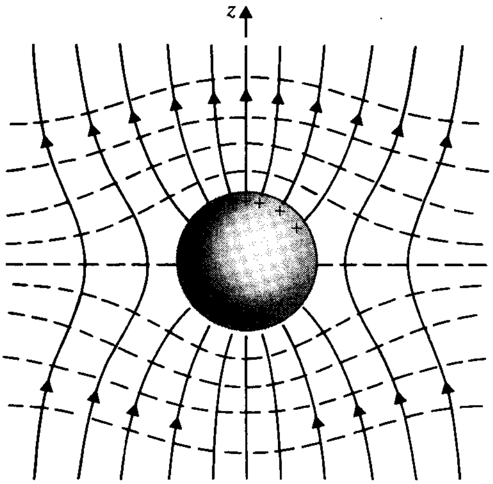


FIGURE 4–21
Conducting sphere in a uniform electric field (Example 4–10).

Legendre polynomials in the solution of a simple boundary-value problem in the following example.

EXAMPLE 4–10 An uncharged conducting sphere of radius b is placed in an initially uniform electric field $\mathbf{E}_0 = \mathbf{a}_z E_0$. Determine (a) the potential distribution $V(R, \theta)$, and (b) the electric field intensity $\mathbf{E}(R, \theta)$ after the introduction of the sphere.

Solution After the conducting sphere is introduced into the electric field, a separation and redistribution of charges will take place in such a way that the surface of the sphere is maintained equipotential. The electric field intensity within the sphere is zero. Outside the sphere the field lines will intersect the surface normally, and the field intensity at points very far away from the sphere will not be affected appreciably. The geometry of this problem is depicted in Fig. 4–21. The potential is, obviously, independent of the azimuthal angle ϕ, and the solution obtained in this section applies.

a) To determine the potential distribution $V(R, \theta)$ for $R \geq b$, we note the following boundary conditions:

$$V(b, \theta) = 0^{\dagger} \tag{4–155a}$$

$$V(R, \theta) = -E_0 z = -E_0 R \cos\theta, \qquad \text{for } R \gg b. \tag{4–155b}$$

Equation (4–155b) is a statement that the original $\mathbf{E}_0$ is not disturbed at points very far away from the sphere. By using Eq. (4–154) we write the general solution

† For this problem it is convenient to assume that $V = 0$ in the equatorial plane ($\theta = \pi/2$), which leads to $V(b, \theta) = 0$, since the surface of the conducting sphere is equipotential. (See Problem P.4–28 for $V(b, \theta) = V_0$.)

as

$$V(R, \theta) = \sum_{n=0}^{\infty} [A_n R^n + B_n R^{-(n+1)}] P_n(\cos\theta), \qquad R \geq b. \tag{4–156}$$

However, in view of Eq. (4–155b), all A_n except A_1 must vanish, and $A_1 = -E_0$. We have, from Eq. (4–156) and Table 4–2,

$$V(R, \theta) = -E_0 R P_1(\cos\theta) + \sum_{n=0}^{\infty} B_n R^{-(n+1)} P_n(\cos\theta)$$

$$= B_0 R^{-1} + (B_1 R^{-2} - E_0 R)\cos\theta + \sum_{n=2}^{\infty} B_n R^{-(n+1)} P_n(\cos\theta), \qquad R \geq b. \tag{4–157}$$

Actually, the first term on the right side of Eq. (4–157) corresponds to the potential of a charged sphere. Since the sphere is uncharged, $B_0 = 0$, and Eq. (4–157) becomes

$$V(R, \theta) = \left(\frac{B_1}{R^2} - E_0 R\right)\cos\theta + \sum_{n=2}^{\infty} B_n R^{-(n+1)} P_n(\cos\theta), \qquad R \geq b. \tag{4–158}$$

Now applying boundary condition (4–155a) at $R = b$, we require

$$0 = \left(\frac{B_1}{b^2} - E_0 b\right)\cos\theta + \sum_{n=2}^{\infty} B_n b^{-(n+1)} P_n(\cos\theta),$$

from which we obtain

$$B_1 = E_0 b^3$$

and

$$B_n = 0, \qquad n \geq 2.$$

We have, finally, from Eq. (4–158),

$$V(R, \theta) = -E_0\left[1 - \left(\frac{b}{R}\right)^3\right] R\cos\theta, \qquad R \geq b. \tag{4–159}$$

b) The electric field intensity $\mathbf{E}(R, \theta)$ for $R \geq b$ can be easily determined from $-\nabla V(R, \theta)$:

$$\mathbf{E}(R, \theta) = \mathbf{a}_R E_R + \mathbf{a}_\theta E_\theta, \tag{4–160}$$

where

$$E_R = -\frac{\partial V}{\partial R} = E_0\left[1 + 2\left(\frac{b}{R}\right)^3\right]\cos\theta, \qquad R \geq b \tag{4–160a}$$

and

$$E_\theta = -\frac{\partial V}{R\,\partial\theta} = -E_0\left[1 - \left(\frac{b}{R}\right)^3\right]\sin\theta, \qquad R \geq b. \tag{4–160b}$$

The surface charge density on the sphere can be found by noting that

$$\rho_s(\theta) = \epsilon_0 E_R\Big|_{R=b} = 3\epsilon_0 E_0 \cos\theta, \tag{4–161}$$

which is proportional to $\cos\theta$, being zero at $\theta = \pi/2$. Some equipotential and field lines are sketched in Fig. 4–21. ∎

It is interesting to note from Eq. (4–159) that the potential is the sum of two terms: $-E_0R\cos\theta$ due to the applied uniform electric field; and $(E_0b^3\cos\theta)/R^2$ due to an electric dipole of a dipole moment:

$$\mathbf{p} = \mathbf{a}_z 4\pi\epsilon_0 b^3 E_0 \tag{4–162}$$

at the center of the sphere. The contribution of the equivalent dipole can be verified by referring to Eq. (3–53). The expressions in Eqs. (4–160a) and (4–160b) for the resultant electric field intensity, being derived from the potential, obviously also represent the combination of the applied uniform field and that of the equivalent dipole, given in Eq. (3–54).

In this chapter we have discussed the analytical solution of electrostatic problems by the method of images and by direct solution of Laplace's equation. The method of images is useful when charges exist near conducting bodies of a simple and compatible geometry: a point charge near a conducting sphere or an infinite conducting plane; and a line charge near a parallel conducting cylinder or a parallel conducting plane. The solution of Laplace's equation by the method of separation of variables requires that the boundaries coincide with coordinate surfaces. These requirements restrict the usefulness of both methods. In practical problems we are often faced with more complicated boundaries, which are not amenable to neat analytical solutions. In such cases we must resort to approximate graphical or numerical methods. These methods are beyond the scope of this book.†

Review Questions

R.4–1 Write Poisson's equation in vector notation
 a) for a simple medium,
 b) for a linear and isotropic but inhomogeneous medium.

R.4–2 Repeat in Cartesian coordinates both parts of Question R.4–1.

R.4–3 Write Laplace's equation for a simple medium
 a) in vector notation, **b)** in Cartesian coordinates.

R.4–4 If $\nabla^2 U = 0$, why does it not follow that U is identically zero?

R.4–5 A fixed voltage is connected across a parallel-plate capacitor.
 a) Does the electric field intensity in the space between the plates depend on the permittivity of the medium?
 b) Does the electric flux density depend on the permittivity of the medium?
Explain.

R.4–6 Assume that fixed charges $+Q$ and $-Q$ are deposited on the plates of an isolated parallel-plate capacitor.
 a) Does the electric field intensity in the space between the plates depend on the permittivity of the medium?
 b) Does the electric flux density depend on the permittivity of the medium?
Explain.

† See, for instance, B. D. Popović, *Introductory Engineering Electromagnetics*, Chapter 5, Addison-Wesley Publishing Co., Reading, Mass., 1971.

R.4–7 Why is the electrostatic potential continuous at a boundary?

R.4–8 State in words the uniqueness theorem of electrostatics.

R.4–9 What is the image of a spherical cloud of electrons with respect to an infinite conducting plane?

R.4–10 Why cannot the point at infinity be used as the point for the zero reference potential for an infinite line charge as it is for a point charge? What is the physical reason for this difference?

R.4–11 What is the image of an infinitely long line charge of density ρ_ℓ with respect to a parallel conducting circular cylinder?

R.4–12 Where is the zero-potential surface of the two-wire transmission line in Fig. 4–6?

R.4–13 In finding the surface charge induced on a grounded sphere by a point charge, can we set $R = a$ in Eq. (4–67) and then evaluate ρ_s by $-\epsilon_0\, \partial V(a, \theta)/\partial R$? Explain.

R.4–14 What is the method of separation of variables? Under what conditions is it useful in solving Laplace's equation?

R.4–15 What are boundary-value problems?

R.4–16 Can all three separation constants (k_x, k_y, and k_z) in Cartesian coordinates be real? Can they all be imaginary? Explain.

R.4–17 Can the separation constant k in the solution of the two-dimensional Laplace's equation (4–120) be imaginary? Explain.

R.4–18 What should we do to modify the solution in Eq. (4–133) for Example 4–8 if the inner conductor of the coaxial cable is grounded and the outer conductor is kept at a potential V_0?

R.4–19 What should we do to modify the solution in Eq. (4–139) for Example 4–9 if the conducting circular cylinder is split vertically in two halves, with $V = V_0$ for $-\pi/2 < \phi < \pi/2$ and $V = -V_0$ for $\pi/2 < \phi < 3\pi/2$?

R.4–20 Can functions $V_1(R, \theta) = C_1 R \cos\theta$ and $V_2(R, \theta) = C_2 R^{-2} \cos\theta$, where C_1 and C_2 are arbitrary constants, be solutions of Laplace's equation in spherical coordinates? Explain.

Problems

P.4–1 The upper and lower conducting plates of a large parallel-plate capacitor are separated by a distance d and maintained at potentials V_0 and 0, respectively. A dielectric slab of dielectric constant 6.0 and uniform thickness $0.8d$ is placed over the lower plate. Assuming negligible fringing effect, determine

a) the potential and electric field distribution in the dielectric slab,

b) the potential and electric field distribution in the air space between the dielectric slab and the upper plate,

c) the surface charge densities on the upper and lower plates.

d) Compare the results in part (b) with those without the dielectric slab.

P.4–2 Prove that the scalar potential V in Eq. (3–61) satisfies Poisson's equation, Eq. (4–6).

P.4–3 Prove that a potential function satisfying Laplace's equation in a given region possesses no maximum or minimum within the region.

P.4–4 Verify that

$$V_1 = C_1/R \qquad \text{and} \qquad V_2 = C_2 z/(x^2 + y^2 + z^2)^{3/2},$$

where C_1 and C_2 are arbitrary constants, are solutions of Laplace's equation.

P.4–5 Assume a point charge Q above an infinite conducting plane at $y = 0$.

a) Prove that $V(x, y, z)$ in Eq. (4–37) satisfies Laplace's equation if the conducting plane is maintained at zero potential.

b) What should the expression for $V(x, y, z)$ be if the conducting plane has a nonzero potential V_0?

c) What is the electrostatic force of attraction between the charge Q and the conducting plane?

P.4–6 Assume that the space between the inner and outer conductors of a long coaxial cylindrical structure is filled with an electron cloud having a volume density of charge $\rho = A/r$ for $a < r < b$, where a and b are, the radii of the inner and outer conductors, respectively. The inner conductor is maintained at a potential V_0, and the outer conductor is grounded. Determine the potential distribution in the region $a < r < b$ by solving Poisson's equation.

P.4–7 A point charge Q exists at a distance d above a large grounded conducting plane. Determine

a) the surface charge density ρ_s,

b) the total charge induced on the conducting plane.

P.4–8 For a positive point charge Q located at distances d_1 and d_2, respectively, from two grounded perpendicular conducting half-planes shown in Fig. 4–4(a), find the expressions for

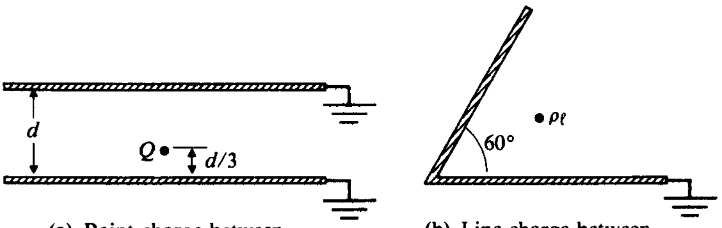
a) the potential and the electric field intensity at an arbitrary point $P(x, y)$ in the first quadrant,

b) the surface charge densities induced on the two half-planes. Sketch the variations of the surface charge densities in the xy-plane.

P.4–9 Determine the systems of image charges that will replace the conducting boundaries that are maintained at zero potential for

a) a point charge Q located between two large, grounded, parallel conducting planes as shown in Fig. 4–22(a),

b) an infinite line charge ρ_ℓ located midway between two large, intersecting conducting planes forming a 60-degree angle, as shown in Fig. 4–22(b).



(a) Point charge between grounded parallel planes.

(b) Line charge between grounded intersecting plane.

FIGURE 4–22
Diagrams for Problem P.4–9.

P.4–10 A straight conducting wire of radius a is parallel to and at height h from the surface of the earth. Assuming that the earth is perfectly conducting, determine the capacitance and the force per unit length between the wire and the earth.

P.4–11 A very long two-wire transmission line, each wire of radius a and separated by a distance d, is supported at a height h above a flat conducting ground. Assuming both d and h to be much larger than a, find the capacitance per unit length of the line.

P.4–12 For the pair of equal and opposite line charges shown in Fig. 4–7,

a) write the expression for electric field intensity **E** at point $P(x, y)$ in Cartesian coordinates,

b) find the equation of the electric field lines sketched in Fig. 4–8.

P.4–13 Determine the capacitance per unit length of a two-wire transmission line with parallel conducting cylinders of different radii a_1 and a_2, their axes being separated by a distance D (where $D > a_1 + a_2$).

P.4–14 A long wire of radius a_1 lies inside a conducting circular tunnel of radius a_2, as shown in Fig. 4–10(a). The distance between their axes is D.

a) Find the capacitance per unit length.

b) Determine the force per unit length on the wire if the wire and the tunnel carry equal and opposite line charges of magnitude ρ_ℓ.

P.4–15 A point charge Q is located inside and at distance d from the center of a grounded spherical conducting shell of radius b (where $b > d$). Use the method of images to determine

a) the potential distribution inside the shell,

b) the charge density ρ_s induced on the inner surface of the shell.

P.4–16 Two conducting spheres of equal radius a are maintained at potentials V_0 and 0, respectively. Their centers are separated by a distance D.

a) Find the image charges and their locations that can electrically replace the two spheres.

b) Find the capacitance between the two spheres.

P.4–17 Two dielectric media with dielectric constants ϵ_1 and ϵ_2 are separated by a plane boundary at $x = 0$, as shown in Fig. 4–23. A point charge Q exists in medium 1 at distance d from the boundary.

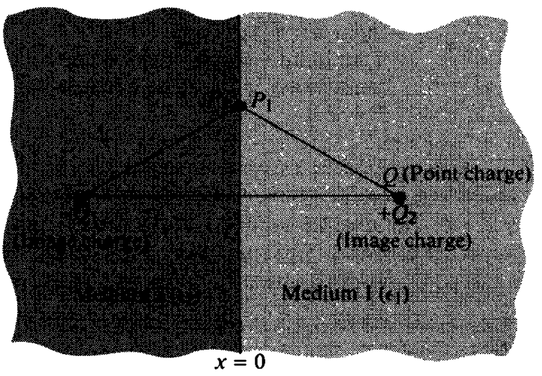


FIGURE 4–23
Image charges in dielectric media (Problem P.4–17).

a) Verify that the field in medium 1 can be obtained from Q and an image charge $-Q_1$, both acting in medium 1.
b) Verify that the field in medium 2 can be obtained from Q and an image charge $+Q_2$ coinciding with Q, both acting in medium 2.
c) Determine Q_1 and Q_2. (*Hint:* Consider neighboring points P_1 and P_2 in media 1 and 2, respectively, and require the continuity of the tangential component of the **E**-field and of the normal component of the **D**-field.)

P.4–18 Describe the geometry of the region in which the potential function can be represented by a single term as follows:

a) $V(x, y) = c_1 xy$,
b) $V(x, y) = c_2 \sin kx \sinh ky$.

Find c_1, c_2, and k in terms of the dimensions and a fixed potential V_0.

P.4–19 In what way should we modify the solution in Eq. (4–114) for Example 4–7 if the boundary conditions on the top, bottom, and right planes in Fig. 4–17 are $\partial V/\partial n = 0$?

P.4–20 In what way should we modify the solution in Eq. (4–114) for Example 4–7 if the top, bottom, and left planes in Fig. 4–17 are grounded ($V = 0$) and an end plate on the right is maintained at a constant potential V_0?

P.4–21 Consider the rectangular region shown in Fig. 4–17 as the cross section of an enclosure formed by four conducting plates. The left and right plates are grounded, and the top and bottom plates are maintained at constant potentials V_1 and V_2, respectively. Determine the potential distribution inside the enclosure.

P.4–22 Consider a metallic rectangular box with sides a and b and height c. The side walls and the bottom surface are grounded. The top surface is isolated and kept at a constant potential V_0. Determine the potential distribution inside the box.

P.4–23 Two infinite insulated conducting planes maintained at potentials 0 and V_0 form a wedge-shaped configuration, as shown in Fig. 4–24. Determine the potential distributions for the regions: (a) $0 < \phi < \alpha$, and (b) $\alpha < \phi < 2\pi$.

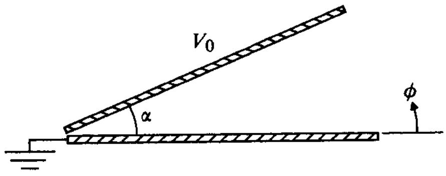


FIGURE 4–24
Two infinite insulated conducting planes maintained at constant potentials (Problem P.4–23).

P.4–24 An infinitely long, thin conducting circular cylinder of radius b is split in four quarter-cylinders, as shown in Fig. 4–25. The quarter-cylinders in the second and fourth quadrants are grounded, and those in the first and third quadrants are kept at potentials V_0 and $-V_0$, respectively. Determine the potential distribution both inside and outside the cylinder.

P.4–25 A long, grounded conducting cylinder of radius b is placed along the z-axis in an initially uniform electric field $\mathbf{E}_0 = \mathbf{a}_x E_0$. Determine potential distribution $V(r, \phi)$ and electric field intensity $\mathbf{E}(r, \phi)$ outside the cylinder. Show that the electric field intensity at the surface of the cylinder may be twice as high as that in the distance, which may cause a

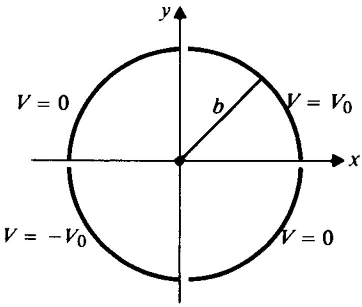


FIGURE 4–25
Cross section of long circular cylinder split in four quarters (Problem P.4–24).

local breakdown or corona. (This phenomenon of corona discharge along the rigging and spars of ships and on airplanes near storms is known as ***St. Elmo's fire.***[†])

P.4–26 A long dielectric cylinder of radius b and dielectric constant ϵ_r is placed along the z-axis in an initially uniform electric field $\mathbf{E}_0 = \mathbf{a}_x E_0$. Determine $V(r, \phi)$ and $\mathbf{E}(r, \phi)$ both inside and outside the dielectric cylinder.

P.4–27 An infinite conducting cone of half-angle α is maintained at potential V_0 and insulated from a grounded conducting plane, as illustrated in Fig. 4–26. Determine

a) the potential distribution $V(\theta)$ in the region $\alpha < \theta < \pi/2$,
b) the electric field intensity in the region $\alpha < \theta < \pi/2$,
c) the charge densities on the cone surface and on the grounded plane.

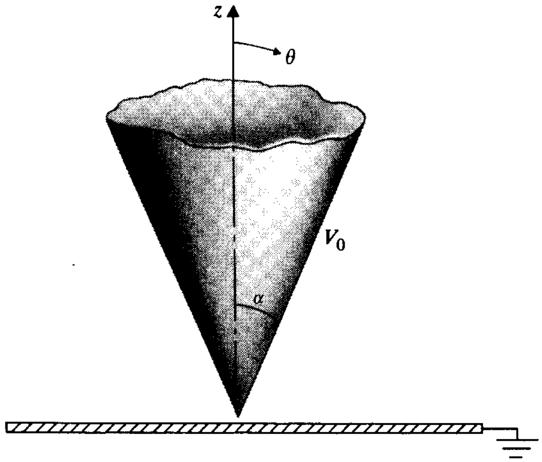


FIGURE 4–26
An infinite conducting cone and a grounded conducting plane (Problem P.4–27).

P.4–28 Rework Example 4–10, assuming that $V(b, \theta) = V_0$ in Eq. (4–155a).

P.4–29 A dielectric sphere of radius b and dielectric constant ϵ_r is placed in an initially uniform electric field, $\mathbf{E}_0 = \mathbf{a}_z E_0$, in air. Determine $V(R, \theta)$ and $\mathbf{E}(R, \theta)$ both inside and outside the dielectric sphere.

[†] R. H. Golde (Ed.), *Lightning*, Academic Press, New York, 1977, vol. 2, Chap. 21.

5

Steady Electric Currents

5–1 Introduction

In Chapters 3 and 4 we dealt with electrostatic problems, field problems associated with electric charges at rest. We now consider the charges in motion that constitute current flow. There are several types of electric currents caused by the *motion of free charges*.† ***Conduction currents*** in conductors and semiconductors are caused by drift motion of conduction electrons and/or holes; ***electrolytic currents*** are the result of migration of positive and negative ions; and ***convection currents*** result from motion of electrons and/or ions in a vacuum. In this chapter we shall pay special attention to conduction currents that are governed by Ohm's law. We will proceed from the point form of Ohm's law that relates current density and electric field intensity and obtain the $V = IR$ relationship in circuit theory. We will also introduce the concept of electromotive force and derive the familiar Kirchhoff's voltage law. Using the principle of ***conservation of charge***, we will show how to obtain a point relationship between current and charge densities, a relationship called the ***equation of continuity*** from which Kirchhoff's current law follows.

When a current flows across the interface between two media of different conductivities, certain boundary conditions must be satisfied, and the direction of current flow is changed. We will discuss these boundary conditions. We will also show that for a homogeneous conducting medium, the current density can be expressed as the gradient of a scalar field, which satisfies Laplace's equation. Hence, an analogous situation exists between steady-current and electrostatic fields that is the basis for mapping the potential distribution of an electrostatic problem in an ***electrolytic tank***.

The electrolyte in an electrolytic tank is essentially a liquid medium with a low conductivity, usually a diluted salt solution. Highly conducting metallic electrodes

† In a time-varying situation there is another type of current caused by bound charges. The time-rate of change of electric displacement leads to a ***displacement current***. This will be discussed in Chapter 7.

are inserted in the solution. When a voltage or potential difference is applied to the electrodes, an electric field is established within the solution, and the molecules of the electrolyte are decomposed into oppositely charged ions by a chemical process called ***electrolysis***. Positive ions move in the direction of the electric field, and negative ions move in a direction opposite to the field, both contributing to a current flow in the direction of the field. An experimental model can be set up in an electrolytic tank, with electrodes of proper geometrical shapes simulating the boundaries in electrostatic problems. The measured potential distribution in the electrolyte is then the solution to Laplace's equation for difficult-to-solve analytic problems having complex boundaries in a homogeneous medium.

Convection currents are the result of the motion of positively or negatively charged particles in a vacuum or rarefied gas. Familiar examples are electron beams in a cathode-ray tube and the violent motions of charged particles in a thunderstorm. Convection currents, the result of hydrodynamic motion involving a mass transport, are not governed by Ohm's law.

The mechanism of conduction currents is different from that of both electrolytic currents and convection currents. In their normal state the atoms of a conductor occupy regular positions in a crystalline structure. The atoms consist of positively charged nuclei surrounded by electrons in a shell-like arrangement. The electrons in the inner shells are tightly bound to the nuclei and are not free to move away. The electrons in the outermost shells of a conductor atom do not completely fill the shells; they are valence or conduction electrons and are only very loosely bound to the nuclei. These latter electrons may wander from one atom to another in a random manner. The atoms, on the average, remain electrically neutral, and there is no net drift motion of electrons. When an external electric field is applied on a conductor, an organized motion of the conduction electrons will result, producing an electric current. The average drift velocity of the electrons is very low (on the order of 10^{-4} or 10^{-3} m/s) even for very good conductors because they collide with the atoms in the course of their motion, dissipating part of their kinetic energy as heat. Even with the drift motion of conduction electrons, a conductor remains electrically neutral. Electric forces prevent excess electrons from accumulating at any point in a conductor. We will show analytically that the charge density in a conductor decreases exponentially with time. In a good conductor the charge density diminishes extremely rapidly toward zero as the state of equilibrium is approached.

5–2 Current Density and Ohm's Law

Consider the steady motion of one kind of charge carriers, each of charge q (which is negative for electrons), across an element of surface Δs with a velocity $\mathbf{u}$, as shown in Fig. 5–1. If N is the number of charge carriers per unit volume, then in time Δt each charge carrier moves a distance $\mathbf{u}\,\Delta t$, and the amount of charge passing through the surface Δs is

$$\Delta Q = Nq\mathbf{u} \cdot \mathbf{a}_n \,\Delta s\,\Delta t \qquad \text{(C)}. \tag{5–1}$$

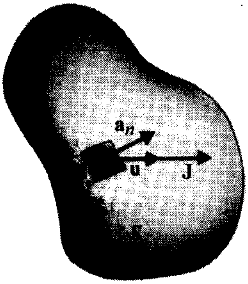


FIGURE 5–1
Conduction current due to drift motion of charge carriers across a surface.

Since current is the time rate of change of charge, we have

$$\Delta I = \frac{\Delta Q}{\Delta t} = Nq\mathbf{u} \cdot \mathbf{a}_n \Delta s = Nq\mathbf{u} \cdot \Delta\mathbf{s} \qquad \text{(A)}. \tag{5-2}$$

In Eq. (5–2) we have written $\Delta\mathbf{s} = \mathbf{a}_n \Delta s$ as a vector quantity. It is convenient to define a vector point function, ***volume current density***, or simply ***current density***, $\mathbf{J}$, in amperes per *square* meter,

$$\mathbf{J} = Nq\mathbf{u} \qquad (\text{A/m}^2), \tag{5-3}$$

so that Eq. (5–2) can be written as

$$\Delta I = \mathbf{J} \cdot \Delta\mathbf{s}. \tag{5-4}$$

The total current I flowing through an arbitrary surface S is then the flux of the $\mathbf{J}$ vector through S:

$$\boxed{I = \int_S \mathbf{J} \cdot d\mathbf{s} \qquad \text{(A)}.} \tag{5-5}$$

Noting that the product Nq is in fact free charge per unit volume, we may rewrite Eq. (5–3) as

$$\boxed{\mathbf{J} = \rho\mathbf{u} \qquad (\text{A/m}^2),} \tag{5-6}$$

which is the relation between the ***convection current density*** and the velocity of the charge carrier.

EXAMPLE 5–1 In vacuum-tube diodes, electrons are emitted from a hot cathode at zero potential and collected by an anode maintained at a potential V_0, resulting in a convection current flow. Assuming that the cathode and the anode are parallel conducting plates and that the electrons leave the cathode with a zero initial velocity (space-charge limited condition), find the relation between the current density J and V_0.

Solution The region between the cathode and the anode is shown in Fig. 5-2, where a cloud of electrons (negative space charge) exists such that the force of repulsion makes the electrons boiled off the hot cathode leave essentially with a zero velocity. In other words, the net electric field at the cathode is zero. Neglecting fringing effects, we have

$$\mathbf{E}(0) = \mathbf{a}_y E_y(0) = -\mathbf{a}_y \left.\frac{dV(y)}{dy}\right|_{y=0} = 0. \tag{5-7}$$

In the steady state the current density is constant, independent of y:

$$\mathbf{J} = -\mathbf{a}_y J = \mathbf{a}_y \rho(y) u(y), \tag{5-8}$$

where the charge density $\rho(y)$ is a negative quantity. The velocity $\mathbf{u} = \mathbf{a}_y u(y)$ is related to the electric field intensity $\mathbf{E}(y) = \mathbf{a}_y E(y)$ by Newton's law of motion:

$$m\frac{du(y)}{dt} = -eE(y) = e\frac{dV(y)}{dy}, \tag{5-9}$$

where $m = 9.11 \times 10^{-31}$ (kg) and $-e = -1.60 \times 10^{-19}$ (C) are the mass and charge, respectively, of an electron. Noting that

$$m\frac{du}{dt} = m\frac{du}{dy}\frac{dy}{dt} = mu\frac{du}{dy}$$
$$= \frac{d}{dy}\left(\frac{1}{2}mu^2\right),$$

we can rewrite Eq. (5-9) as

$$\frac{d}{dy}\left(\frac{1}{2}mu^2\right) = e\frac{dV}{dy}. \tag{5-10}$$

Integration of Eq. (5-10) gives

$$\tfrac{1}{2}mu^2 = eV, \tag{5-11}$$

where the constant of integration has been set to zero because at $y = 0$, $u(0) = V(0) = 0$. From Eq. (5-11) we obtain

$$u = \left(\frac{2e}{m}V\right)^{1/2}. \tag{5-12}$$

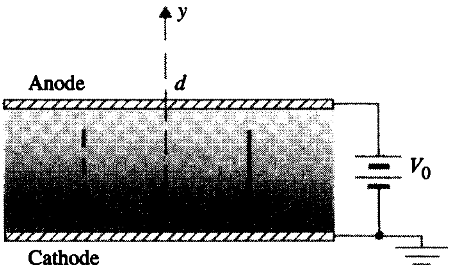


FIGURE 5-2
Space-charge-limited vacuum diode (Example 5-1).

In order to find $V(y)$ in the interelectrode region we must solve Poisson's equation with ρ expressed in terms of $V(y)$ from Eq. (5–8):

$$\rho = -\frac{J}{u} = -J\sqrt{\frac{m}{2e}}\,V^{-1/2}. \tag{5–13}$$

We have, from Eq. (4–6),

$$\frac{d^2V}{dy^2} = -\frac{\rho}{\epsilon_0} = \frac{J}{\epsilon_0}\sqrt{\frac{m}{2e}}\,V^{-1/2}. \tag{5–14}$$

Equation (5–14) can be integrated if both sides are first multiplied by $2\,dV/dy$. The result is

$$\left(\frac{dV}{dy}\right)^2 = \frac{4J}{\epsilon_0}\sqrt{\frac{m}{2e}}\,V^{1/2} + c. \tag{5–15}$$

At $y = 0$, $V = 0$, and $dV/dy = 0$ from Eq. (5–7), so $c = 0$. Equation (5–15) becomes

$$V^{-1/4}\,dV = 2\sqrt{\frac{J}{\epsilon_0}}\left(\frac{m}{2e}\right)^{1/4} dy. \tag{5–16}$$

Integrating the left side of Eq. (5–16) from $V = 0$ to V_0 and the right side from $y = 0$ to d, we obtain

$$\frac{4}{3}\,V_0^{3/4} = 2\sqrt{\frac{J}{\epsilon_0}}\left(\frac{m}{2e}\right)^{1/4} d,$$

or

$$J = \frac{4\epsilon_0}{9d^2}\sqrt{\frac{2e}{m}}\,V_0^{3/2} \qquad (\text{A/m}^2). \tag{5–17}$$

Equation (5–17) states that the convection current density in a space-charge limited vacuum diode is proportional to the three-halves power of the potential difference between the anode and the cathode. This nonlinear relation is known as the ***Child-Langmuir law.*** ■

In the case of conduction currents there may be more than one kind of charge carriers (electrons, holes, and ions) drifting with different velocities. Equation (5–3) should be generalized to read

$$\mathbf{J} = \sum_i N_i q_i \mathbf{u}_i \qquad (\text{A/m}^2). \tag{5–18}$$

As indicated in Section 5–1, conduction currents are the result of the drift motion of charge carriers under the influence of an applied electric field. The atoms remain neutral ($\rho = 0$). It can be justified analytically that for most conducting materials the average drift velocity is directly proportional to the electric field intensity. For metallic conductors we write

$$\mathbf{u} = -\mu_e \mathbf{E} \qquad (\text{m/s}), \tag{5–19}$$

where μ_e is the electron ***mobility*** measured in ($\text{m}^2/\text{V}\cdot\text{s}$). The electron mobility for copper is 3.2×10^{-3} ($\text{m}^2/\text{V}\cdot\text{s}$). It is 1.4×10^{-4} ($\text{m}^2/\text{V}\cdot\text{s}$) for aluminum and 5.2×10^{-3}

(m^2/V·s) for silver. From Eqs. (5–3) and (5–19) we have

$$\mathbf{J} = -\rho_e \mu_e \mathbf{E}, \tag{5–20}$$

where $\rho_e = -Ne$ is the charge density of the drifting electrons and is a negative quantity. Equation (5–20) can be rewritten as

$$\boxed{\mathbf{J} = \sigma \mathbf{E} \qquad (\text{A/m}^2),} \tag{5–21}$$

where the proportionality constant, $\sigma = -\rho_e \mu_e$, is a macroscopic constitutive parameter of the medium called ***conductivity***.

For semiconductors, conductivity depends on the concentration and mobility of both electrons and holes:

$$\sigma = -\rho_e \mu_e + \rho_h \mu_h, \tag{5–22}$$

where the subscript h denotes hole. In general, $\mu_e \neq \mu_h$. For germanium, typical values are $\mu_e = 0.38$, $\mu_h = 0.18$; for silicon, $\mu_e = 0.12$, $\mu_h = 0.03$ (m^2/V·s).

Equation (5–21) is a constitutive relation of a conducting medium. Isotropic materials for which the linear relation Eq. (5–21) holds are called ***ohmic media***. The unit for σ is ampere per volt-meter (A/V·m) or siemens per meter (S/m). Copper, the most commonly used conductor, has a conductivity 5.80×10^7 (S/m). On the other hand, the conductivity of germanium is around 2.2 (S/m), and that of silicon is 1.6×10^{-3} (S/m). The conductivity of semiconductors is highly dependent of (increases with) temperature. Hard rubber, a good insulator, has a conductivity of only 10^{-15} (S/m). Appendix B–4 lists the conductivities of some other frequently used materials. However, note that, unlike the dielectric constant, the conductivity of materials varies over an extremely wide range. The reciprocal of conductivity is called ***resistivity***, in ohm-meters (Ω·m). We prefer to use conductivity; there is really no compelling need to use both conductivity and resistivity.

We recall ***Ohm's law*** from circuit theory that the voltage V_{12} across a resistance R, in which a current I flows from point 1 to point 2, is equal to RI; that is,

$$V_{12} = RI. \tag{5–23}$$

Here R is usually a piece of conducting material of a given length; V_{12} is the voltage between two terminals 1 and 2; and I is the total current flowing from terminal 1 to terminal 2 through a finite cross section.

Equation (5–23) is *not* a point relation. Although there is little resemblance between Eq. (5–21) and Eq. (5–23), the former is generally referred to as the ***point form of Ohm's law***. It holds at all points in space, and σ can be a function of space coordinates.

Let us use the point form of Ohm's law to derive the voltage-current relationship of a piece of homogeneous material of conductivity σ, length ℓ, and uniform cross section S, as shown in Fig. 5–3. Within the conducting material, $\mathbf{J} = \sigma\mathbf{E}$, where both $\mathbf{J}$ and $\mathbf{E}$ are in the direction of current flow. The potential difference or voltage

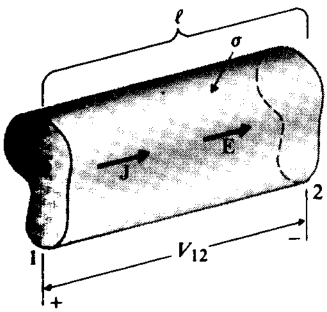


FIGURE 5-3
Homogeneous conductor with a constant cross section.

between terminals 1 and 2 is†

$$V_{12} = E\ell$$

or

$$E = \frac{V_{12}}{\ell}. \tag{5-24}$$

The total current is

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} = JS$$

or

$$J = \frac{I}{S}. \tag{5-25}$$

Using Eqs. (5–24) and (5–25) in Eq. (5–21), we obtain

$$\frac{I}{S} = \sigma \frac{V_{12}}{\ell}$$

or

$$V_{12} = \left(\frac{\ell}{\sigma S}\right) I = RI, \tag{5-26}$$

which is the same as Eq. (5–23). From Eq. (5–26) we have the formula for the ***resistance*** of a straight piece of homogeneous material of a uniform cross section for steady current (d.c.):

$$\boxed{R = \frac{\ell}{\sigma S} \quad (\Omega).} \tag{5-27}$$

We could have started with Eq. (5–23) as the experimental Ohm's law and applied it to a homogeneous conductor of length ℓ and uniform cross-section S. Using the formula in Eq. (5–27), we could derive the point relationship in Eq. (5–21).

† We will discuss the significance of V_{12} and E more in detail in Section 5–3.

EXAMPLE 5–2 Determine the d-c resistance of 1-(km) of wire having a 1-(mm) radius (a) if the wire is made of copper, and (b) if the wire is made of aluminum.

Solution Since we are dealing with conductors of a uniform cross section, Eq. (5–27) applies.

a) For copper wire, $\sigma_{cu} = 5.80 \times 10^7$ (S/m):

$$\ell = 10^3 \text{ (m)}, \qquad S = \pi(10^{-3})^2 = 10^{-6}\pi \quad (\text{m}^2).$$

We have

$$R_{cu} = \frac{\ell}{\sigma_{cu}S} = \frac{10^3}{5.80 \times 10^7 \times 10^{-6}\pi} = 5.49 \quad (\Omega).$$

b) For aluminum wire, $\sigma_{al} = 3.54 \times 10^7$ (S/m):

$$R_{al} = \frac{\ell}{\sigma_{al}S} = \frac{\sigma_{cu}}{\sigma_{al}} R_{cu} = \frac{5.80}{3.54} \times 5.49 = 8.99 \quad (\Omega).$$

The ***conductance***, G, or the reciprocal of resistance, is useful in combining resistances in parallel. The unit for conductance is (Ω^{-1}), or siemens (S).

$$G = \frac{1}{R} = \sigma \frac{S}{\ell} \quad (\text{S}). \tag{5–28}$$

From circuit theory we know the following:

a) When resistances R_1 and R_2 are connected in series (same current), the total resistance R is

$$\boxed{R_{sr} = R_1 + R_2.} \tag{5–29}$$

b) When resistances R_1 and R_2 are connected in parallel (same voltage), we have

$$\boxed{\frac{1}{R_{||}} = \frac{1}{R_1} + \frac{1}{R_2}} \tag{5–30a}$$

or

$$\boxed{G_{||} = G_1 + G_2.} \tag{5–30b}$$

5–3 Electromotive Force and Kirchhoff's Voltage Law

In Section 3–2 we pointed out that static electric field is conservative and that the scalar line integral of static electric intensity around any closed path is zero; that is,

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0. \tag{5–31}$$

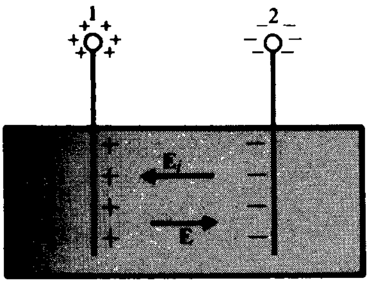


FIGURE 5–4
Electric fields inside an electric battery.

For an ohmic material $\mathbf{J} = \sigma\mathbf{E}$, Eq. (5–31) becomes

$$\oint_C \frac{1}{\sigma}\mathbf{J} \cdot d\boldsymbol{\ell} = 0. \tag{5–32}$$

Equation (5–32) tells us that ***a steady current cannot be maintained in the same direction in a closed circuit by an electrostatic field.*** A steady current in a circuit is the result of the motion of charge carriers, which, in their paths, collide with atoms and dissipate energy in the circuit. This energy must come from a nonconservative field, since a charge carrier completing a closed circuit in a conservative field neither gains nor loses energy. The source of the nonconservative field may be electric batteries (conversion of chemical energy to electric energy), electric generators (conversion of mechanical energy to electric energy), thermocouples (conversion of thermal energy to electric energy), photovoltaic cells (conversion of light energy to electric energy), or other devices. These electrical energy sources, when connected in an electric circuit, provide a driving force for the charge carriers. This force manifests itself as an equivalent ***impressed electric field intensity*** $\mathbf{E}_i$.

Consider an electric battery with electrodes 1 and 2, shown schematically in Fig. 5–4. Chemical action creates a cumulation of positive and negative charges at electrodes 1 and 2, respectively. These charges give rise to an electrostatic field intensity $\mathbf{E}$ both outside and inside the battery. Inside the battery, $\mathbf{E}$ must be equal in magnitude and opposite in direction to the nonconservative $\mathbf{E}_i$ produced by chemical action, since no current flows in the open-circuited battery and the net force acting on the charge carriers must vanish. The line integral of the impressed field intensity $\mathbf{E}_i$ from the negative to the positive electrode (from electrode 2 to electrode 1 in Fig. 5–4) inside the battery is customarily called the ***electromotive force***† (emf) of the battery. The SI unit for emf is volt, and an emf is *not* a force in newtons. Denoted by $\mathscr{V}$, the electromotive force is a measure of the strength of the nonconservative source. We have

$$\mathscr{V} = \int_2^1 \mathbf{E}_i \cdot d\boldsymbol{\ell} = -\underset{\substack{\text{Inside}\\ \text{the source}}}{\int_2^1} \mathbf{E} \cdot d\boldsymbol{\ell}. \tag{5–33}$$

† Also called ***electromotance***.

The conservative electrostatic field intensity **E** satisfies Eq. (5–31):

$$\oint_C \mathbf{E}\cdot d\boldsymbol{\ell} = \underset{\substack{\text{Outside}\\ \text{the source}}}{\int_1^2 \mathbf{E}\cdot d\boldsymbol{\ell}} + \underset{\substack{\text{Inside}\\ \text{the source}}}{\int_2^1 \mathbf{E}\cdot d\boldsymbol{\ell}} = 0. \tag{5-34}$$

Combining Eqs. (5–33) and (5–34), we have

$$\mathscr{V} = \underset{\substack{\text{Outside}\\ \text{the source}}}{\int_1^2 \mathbf{E}\cdot d\boldsymbol{\ell}} \tag{5-35}$$

or

$$\mathscr{V} = V_{12} = V_1 - V_2. \tag{5-36}$$

In Eqs. (5–35) and (5–36) we have expressed the emf of the source as a line integral of the conservative **E** and interpreted it as a ***voltage rise***. In spite of the nonconservative nature of $\mathbf{E}_i$, the emf can be expressed as a potential difference between the positive and negative terminals. This was what we did in arriving at Eq. (5–24).

When a resistor in the form of Fig. 5–3 is connected between terminals 1 and 2 of the battery, completing the circuit, the *total* electric field intensity (electrostatic **E** caused by charge cumulation, as well as impressed $\mathbf{E}_i$ caused by chemical action), must be used in the point form of Ohm's law. We have, instead of Eq. (5–21),

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}_i), \tag{5-37}$$

where $\mathbf{E}_i$ exists inside the battery only, while **E** has a nonzero value both inside and outside the source. From Eq. (5–37) we obtain

$$\mathbf{E} + \mathbf{E}_i = \frac{\mathbf{J}}{\sigma}. \tag{5-38}$$

The scalar line integral of Eq. (5–38) around the closed circuit yields, in view of Eqs. (5–31) and (5–33),

$$\mathscr{V} = \oint_C (\mathbf{E} + \mathbf{E}_i)\cdot d\boldsymbol{\ell} = \oint_C \frac{1}{\sigma}\mathbf{J}\cdot d\boldsymbol{\ell}. \tag{5-39}$$

Equation (5–39) should be compared to Eq. (5–32), which holds when there is no source of nonconservative field. If the resistor has a conductivity σ, length ℓ, and uniform cross section S, $J = I/S$ and the right side of Eq. (5–39) becomes RI. We have†

$$\mathscr{V} = RI. \tag{5-40}$$

If there are more than one source of electromotive force and more than one resistor (including the internal resistances of the sources) in the closed path, we generalize

† We assume the battery to have a negligible internal resistance; otherwise, its effect must be included in Eq. (5–40). An ***ideal voltage source*** is one whose terminal voltage is equal to its emf and is independent of the current flowing through it. This implies that an ideal voltage source has a zero internal resistance.

Eq. (5–40) to

$$\boxed{\sum_j \mathscr{V}_j = \sum_k R_k I_k \qquad \text{(V)}.} \tag{5–41}$$

Equation (5–41) is an expression of ***Kirchhoff's voltage law***. It states that, ***around a closed path in an electric circuit, the algebraic sum of the emf's (voltage rises) is equal to the algebraic sum of the voltage drops across the resistances.*** It applies to *any closed path* in a network. The direction of tracing the path can be arbitrarily assigned, and the currents in the different resistances need not be the same. Kirchhoff's voltage law is the basis for loop analysis in circuit theory.

5–4 Equation of Continuity and Kirchhoff's Current Law

The ***principle of conservation of charge*** is one of the fundamental postulates of physics. Electric charges may not be created or destroyed; all charges either at rest or in motion must be accounted for at all times. Consider an arbitrary volume V bounded by surface S. A net charge Q exists within this region. If a net current I flows across the surface *out* of this region, the charge in the volume must *decrease* at a rate that equals the current. Conversely, if a net current flows across the surface *into* the region, the charge in the volume must *increase* at a rate equal to the current. The current leaving the region is the total outward flux of the current density vector through the surface S. We have

$$I = \oint_S \mathbf{J} \cdot d\mathbf{s} = -\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho\, dv. \tag{5–42}$$

Divergence theorem, Eq. (2–115), may be invoked to convert the surface integral of $\mathbf{J}$ to the volume integral of $\nabla \cdot \mathbf{J}$. We obtain, for a stationary volume,

$$\int_V \nabla \cdot \mathbf{J}\, dv = -\int_V \frac{\partial \rho}{\partial t}\, dv. \tag{5–43}$$

In moving the time derivative of ρ inside the volume integral, it is necessary to use partial differentiation because ρ may be a function of time as well as of space coordinates. Since Eq. (5–43) must hold regardless of the choice of V, the integrands must be equal. Thus we have

$$\boxed{\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \qquad (\text{A/m}^3).} \tag{5–44}$$

This point relationship derived from the principle of conservation of charge is called the ***equation of continuity***.

For steady currents, charge density does not vary with time, $\partial\rho/\partial t = 0$. Equation (5–44) becomes

$$\nabla \cdot \mathbf{J} = 0. \tag{5–45}$$

Thus, steady electric currents are divergenceless or solenoidal. Equation (5–45) is a point relationship and holds also at points where $\rho = 0$ (no flow source). It means that the field lines or streamlines of steady currents close upon themselves, unlike those of electrostatic field intensity that originate and end on charges. Over any enclosed surface, Eq. (5–45) leads to the following integral form:

$$\oint_S \mathbf{J} \cdot d\mathbf{s} = 0, \tag{5–46}$$

which can be written as

$$\boxed{\sum_j I_j = 0 \qquad \text{(A).}} \tag{5–47}$$

Equation (5–47) is an expression of ***Kirchhoff's current law***. It states that ***the algebraic sum of all the currents flowing out of a junction in an electric circuit is zero.***† Kirchhoff's current law is the basis for node analysis in circuit theory.

In Section 3–6, we stated that charges introduced in the interior of a conductor will move to the conductor surface and redistribute themselves in such a way as to make $\rho = 0$ and $\mathbf{E} = 0$ inside under equilibrium conditions. We are now in a position to prove this statement and to calculate the time it takes to reach an equilibrium. Combining Ohm's law, Eq. (5–21), with the equation of continuity and assuming a constant σ, we have

$$\sigma \nabla \cdot \mathbf{E} = -\frac{\partial \rho}{\partial t}. \tag{5–48}$$

In a simple medium, $\nabla \cdot \mathbf{E} = \rho/\epsilon$, and Eq. (5–48) becomes

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon}\rho = 0. \tag{5–49}$$

The solution of Eq. (5–49) is

$$\boxed{\rho = \rho_0 e^{-(\sigma/\epsilon)t} \qquad (\text{C/m}^3),} \tag{5–50}$$

where ρ_0 is the initial charge density at $t = 0$. Both ρ and ρ_0 can be functions of the space coordinates, and Eq. (5–50) says that the charge density at a given location will decrease with time exponentially. An initial charge density ρ_0 will decay to $1/e$

† This includes the currents of current generators at the junction, if any. An ***ideal current generator*** is one whose current is independent of its terminal voltage. This implies that an ideal current source has an infinite internal resistance.

or 36.8% of its value in a time equal to

$$\tau = \frac{\epsilon}{\sigma} \quad \text{(s)}. \tag{5-51}$$

The time constant τ is called the ***relaxation time***. For a good conductor such as copper—$\sigma = 5.80 \times 10^7$ (S/m), $\epsilon \cong \epsilon_0 = 8.85 \times 10^{-12}$ (F/m)—τ equals 1.52×10^{-19} (s), a very short time indeed. The transient time is so brief that, for all practical purposes, ρ can be considered zero in the interior of a conductor—see Eq. (3–69) in Section 3–6. The relaxation time for a good insulator is not infinite but can be hours or days.

5–5 Power Dissipation and Joule's Law

In Section 5–1 we indicated that under the influence of an electric field, conduction electrons in a conductor undergo a drift motion macroscopically. Microscopically, these electrons collide with atoms on lattice sites. Energy is thus transmitted from the electric field to the atoms in thermal vibration. The work Δw done by an electric field $\mathbf{E}$ in moving a charge q a distance $\Delta \ell$ is $q\mathbf{E} \cdot (\Delta \ell)$, which corresponds to a power

$$p = \lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t} = q\mathbf{E} \cdot \mathbf{u}, \tag{5-52}$$

where $\mathbf{u}$ is the drift velocity. The total power delivered to all the charge carriers in a volume dv is

$$dP = \sum_i p_i = \mathbf{E} \cdot \left(\sum_i N_i q_i \mathbf{u}_i \right) dv,$$

which, by virtue of Eq. (5–18), is

$$dP = \mathbf{E} \cdot \mathbf{J}\, dv$$

or

$$\frac{dP}{dv} = \mathbf{E} \cdot \mathbf{J} \quad (\text{W/m}^3). \tag{5-53}$$

Thus the point function $\mathbf{E} \cdot \mathbf{J}$ is a ***power density*** under steady-current conditions. For a given volume V the total electric power converted into heat is

$$\boxed{P = \int_V \mathbf{E} \cdot \mathbf{J}\, dv \quad \text{(W)}.} \tag{5-54}$$

This is known as ***Joule's law***. (Note that the SI unit for P is watt, not joule, which is the unit for energy or work.) Equation (5–53) is the corresponding point relationship.

In a conductor of a constant cross section, $dv = ds\, d\ell$, with $d\ell$ measured in the direction $\mathbf{J}$. Equation (5–54) can be written as

$$P = \int_L E\, d\ell \int_S J\, ds = VI,$$

where I is the current in the conductor. Since $V = RI$, we have

$$\boxed{P = I^2 R \qquad \text{(W)}.} \tag{5–55}$$

Equation (5–55) is, of course, the familiar expression for ohmic power representing the heat dissipated in resistance R per unit time.

5–6 Boundary Conditions for Current Density

When current obliquely crosses an interface between two media with different conductivities, the current density vector changes both in direction and in magnitude. A set of boundary conditions can be derived for **J** in a way similar to that used in Section 3–9 for obtaining the boundary conditions for **D** and **E**. The governing equations for steady current density **J** in the absence of nonconservative energy sources are

Governing Equations for Steady Current Density		
Differential Form	Integral Form	
$\nabla \cdot \mathbf{J} = 0$	$\oint_S \mathbf{J} \cdot d\mathbf{s} = 0$	(5–56)
$\nabla \times \left(\dfrac{\mathbf{J}}{\sigma}\right) = 0$	$\oint_C \dfrac{1}{\sigma} \mathbf{J} \cdot d\boldsymbol{\ell} = 0$	(5–57)

The divergence equation is the same as Eq. (5–45), and the curl equation is obtained by combining Ohm's law ($\mathbf{J} = \sigma\mathbf{E}$) with $\nabla \times \mathbf{E} = 0$. By applying Eqs. (5–56) and (5–57) at the interface between two ohmic media with conductivities σ_1 and σ_2, we obtain the boundary conditions for the normal and tangential components of **J**.

Without actually constructing a pillbox at the interface as was done in Fig. 3–23, we know from Section 3–9 that ***the normal component of a divergenceless vector field is continuous.*** Hence from $\nabla \cdot \mathbf{J} = 0$ we have

$$\boxed{J_{1n} = J_{2n} \qquad (\text{A/m}^2).} \tag{5–58}$$

Similarly, ***the tangential component of a curl-free vector field is continuous across an interface.*** We conclude from $\nabla \times (\mathbf{J}/\sigma) = 0$ that

$$\boxed{\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}.} \tag{5–59}$$

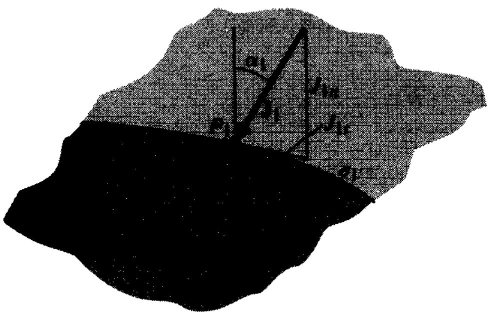


FIGURE 5–5
Boundary conditions at interface between two conducting media (Example 5–3).

Equation (5–59) states that ***the ratio of the tangential components of* J *at two sides of an interface is equal to the ratio of the conductivities.*** Comparing the boundary conditions Eqs. (5–58) and (5–59) for steady current density in ohmic media with the boundary conditions Eqs. (3–123) and (3–119), respectively, for electrostatic flux density at an interface of dielectric media where there are no free charges, we note an exact analogy of **J** and σ with **D** and ϵ.

EXAMPLE 5–3 Two conducting media with conductivities σ_1 and σ_2 are separated by an interface, as shown in Fig. 5–5. The steady current density in medium 1 at point P_1 has a magnitude J_1 and makes an angle α_1 with the normal. Determine the magnitude and direction of the current density at point P_2 in medium 2.

Solution Using Eqs. (5–58) and (5–59), we have

$$J_1 \cos \alpha_1 = J_2 \cos \alpha_2 \tag{5–60}$$

and

$$\sigma_2 J_1 \sin \alpha_1 = \sigma_1 J_2 \sin \alpha_2. \tag{5–61}$$

Division of Eq. (5–61) by Eq. (5–60) yields

$$\boxed{\frac{\tan \alpha_2}{\tan \alpha_1} = \frac{\sigma_2}{\sigma_1}.} \tag{5–62}$$

If medium 1 is a much better conductor than medium 2 ($\sigma_1 \gg \sigma_2$ or $\sigma_2/\sigma_1 \to 0$), α_2 approaches zero, and $\mathbf{J}_2$ emerges almost perpendicularly to the interface (normal to the surface of the good conductor). The magnitude of $\mathbf{J}_2$ is

$$\begin{aligned} J_2 &= \sqrt{J_{2t}^2 + J_{2n}^2} = \sqrt{(J_2 \sin \alpha_2)^2 + (J_2 \cos \alpha_2)^2} \\ &= \left[\left(\frac{\sigma_2}{\sigma_1} J_1 \sin \alpha_1\right)^2 + (J_1 \cos \alpha_1)^2\right]^{1/2} \end{aligned}$$

or

$$\boxed{J_2 = J_1\left[\left(\frac{\sigma_2}{\sigma_1}\sin\alpha_1\right)^2 + \cos^2\alpha_1\right]^{1/2}.} \tag{5–63}$$

By examining Fig. 5–5, can you tell whether medium 1 or medium 2 is the better conductor? ■

For a homogeneous conducting medium the differential form of Eq. (5–57) simplifies to

$$\nabla \times \mathbf{J} = 0. \tag{5–64}$$

From Section 2–11 we know that a curl-free vector field can be expressed as the gradient of a scalar potential field. Let us write

$$\mathbf{J} = -\nabla\psi. \tag{5–65}$$

Substitution of Eq. (5–65) into $\nabla \cdot \mathbf{J} = 0$ yields a Laplace's equation in ψ; that is,

$$\nabla^2\psi = 0. \tag{5–66}$$

A problem in steady-current flow can therefore be solved by determining ψ (A/m) from Eq. (5–66), subject to appropriate boundary conditions and then by finding **J** from its negative gradient in exactly the same way as a problem in electrostatics is solved. As a matter of fact, ψ and electrostatic potential are simply related: $\psi = \sigma V$. As indicated in Section 5–1, this similarity between electrostatic and steady-current fields is the basis for using an electrolytic tank to map the potential distribution of difficult-to-solve electrostatic boundary-value problems.†

When a steady current flows across the boundary between two different lossy dielectrics (dielectrics with permittivities ϵ_1 and ϵ_2 and finite conductivities σ_1 and σ_2), the tangential component of the electric field is continuous across the interface as usual; that is, $E_{2t} = E_{1t}$, which is equivalent to Eq. (5–59). The normal component of the electric field, however, must simultaneously satisfy both Eq. (5–58) and Eq. (3–121b). We require

$$J_{1n} = J_{2n} \rightarrow \sigma_1 E_{1n} = \sigma_2 E_{2n} \tag{5–67}$$

$$D_{1n} - D_{2n} = \rho_s \rightarrow \epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s, \tag{5–68}$$

where the reference unit normal is *outward from medium 2*. Hence, unless $\sigma_2/\sigma_1 = \epsilon_2/\epsilon_1$, a surface charge must exist at the interface. From Eqs. (5–67) and (5–68) we find

$$\rho_s = \left(\epsilon_1 \frac{\sigma_2}{\sigma_1} - \epsilon_2\right)E_{2n} = \left(\epsilon_1 - \epsilon_2 \frac{\sigma_1}{\sigma_2}\right)E_{1n}. \tag{5–69}$$

† See, for instance, E. Weber, *Electromagnetic Fields*, Vol. I: *Mapping of Fields*, pp. 187–193, John Wiley and Sons, 1950.

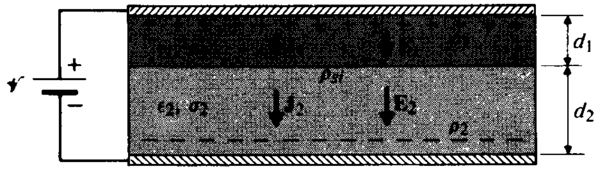


FIGURE 5–6
Parallel-plate capacitor with two lossy dielectrics (Example 5–4).

Again, if medium 2 is a much better conductor than medium 1 ($\sigma_2 \gg \sigma_1$ or $\sigma_1/\sigma_2 \to 0$), Eq. (5–69) becomes approximately

$$\rho_s = \epsilon_1 E_{1n} = D_{1n}, \tag{5–70}$$

which is the same as Eq. (3–122).

EXAMPLE 5–4 An emf $\mathscr{V}$ is applied across a parallel-plate capacitor of area S. The space between the conducting plates is filled with two different lossy dielectrics of thicknesses d_1 and d_2, permittivities ϵ_1 and ϵ_2, and conductivities σ_1 and σ_2, respectively. Determine (a) the current density between the plates, (b) the electric field intensities in both dielectrics, and (c) the surface charge densities on the plates and at the interface.

Solution Refer to Fig. 5–6.

a) The continuity of the normal component of **J** assures that the current densities and therefore the currents in both media are the same. By Kirchhoff's voltage law we have

$$\mathscr{V} = (R_1 + R_2)I = \left(\frac{d_1}{\sigma_1 S} + \frac{d_2}{\sigma_2 S}\right)I.$$

Hence,

$$J = \frac{I}{S} = \frac{\mathscr{V}}{(d_1/\sigma_1) + (d_2/\sigma_2)} = \frac{\sigma_1\sigma_2\mathscr{V}}{\sigma_2 d_1 + \sigma_1 d_2} \quad (\text{A/m}^2). \tag{5–71}$$

b) To determine the electric field intensities E_1 and E_2 in both media, two equations are needed. Neglecting fringing effect at the edges of the plates, we have

$$\mathscr{V} = E_1 d_1 + E_2 d_2 \tag{5–72}$$

and

$$\sigma_1 E_1 = \sigma_2 E_2. \tag{5–73}$$

Equation (5–73) comes from $J_1 = J_2$. Solving Eqs. (5–72) and (5–73), we obtain

$$E_1 = \frac{\sigma_2\mathscr{V}}{\sigma_2 d_1 + \sigma_1 d_2} \quad (\text{V/m}) \tag{5–74}$$

and

$$E_2 = \frac{\sigma_1\mathscr{V}}{\sigma_2 d_1 + \sigma_1 d_2} \quad (\text{V/m}). \tag{5–75}$$

c) The surface charge densities on the upper and lower plates can be determined by using Eq. (5–70):

$$\rho_{s1} = \epsilon_1 E_1 = \frac{\epsilon_1 \sigma_2 \mathscr{V}}{\sigma_2 d_1 + \sigma_1 d_2} \quad (\text{C/m}^2) \tag{5-76}$$

$$\rho_{s2} = -\epsilon_2 E_2 = -\frac{\epsilon_2 \sigma_1 \mathscr{V}}{\sigma_2 d_1 + \sigma_1 d_2} \quad (\text{C/m}^2). \tag{5-77}$$

The negative sign in Eq. (5–77) comes about because $\mathbf{E}_2$ and the *outward* normal at the lower plate are in opposite directions.

Equation (5–69) can be used to find the surface charge density at the interface of the dielectrics. We have

$$\begin{aligned} \rho_{si} &= \left(\epsilon_2 \frac{\sigma_1}{\sigma_2} - \epsilon_1\right) \frac{\sigma_2 \mathscr{V}}{\sigma_2 d_1 + \sigma_1 d_2} \\ &= \frac{(\epsilon_2 \sigma_1 - \epsilon_1 \sigma_2)\mathscr{V}}{\sigma_2 d_1 + \sigma_1 d_2} \quad (\text{C/m}^2). \end{aligned} \tag{5-78}$$

From these results we see that $\rho_{s2} \neq -\rho_{s1}$ but that $\rho_{s1} + \rho_{s2} + \rho_{si} = 0$. ■

In Example 5–4 we encounter a situation in which both static charges and a steady current exist. As we shall see in Chapter 6, a steady current gives rise to a steady magnetic field. We have, then, both a static electric field and a steady magnetic field. They constitute an ***electromagnetostatic field.*** The electric and magnetic fields of an electromagnetostatic field are coupled through the constitutive relation $\mathbf{J} = \sigma \mathbf{E}$ of the conducting medium.

5–7 Resistance Calculations

In Section 3–10 we discussed the procedure for finding the capacitance between two conductors separated by a dielectric medium. These conductors may be of arbitrary shapes, as was shown in Fig. 3–27, which is reproduced here as Fig. 5–7. In terms of electric field quantities the basic formula for capacitance can be written as

$$C = \frac{Q}{V} = \frac{\oint_S \mathbf{D} \cdot d\mathbf{s}}{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}} = \frac{\oint_S \epsilon \mathbf{E} \cdot d\mathbf{s}}{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}}, \tag{5-79}$$

where the surface integral in the numerator is carried out over a surface enclosing the positive conductor and the line integral in the denominator is from the negative (lower-potential) conductor to the positive (higher-potential) conductor (see Eq. 5–35).

When the dielectric medium is lossy (having a small but nonzero conductivity), a current will flow from the positive to the negative conductor, and a current-density field will be established in the medium. Ohm's law, $\mathbf{J} = \sigma \mathbf{E}$, ensures that the streamlines for $\mathbf{J}$ and $\mathbf{E}$ will be the same in an isotropic medium. The resistance between

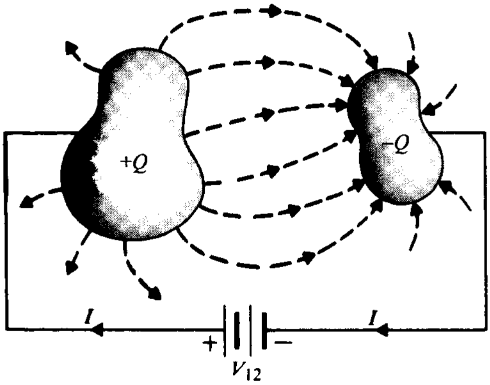


FIGURE 5–7
Two conductors in a lossy dielectric medium.

the conductors is

$$R = \frac{V}{I} = \frac{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}}{\oint_S \mathbf{J} \cdot d\mathbf{s}} = \frac{-\int_L \mathbf{E} \cdot d\boldsymbol{\ell}}{\oint_S \sigma\mathbf{E} \cdot d\mathbf{s}}, \tag{5-80}$$

where the line and surface integrals are taken over the same L and S as those in Eq. (5–79). Comparison of Eqs. (5–79) and (5–80) shows the following interesting relationship:

$$\boxed{RC = \frac{C}{G} = \frac{\epsilon}{\sigma}.} \tag{5-81}$$

Equation (5–81) holds if ϵ and σ of the medium have the same space dependence or if the medium is homogeneous (independent of space coordinates). In these cases, if the capacitance between two conductors is known, the resistance (or conductance) can be obtained directly from the ϵ/σ ratio without recomputation.

EXAMPLE 5–5 Find the leakage resistance per unit length (a) between the inner and outer conductors of a coaxial cable that has an inner conductor of radius a, an outer conductor of inner radius b, and a medium with conductivity σ, and (b) of a parallel-wire transmission line consisting of wires of radius a separated by a distance D in a medium with conductivity σ.

Solution

a) The capacitance per unit length of a coaxial cable has been obtained as Eq. (3–139) in Example 3–18:

$$C_1 = \frac{2\pi\epsilon}{\ln(b/a)} \quad \text{(F/m).}$$

Hence the leakage resistance per unit length is, from Eq. (5–81),

$$R_1 = \frac{\epsilon}{\sigma}\left(\frac{1}{C_1}\right) = \frac{1}{2\pi\sigma}\ln\left(\frac{b}{a}\right) \qquad (\Omega\cdot\text{m}). \tag{5–82}$$

The conductance per unit length is $G_1 = 1/R_1$.

b) For the parallel-wire transmission line, Eq. (4–47) in Example 4–4 gives the capacitance per unit length:

$$C_1' = \frac{\pi\epsilon}{\cosh^{-1}\left(\dfrac{D}{2a}\right)} \qquad (\text{F/m}).$$

Therefore the leakage resistance per unit length is, without further ado,

$$\begin{aligned} R_1' &= \frac{\epsilon}{\sigma}\left(\frac{1}{C_1'}\right) = \frac{1}{\pi\sigma}\cosh^{-1}\left(\frac{D}{2a}\right) \\ &= \frac{1}{\pi\sigma}\ln\left[\frac{D}{2a} + \sqrt{\left(\frac{D}{2a}\right)^2 - 1}\right] \qquad (\Omega\cdot\text{m}). \end{aligned} \tag{5–83}$$

The conductance per unit length is $G_1' = 1/R_1'$. ▬

It must be emphasized here that the resistance *between* the conductors for a length ℓ of the coaxial cable is R_1/ℓ, not ℓR_1; similarly, the leakage resistance of a length ℓ of the parallel-wire transmission line is R_1'/ℓ, not $\ell R_1'$. Do you know why?

In certain situations, electrostatic and steady-current problems are not exactly analogous, even when the geometrical configurations are the same. This is because current flow can be confined strictly within a conductor (which has a *very large* σ in comparison to that of the surrounding medium), whereas electric flux usually cannot be contained within a dielectric slab of finite dimensions. The range of the dielectric constant of available materials is very limited (see Appendix B–3), and the flux-fringing around conductor edges makes the computation of capacitance less accurate.

The procedure for computing the resistance of a piece of conducting material between specified equipotential surfaces (or terminals) is as follows:

1. Choose an appropriate coordinate system for the given geometry.
2. Assume a potential difference V_0 between conductor terminals.
3. Find electric field intensity **E** within the conductor. (If the material is homogeneous, having a *constant* conductivity, the general method is to solve Laplace's equation $\nabla^2 V = 0$ for V in the chosen coordinate system, and then obtain $\mathbf{E} = -\nabla V$.)
4. Find the total current

$$I = \int_S \mathbf{J}\cdot d\mathbf{s} = \int_S \sigma\mathbf{E}\cdot d\mathbf{s},$$

where S is the cross-sectional area over which I flows.

5. Find resistance R by taking the ratio V_0/I.

It is important to note that if the conducting material is inhomogeneous and if the conductivity is a function of space coordinates, Laplace's equation for V does not hold. Can you explain why and indicate how $\mathbf{E}$ can be determined under these circumstances?

When the given geometry is such that $\mathbf{J}$ can be determined easily from a total current I, we may start the solution by assuming an I. From I, $\mathbf{J}$ and $\mathbf{E} = \mathbf{J}/\sigma$ are found. Then the potential difference V_0 is determined from the relation

$$V_0 = -\int \mathbf{E} \cdot d\ell,$$

where the integration is from the low-potential terminal to the high-potential terminal. The resistance $R = V_0/I$ is independent of the assumed I, which will be canceled in the process.

EXAMPLE 5–6 A conducting material of uniform thickness h and conductivity σ has the shape of a quarter of a flat circular washer, with inner radius a and outer radius b, as shown in Fig. 5–8. Determine the resistance between the end faces.

Solution Obviously, the appropriate coordinate system to use for this problem is the cylindrical coordinate system. Following the foregoing procedure, we first assume a potential difference V_0 between the end faces, say $V = 0$ on the end face at $y = 0$ ($\phi = 0$) and $V = V_0$ on the end face at $x = 0$ ($\phi = \pi/2$). We are to solve Laplace's equation in V subject to the following boundary conditions:

$$V = 0 \quad \text{at} \quad \phi = 0, \tag{5–84a}$$

$$V = V_0 \quad \text{at} \quad \phi = \pi/2. \tag{5–84b}$$

Since potential V is a function of ϕ only, Laplace's equation in cylindrical coordinates simplifies to

$$\frac{d^2V}{d\phi^2} = 0. \tag{5–85}$$

The general solution of Eq. (5–85) is

$$V = c_1\phi + c_2,$$

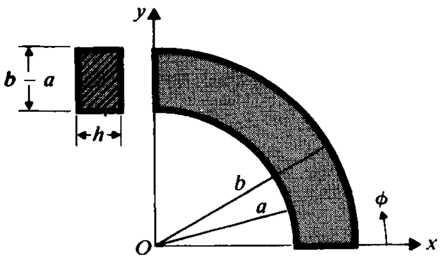


FIGURE 5–8
A quarter of a flat conducting circular washer (Example 5–6).

which, upon using the boundary conditions in Eqs. (5–84a) and (5–84b), becomes

$$V = \frac{2V_0}{\pi}\phi. \tag{5–86}$$

The current density is

$$\begin{aligned}\mathbf{J} &= \sigma\mathbf{E} = -\sigma\nabla V \\ &= -\mathbf{a}_\phi\sigma\frac{\partial V}{r\partial\phi} = -\mathbf{a}_\phi\frac{2\sigma V_0}{\pi r}.\end{aligned} \tag{5–87}$$

The total current I can be found by integrating $\mathbf{J}$ over the $\phi = \pi/2$ surface at which $d\mathbf{s} = -\mathbf{a}_\phi h\,dr$. We have

$$\begin{aligned}I &= \int_S \mathbf{J}\cdot d\mathbf{s} = \frac{2\sigma V_0}{\pi}h\int_a^b \frac{dr}{r} \\ &= \frac{2\sigma h V_0}{\pi}\ln\frac{b}{a}.\end{aligned} \tag{5–88}$$

Therefore,

$$R = \frac{V_0}{I} = \frac{\pi}{2\sigma h \ln(b/a)}. \tag{5–89}$$

Note that, for this problem, it is not convenient to begin by assuming a total current I because it is not obvious how $\mathbf{J}$ varies with r for a given I. Without $\mathbf{J}$, $\mathbf{E}$ and V_0 cannot be determined. ▬

Review Questions

R.5–1 Explain the difference between conduction and convection currents.

R.5–2 Explain the operation of an electrolytic tank. In what ways do electrolytic currents differ from conduction and convection currents?

R.5–3 Define *mobility* of the electron in a conductor. What is its SI unit?

R.5–4 What is the *Child-Langmuir law*?

R.5–5 What is the point form for *Ohm's law*?

R.5–6 Define *conductivity*. What is its SI unit?

R.5–7 Why does the resistance formula in Eq. (5–27) require that the material be homogeneous and straight and that it have a uniform cross section?

R.5–8 Prove Eqs. (5–29) and (5–30b).

R.5–9 Define *electromotive force* in words.

R.5–10 What is the difference between impressed and electrostatic field intensities?

R.5–11 State *Kirchhoff's voltage law* in words.

R.5–12 What are the characteristics of an ideal voltage source?

R.5–13 Can the currents in different branches (resistors) of a closed loop in an electric network flow in opposite directions? Explain.

R.5–14 What is the physical significance of the *equation of continuity*?

R.5–15 State *Kirchhoff's current law* in words.

R.5–16 What are the characteristics of an ideal current source?

R.5–17 Define *relaxation time*. What is the order of magnitude of the relaxation time in copper?

R.5–18 In what ways should Eq. (5–48) be modified when σ is a function of space coordinates?

R.5–19 State Joule's law. Express the power dissipated in a volume

a) in terms of **E** and σ,

b) in terms of **J** and σ.

R.5–20 Does the relation $\nabla \times \mathbf{J} = 0$ hold in a medium whose conductivity is not constant? Explain.

R.5–21 What are the boundary conditions of the normal and tangential components of steady current at the interface of two media with different conductivities?

R.5–22 What quantities in electrostatics are analogous to the steady current density vector and conductivity in an ohmic medium?

R.5–23 What is the basis of using an electrolytic tank to map the potential distribution of electrostatic boundary-value problems?

R.5–24 What is the relation between the resistance and the capacitance formed by two conductors immersed in a lossy dielectric medium that has permittivity ϵ and conductivity σ?

R.5–25 Under what situations will the relation between R and C in R.5–24 be only approximately correct? Give a specific example.

Problems

P.5–1 Assuming S to be the area of the electrodes in the space-charge-limited vacuum diode in Fig. 5–2, find

a) $V(y)$ and $E(y)$ within the interelectrode region,

b) the total amount of charge in the interelectrode region,

c) the total surface charge on the cathode and on the anode,

d) the transit time of an electron from the cathode to the anode with $V_0 = 200$ (V) and $d = 1$ (cm).

P.5–2 Starting with Ohm's law as expressed in Eq. (5–26) applied to a resistor of length ℓ, conductivity σ, and uniform cross-section S, verify the point form of Ohm's law represented by Eq. (5–21).

P.5–3 A long, round wire of radius a and conductivity σ is coated with a material of conductivity 0.1σ.

a) What must be the thickness of the coating so that the resistance per unit length of the uncoated wire is reduced by 50%?

b) Assuming a total current I in the coated wire, find **J** and **E** in both the core and the coating material.

P.5–4 Find the current and the heat dissipated in each of the five resistors in the network shown in Fig. 5–9 if

$$R_1 = \tfrac{1}{3}\,(\Omega), \quad R_2 = 20\,(\Omega), \quad R_3 = 30\,(\Omega), \quad R_4 = 8\,(\Omega), \quad R_5 = 10\,(\Omega),$$

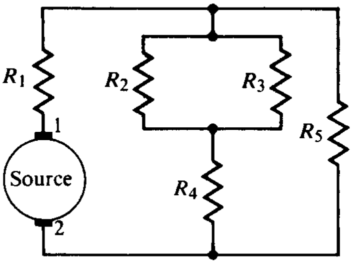


FIGURE 5–9
A network problem (Problem P.5–4).

and if the source is an ideal d-c voltage generator of 0.7 (V) with its positive polarity at terminal 1. What is the total resistance seen by the source at terminal pair 1–2?

P.5–5 Solve Problem P.5–4, assuming that the source is an ideal current generator that supplies a direct current of 0.7 (A) out of terminal 1.

P.5–6 Lightning strikes a lossy dielectric sphere—$\epsilon = 1.2\,\epsilon_0$, $\sigma = 10$ (S/m)—of radius 0.1 (m) at time $t = 0$, depositing uniformly in the sphere a total charge 1 (mC). Determine, for all t,

a) the electric field intensity both inside and outside the sphere,
b) the current density in the sphere.

P.5–7 Refer to Problem P.5–6.

a) Calculate the time it takes for the charge density in the sphere to diminish to 1% of its initial value.
b) Calculate the change in the electrostatic energy stored in the sphere as the charge density diminishes from the initial value to 1% of its value. What happens to this energy?
c) Determine the electrostatic energy stored in the space outside the sphere. Does this energy change with time?

P.5–8 A d-c voltage of 6 (V) applied to the ends of 1 (km) of a conducting wire of 0.5 (mm) radius results in a current of 1/6 (A). Find

a) the conductivity of the wire,
b) the electric field intensity in the wire,
c) the power dissipated in the wire,
d) the electron drift velocity, assuming electron mobility in the wire to be 1.4×10^{-3} (m^2/V·s).

P.5–9 Two lossy dielectric media with permittivities and conductivities (ϵ_1, σ_1) and (ϵ_2, σ_2) are in contact. An electric field with a magnitude E_1 is incident from medium 1 upon the interface at an angle α_1 measured from the common normal, as in Fig. 5–10.

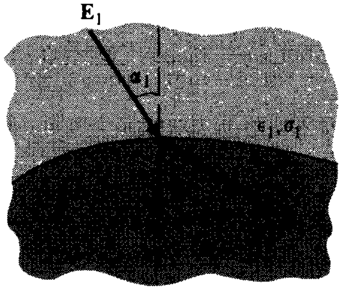


FIGURE 5–10
Boundary between two lossy dielectric media (Problem P.5–9).

a) Find the magnitude and direction of $\mathbf{E}_2$ in medium 2.
b) Find the surface charge density at the interface.
c) Compare the results in parts (a) and (b) with the case in which both media are perfect dielectrics.

P.5–10 The space between two parallel conducting plates each having an area S is filled with an inhomogeneous ohmic medium whose conductivity varies linearly from σ_1 at one plate ($y = 0$) to σ_2 at the other plate ($y = d$). A d-c voltage V_0 is applied across the plates as in Fig. 5–11. Determine
a) the total resistance between the plates,
b) the surface charge densities on the plates,
c) the volume charge density and the total amount of charge between the plates.

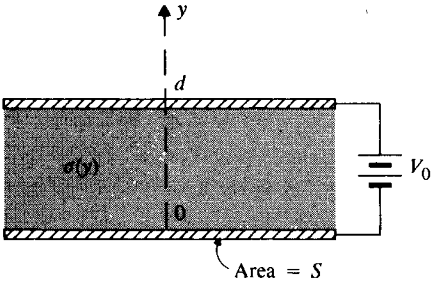


FIGURE 5–11
Inhomogeneous ohmic medium with conductivity $\sigma(y)$ (Problem P.5–10).

P.5–11 Refer to Example 5–4.
a) Draw the equivalent circuit of the two-layer, parallel-plate capacitor with lossy dielectrics, and identify the magnitude of each component.
b) Determine the power dissipated in the capacitor.

P.5–12 Refer again to Example 5–4. Assuming that a voltage V_0 is applied across the parallel-plate capacitor with the two layers of different lossy dielectrics at $t = 0$,
a) express the surface charge density ρ_{si} at the dielectric interface as a function of t,
b) express the electric field intensities $\mathbf{E}_1$ and $\mathbf{E}_2$ as functions of t.

P.5–13 A d-c voltage V_0 is applied across a cylindrical capacitor of length L. The radii of the inner and outer conductors are a and b, respectively. The space between the conductors is filled with two different lossy dielectrics having, respectively, permittivity ϵ_1 and conductivity σ_1 in the region $a < r < c$, and permittivity ϵ_2 and conductivity σ_2 in the region $c < r < b$. Determine
a) the current density in each region,
b) the surface charge densities on the inner and outer conductors and at the interface between the two dielectrics.

P.5–14 Refer to the flat conducting quarter-circular washer in Example 5–6 and Fig. 5–8. Find the resistance between the curved sides.

P.5–15 Find the resistance between two concentric spherical surfaces of radii R_1 and R_2 ($R_1 < R_2$) if the space between the surfaces is filled with a homogeneous and isotropic material having a conductivity σ.

P.5–16 Determine the resistance between two concentric spherical surfaces of radii R_1 and R_2 ($R_1 < R_2$), assuming that a material of conductivity $\sigma = \sigma_0(1 + k/R)$ fills the space between them. (*Note:* Laplace's equation for V does not apply here.)

P.5–17 A homogeneous material of uniform conductivity σ is shaped like a truncated conical block and defined in spherical coordinates by

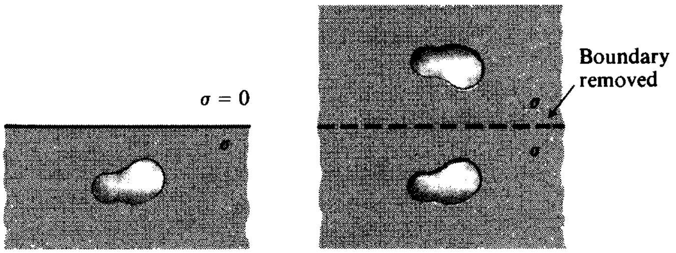
$$R_1 \le R \le R_2 \qquad \text{and} \qquad 0 \le \theta \le \theta_0.$$

Determine the resistance between the $R = R_1$ and $R = R_2$ surfaces.

P.5–18 Redo Problem P.5–17, assuming that the truncated conical block is composed of an inhomogeneous material with a nonuniform conductivity $\sigma(R) = \sigma_0 R_1/R$, where $R_1 \le R \le R_2$.

P.5–19 Two conducting spheres of radii b_1 and b_2 that have a very high conductivity are immersed in a poorly conducting medium (for example, they are buried very deep in the ground) of conductivity σ and permittivity ϵ. The distance, d, between the spheres is very large in comparison with the radii. Determine the resistance between the conducting spheres. (*Hint:* Find the capacitance between the spheres by following the procedure in Section 3–10 and using Eq. (5–81).)

P.5–20 Justify the statement that the steady-current problem associated with a conductor buried in a poorly conducting medium near a plane boundary with air, as shown in Fig. 5–12(a), can be replaced by that of the conductor and its image, both immersed in the poorly conducting medium as shown in Fig. 5–12(b).



(a) Conductor in a poorly conducting medium near a plane boundary.

(b) Image conductor in conducting medium replacing the plane boundary.

FIGURE 5–12
Steady-current problem with a plane boundary (Problem P.5–20).

P.5–21 A ground connection is made by burying a hemispherical conductor of radius 25 (mm) in the earth with its base up, as shown in Fig. 5–13. Assuming the earth conductivity to be 10^{-6} S/m, find the resistance of the conductor to far-away points in the ground. (*Hint:* Use the image method in P.5–20.)

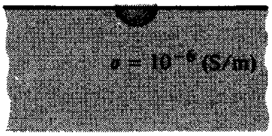


FIGURE 5–13
Hemispherical conductor in ground (Problem P.5–21).

P.5–22 Assume a rectangular conducting sheet of conductivity σ, width a, and height b. A potential difference V_0 is applied to the side edges, as shown in Fig. 5–14. Find

a) the potential distribution,

b) the current density everywhere within the sheet. (*Hint:* Solve Laplace's equation in Cartesian coordinates subject to appropriate boundary conditions.)

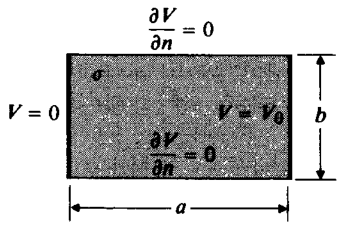


FIGURE 5–14
A conducting sheet (Problem P.5–22).

P.5–23 A uniform current density $\mathbf{J} = \mathbf{a}_x J_0$ flows in a very large rectangular block of homogeneous material of a uniform thickness having a conductivity σ. A hole of radius b is drilled in the material. Find the new current density $\mathbf{J}'$ in the conducting material. (*Hint:* Solve Laplace's equation in cylindrical coordinates and note that V approaches $-(J_0 r/\sigma) \cos \phi$ as $r \to \infty$, where ϕ is the angle measured from the x-axis.)

6

Static Magnetic Fields

6–1 Introduction

In Chapter 3 we dealt with static electric fields caused by electric charges at rest. We saw that electric field intensity **E** is the only fundamental vector field quantity required for the study of electrostatics in free space. In a material medium it is convenient to define a second vector field quantity, the electric flux density (or electric displacement) **D**, to account for the effect of polarization. The following two equations form the basis of the electrostatic model:

$$\nabla \cdot \mathbf{D} = \rho, \tag{6–1}$$

$$\nabla \times \mathbf{E} = 0. \tag{6–2}$$

The electrical property of the medium determines the relation between **D** and **E**. If the medium is linear and isotropic, we have the simple ***constitutive relation*** $\mathbf{D} = \epsilon \mathbf{E}$, where the permittivity ϵ is a scalar.

When a small test charge q is placed in an electric field **E**, it experiences an ***electric force*** $\mathbf{F}_e$, which is a function of the position of q. We have

$$\boxed{\mathbf{F}_e = q\mathbf{E} \qquad \text{(N)}.} \tag{6–3}$$

When the test charge is in motion in a magnetic field (to be defined presently), experiments show that it experiences another force, $\mathbf{F}_m$, which has the following characteristics: (1) The magnitude of $\mathbf{F}_m$ is proportional to q; (2) the direction of $\mathbf{F}_m$ at any point is at right angles to the velocity vector of the test charge as well as to a fixed direction at that point; and (3) the magnitude of $\mathbf{F}_m$ is also proportional to the component of the velocity at right angles to this fixed direction. The force $\mathbf{F}_m$ is a ***magnetic force***; it cannot be expressed in terms of **E** or **D**. The characteristics of $\mathbf{F}_m$ can be described by defining a new vector field quantity, the ***magnetic flux density*** **B**, that specifies both the fixed direction and the constant of proportionality. In SI units the magnetic force can be expressed as

$$\boxed{\mathbf{F}_m = q\mathbf{u} \times \mathbf{B} \qquad \text{(N)},} \tag{6-4}$$

where $\mathbf{u}$ (m/s) is the velocity vector, and $\mathbf{B}$ is measured in webers per square meter (Wb/m^2) or teslas (T).† The total ***electromagnetic force*** on a charge q is, then, $\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m$; that is,

$$\boxed{\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \qquad \text{(N)},} \tag{6-5}$$

which is called ***Lorentz's force equation.*** Its validity has been unquestionably established by experiments. We may consider $\mathbf{F}_e/q$ for a small q as the definition for electric field intensity $\mathbf{E}$ (as we did in Eq. 3–2), and $\mathbf{F}_m/q = \mathbf{u} \times \mathbf{B}$ as the defining relation for magnetic flux density $\mathbf{B}$. Alternatively, we may consider Lorentz's force equation as a fundamental postulate of our electromagnetic model; it cannot be derived from other postulates.

We begin the study of static magnetic fields in free space by two postulates specifying the divergence and the curl of **B**. From the solenoidal character of **B** a vector magnetic potential is defined, which is shown to obey a vector Poisson's equation. Next we derive the Biot-Savart law, which can be used to determine the magnetic field of a current-carrying circuit. The postulated curl relation leads directly to Ampère's circuital law, which is particularly useful when symmetry exists.

The macroscopic effect of magnetic materials in a magnetic field can be studied by defining a magnetization vector. Here we introduce a fourth vector field quantity, the magnetic field intensity **H**. From the relation between **B** and **H** we define the permeability of the material, following which we discuss magnetic circuits and the microscopic behavior of magnetic materials. We then examine the boundary conditions of **B** and **H** at the interface of two different magnetic media; self- and mutual inductances; and magnetic energy, forces, and torques.

6–2 Fundamental Postulates of Magnetostatics in Free Space

To study magnetostatics (steady magnetic fields) in free space, we need only consider the magnetic flux density vector, **B**. The two fundamental postulates of magnetostatics that specify the divergence and the curl of **B** *in free space* are

$$\boxed{\nabla \cdot \mathbf{B} = 0,} \tag{6-6}$$

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.} \tag{6-7}$$

† One weber per square meter or one tesla equals 10^4 gauss in CGS units. The earth magnetic field is about $\frac{1}{2}$ gauss or 0.5×10^{-4} T. (A weber is the same as a volt-second.)

In Eq. (6–7), μ_0 is the permeability of free space:

$$\mu_0 = 4\pi \times 10^{-7} \quad \text{(H/m)}$$

(see Eq. 1–9), and **J** is the current density. Since the divergence of the curl of any vector field is zero (see Eq. 2–149), we obtain from Eq. (6–7)

$$\nabla \cdot \mathbf{J} = 0, \tag{6–8}$$

which is consistent with Eq. (5–44) for steady currents.

Comparison of Eq. (6–6) with the analogous equation for electrostatics in free space, $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ (Eq. 3–4), leads us to conclude that there is no magnetic analogue for electric charge density ρ. Taking the volume integral of Eq. (6–6) and applying the divergence theorem, we have

$$\boxed{\oint_S \mathbf{B} \cdot d\mathbf{s} = 0,} \tag{6–9}$$

where the surface integral is carried out over the bounding surface of an arbitrary volume. Comparing Eq. (6–9) with Eq. (3–7), we again deny the existence of isolated magnetic charges. ***There are no magnetic flow sources, and the magnetic flux lines always close upon themselves.*** Equation (6–9) is also referred to as an expression for ***the law of conservation of magnetic flux*** because it states that the total outward magnetic flux through any closed surface is zero.

The traditional designation of north and south poles in a permanent bar magnet does not imply that an isolated positive magnetic charge exists at the north pole and a corresponding amount of isolated negative magnetic charge exists at the south pole. Consider the bar magnet with north and south poles in Fig. 6–1(a). If this magnet is cut into two segments, new south and north poles appear, and we have two shorter magnets as in Fig. 6–1(b). If each of the two shorter magnets is cut again into two segments, we have four magnets, each with a north pole and a south pole as in Fig. 6–1(c). This process could be continued until the magnets are of atomic dimensions; but each infinitesimally small magnet would still have a north pole and a south pole. Obviously, then, magnetic poles cannot be isolated. The magnetic flux lines follow closed paths from one end of a magnet to the other end outside the magnet and then

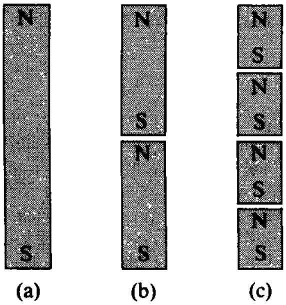


FIGURE 6–1
Successive division of a bar magnet.

continue inside the magnet back to the first end. The designation of north and south poles is in accordance with the fact that the respective ends of a bar magnet freely suspended in the earth's magnetic field will seek the north and south directions.†

The integral form of the curl relation in Eq. (6–7) can be obtained by integrating both sides over an open surface and applying Stokes's theorem. We have

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{s}$$

or

$$\boxed{\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I,} \tag{6–10}$$

where the path C for the line integral is the contour bounding the surface S, and I is the total current through S. The sense of tracing C and the direction of current flow follow the right-hand rule. Equation (6–10) is a form of ***Ampère's circuital law***, which states that ***the circulation of the magnetic flux density in free space around any closed path is equal to μ_0 times the total current flowing through the surface bounded by the path.*** Ampère's circuital law is very useful in determining the magnetic flux density $\mathbf{B}$ caused by a current I when there is a closed path C around the current such that the magnitude of $\mathbf{B}$ is constant over the path.

The following is a summary of the two fundamental postulates of magnetostatics in free space:

Postulates of Magnetostatics in Free Space	
Differential Form	Integral Form
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$
$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$	$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I$

EXAMPLE 6–1 An infinitely long, straight conductor with a circular cross section of radius b carries a steady current I. Determine the magnetic flux density both inside and outside the conductor.

† We note here parenthetically that examination of some prehistoric rock formations has led to the belief that there have been dramatic reversals of the earth's magnetic field every ten million years or so. The earth's magnetic field is thought to be produced by the rolling motions of the molten iron in the earth's outer core, but the exact reasons for the field reversals are still not well understood. The next such reversal is predicted to be only about 2000 years from now. One cannot conjecture all the dire consequences of such a reversal, but among them would be disruptions in global navigation and drastic changes in the migratory patterns of birds.

Solution First we note that this is a problem with cylindrical symmetry and that Ampère's circuital law can be used to advantage. If we align the conductor along the z-axis, the magnetic flux density **B** will be ϕ-directed and will be constant along any circular path around the z-axis. Figure 6–2(a) shows a cross section of the conductor and the two circular paths of integration, C_1 and C_2, inside and outside, respectively, the current-carrying conductor. Note again that the directions of C_1 and C_2 and the direction of I follow the right-hand rule. (When the fingers of the right hand follow the directions of C_1 and C_2, the thumb of the right hand points to the direction of I.)

a) *Inside the conductor:*

$$\mathbf{B}_1 = \mathbf{a}_\phi B_{\phi 1}, \qquad d\boldsymbol{\ell} = \mathbf{a}_\phi r_1 d\phi$$

$$\oint_{C_1} \mathbf{B}_1 \cdot d\boldsymbol{\ell} = \int_0^{2\pi} B_{\phi 1} r_1 \, d\phi = 2\pi r_1 B_{\phi 1}.$$

The current through the area enclosed by C_1 is

$$I_1 = \frac{\pi r_1^2}{\pi b^2} I = \left(\frac{r_1}{b}\right)^2 I.$$

Therefore, from Ampère's circuital law,

$$\mathbf{B}_1 = \mathbf{a}_\phi B_{\phi 1} = \mathbf{a}_\phi \frac{\mu_0 r_1 I}{2\pi b^2}, \qquad r_1 \le b. \tag{6–11a}$$

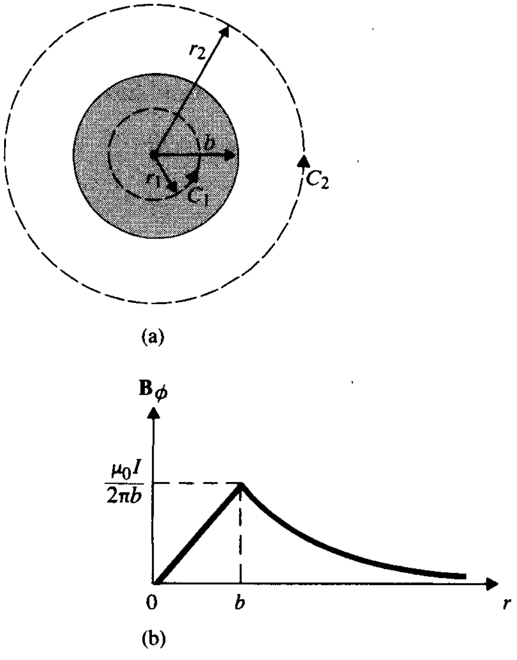


FIGURE 6–2
Magnetic flux density of an infinitely long circular conductor carrying a current I out of paper (Example 6–1).

b) *Outside the conductor:*

$$\mathbf{B}_2 = \mathbf{a}_\phi B_{\phi 2}, \qquad d\boldsymbol{\ell} = \mathbf{a}_\phi r_2\, d\phi$$

$$\oint_{C_2} \mathbf{B}_2 \cdot d\boldsymbol{\ell} = 2\pi r_2 B_{\phi 2}.$$

Path C_2 outside the conductor encloses the total current I. Hence

$$\mathbf{B}_2 = \mathbf{a}_\phi B_{\phi 2} = \mathbf{a}_\phi \frac{\mu_0 I}{2\pi r_2}, \qquad r_2 \geq b. \tag{6-11b}$$

Examination of Eqs. (6–11a) and (6–11b) reveals that the magnitude of **B** increases linearly with r_1 from 0 until $r_1 = b$, after which it decreases inversely with r_2. The variation of B_ϕ versus r is sketched in Fig. 6–2(b). ∎

If the problem is not that of a solid cylindrical conductor carrying a total steady current I, but that of a very thin circular tube carrying a surface current, then it is obvious from Ampère's circuital law that $\mathbf{B} = 0$ inside the tube. Outside the tube, Eq. (6–11b) still applies with I = total current flowing in the tube. Thus, for an infinitely long, hollow cylinder carrying a surface current density $\mathbf{J}_s = \mathbf{a}_z J_s$ (A/m), $I = 2\pi b J_s$, we have

$$B = \begin{cases} 0, & r < b, \\ \mathbf{a}_\phi \dfrac{\mu_0 b}{r} J_s, & r > b. \end{cases} \tag{6-12}$$

EXAMPLE 6–2 Determine the magnetic flux density inside a closely wound toroidal coil with an air core having N turns and carrying a current I. The toroid has a mean radius b, and the radius of each turn is a.

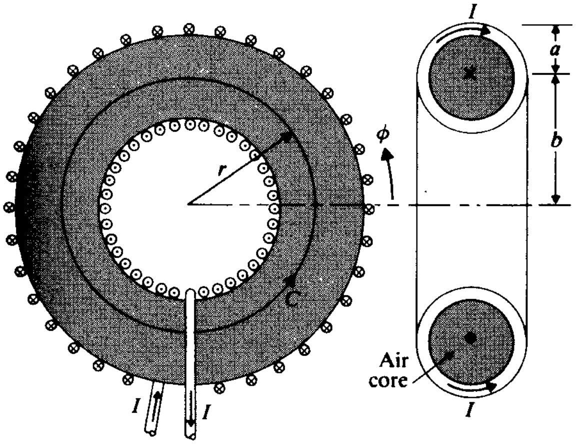


FIGURE 6–3
A current-carrying toroidal oil (Example 6–2).

Solution Figure 6-3 depicts the geometry of this problem. Cylindrical symmetry ensures that **B** has only a ϕ-component and is constant along any circular path about the axis of the toroid. We construct a circular contour C with radius r as shown. For $(b - a) < r < b + a$, Eq. (6-10) leads directly to

$$\oint \mathbf{B} \cdot d\boldsymbol{\ell} = 2\pi r B_\phi = \mu_0 N I,$$

where we have assumed that the toroid has an air core with permeability μ_0. Therefore,

$$\mathbf{B} = \mathbf{a}_\phi B_\phi = \mathbf{a}_\phi \frac{\mu_0 N I}{2\pi r}, \qquad (b - a) < r < (b + a). \tag{6-13}$$

It is apparent that $\mathbf{B} = 0$ for $r < (b - a)$ and $r > (b + a)$, since the net total current enclosed by a contour constructed in these two regions is zero. ■

EXAMPLE 6-3 Determine the magnetic flux density inside an infinitely long solenoid with air core having n closely wound turns per unit length and carrying a current I as shown in Fig. 6-4.

Solution This problem can be solved in two ways.

a) *As a direct application of Ampère's circuital law.* It is clear that there is no magnetic field outside of the solenoid. To determine the **B**-field inside, we construct a rectangular contour C of length L that is partially inside and partially outside the solenoid. By reason of symmetry the **B**-field inside must be parallel to the axis. Applying Ampère's circuital law, we have

$$BL = \mu_0 n L I$$

or

$$B = \mu_0 n I. \tag{6-14}$$

The direction of **B** goes from right to left, conforming to the right-hand rule with respect to the direction of the current I in the solenoid, as indicated in Fig. 6-4.

b) *As a special case of toroid.* The straight solenoid may be regarded as a special case of the toroidal coil in Example 6-2 with an infinite radius ($b \to \infty$). In such

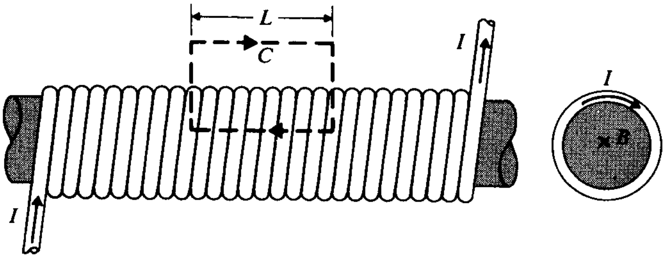


FIGURE 6-4
A current-carrying long solenoid (Example 6-3).

a case the dimensions of the cross section of the core are very small in comparison with b, and the magnetic flux density inside the core is approximately constant. We have, from Eq. (6–13),

$$B = \mu_0\left(\frac{N}{2\pi b}\right)I = \mu_0 nI,$$

which is the same as Eq. (6–14). ∎

6–3 Vector Magnetic Potential

The divergence-free postulate of **B** in Eq. (6–6), $\nabla \cdot \mathbf{B} = 0$, assures that **B** is solenoidal. As a consequence, **B** can be expressed as the curl of another vector field, say **A**, such that (see Identity II, Eq. (2–149), in Section 2–11)

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A} \qquad \text{(T)}.} \tag{6–15}$$

The vector field A so defined is called the ***vector magnetic potential.*** Its SI unit is weber per meter (Wb/m). Thus if we can find **A** of a current distribution, **B** can be obtained from **A** by a differential (curl) operation. This is quite similar to the introduction of the scalar electric potential V for the curl-free **E** in electrostatics (Section 3–5) and the obtaining of **E** from the relation $\mathbf{E} = -\nabla V$. However, the definition of a vector requires the specification of both its curl and its divergence. Hence Eq. (6–15) alone is not sufficient to define **A**; we must still specify its divergence.

How do we choose $\nabla \cdot \mathbf{A}$? Before we answer this question, let us take the curl of **B** in Eq. (6–15) and substitute it in Eq. (6–7). We have

$$\nabla \times \nabla \times \mathbf{A} = \mu_0 \mathbf{J}. \tag{6–16}$$

Here we digress to introduce a formula for the curl curl of a vector:

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{6–17a}$$

or

$$\nabla^2 \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A}. \tag{6–17b}$$

Equation (6–17a)† or (6–17b) can be regarded as the definition of $\nabla^2 \mathbf{A}$, the Laplacian of **A**. For Cartesian coordinates it can be readily verified by direct substitution (Problem P.6–16) that

$$\nabla^2 \mathbf{A} = \mathbf{a}_x \nabla^2 A_x + \mathbf{a}_y \nabla^2 A_y + \mathbf{a}_z \nabla^2 A_z. \tag{6–18}$$

Thus, for Cartesian coordinates the Laplacian of a vector field **A** is another vector field whose components are the Laplacian (the divergence of the gradient) of the

† Equation (6–17a) can also be obtained heuristically from the vector triple product formula in Eq. (2–20) by considering the del operator, ∇, a vector:

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - (\nabla \cdot \nabla)\mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}.$$

corresponding components of **A**. This, however, is not true for other coordinate systems.

We now expand $\nabla \times \nabla \times \mathbf{A}$ in Eq. (6–16) according to Eq. (6–17a) and obtain

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}. \tag{6–19}$$

With the purpose of simplifying Eq. (6–19) to the greatest extent possible we choose

$$\boxed{\nabla \cdot \mathbf{A} = 0,} \tag{6–20}^{\dagger}$$

and Eq. (6–19) becomes

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}.} \tag{6–21}$$

This is a ***vector Poisson's equation***. In Cartesian coordinates, Eq. (6–21) is equivalent to three scalar Poisson's equations:

$$\nabla^2 A_x = -\mu_0 J_x, \tag{6–22a}$$
$$\nabla^2 A_y = -\mu_0 J_y, \tag{6–22b}$$
$$\nabla^2 A_z = -\mu_0 J_z. \tag{6–22c}$$

Each of these three equations is mathematically the same as the scalar Poisson's equation, Eq. (4–6), in electrostatics. In free space the equation

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

has a particular solution (see Eq. 3–61),

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho}{R}\, dv'.$$

Hence the solution for Eq. (6–22a) is

$$A_x = \frac{\mu_0}{4\pi} \int_{V'} \frac{J_x}{R}\, dv'.$$

We can write similar solutions for A_y and A_z. Combining the three components, we have the solution for Eq. (6–21):

$$\boxed{\mathbf{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J}}{R}\, dv' \qquad \text{(Wb/m).}} \tag{6–23}$$

† This relation is called ***Coulomb condition*** or ***Coulomb gauge***.

Equation (6–23) enables us to find the vector magnetic potential **A** from the volume current density **J**. The magnetic flux density **B** can then be obtained from $\nabla \times \mathbf{A}$ by differentiation, in a way similar to that of obtaining the static electric field **E** from $-\nabla V$.

Vector potential **A** relates to the magnetic flux Φ through a given area S that is bounded by contour C in a simple way:

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s}. \tag{6–24}$$

The SI unit for magnetic flux is weber (Wb), which is equivalent to tesla-square meter ($\text{T}\cdot\text{m}^2$). Using Eq. (6–15) and Stokes's theorem, we have

$$\Phi = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_C \mathbf{A} \cdot d\boldsymbol{\ell} \qquad \text{(Wb)}. \tag{6–25}$$

Thus, vector magnetic potential **A** does have physical significance in that its line integral around any closed path equals the total magnetic flux passing through the area enclosed by the path.

6–4 The Biot-Savart Law and Applications

In many applications we are interested in determining the magnetic field due to a current-carrying circuit. For a thin wire with cross-sectional area S, dv' equals $S\,d\ell'$, and the current flow is entirely along the wire. We have

$$\mathbf{J}\,dv' = JS\,d\boldsymbol{\ell}' = I\,d\boldsymbol{\ell}', \tag{6–26}$$

and Eq. (6–23) becomes

$$\boxed{\mathbf{A} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\boldsymbol{\ell}'}{R} \qquad \text{(Wb/m)},} \tag{6–27}$$

where a circle has been put on the integral sign because the current I must flow in a closed path,† which is designated C'. The magnetic flux density is then

$$\begin{aligned} \mathbf{B} = \nabla \times \mathbf{A} &= \nabla \times \left[\frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\boldsymbol{\ell}'}{R}\right] \\ &= \frac{\mu_0 I}{4\pi} \oint_{C'} \nabla \times \left(\frac{d\boldsymbol{\ell}'}{R}\right). \end{aligned} \tag{6–28}$$

† We are now dealing with direct (non-time-varying) currents that give rise to steady magnetic fields. Circuits containing time-varying sources may send time-varying currents along an open wire and deposit charges at its ends. Antennas are examples.

It is very important to note in Eq. (6–28) that the *unprimed* curl operation implies differentiations with respect to the space coordinates of the *field point*, and that the integral operation is with respect to the *primed source coordinates*. The integrand in Eq. (6–28) can be expanded into two terms by using the following identity (see Problem P.2–37):

$$\nabla \times (f\mathbf{G}) = f\nabla \times \mathbf{G} + (\nabla f) \times \mathbf{G}. \tag{6–29}$$

We have, with $f = 1/R$ and $\mathbf{G} = d\boldsymbol{\ell}'$,

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \left[\frac{1}{R} \nabla \times d\boldsymbol{\ell}' + \left(\nabla \frac{1}{R} \right) \times d\boldsymbol{\ell}' \right]. \tag{6–30}$$

Now, since the unprimed and primed coordinates are independent, $\nabla \times d\boldsymbol{\ell}'$ equals 0, and the first term on the right side of Eq. (6–30) vanishes. The distance R is measured from $d\boldsymbol{\ell}'$ at (x', y', z') to the field point at (x, y, z). Thus we have

$$\begin{aligned} \frac{1}{R} &= [(x - x')^2 + (y - y')^2 + (z - z')^2]^{-1/2}; \\ \nabla\left(\frac{1}{R}\right) &= \mathbf{a}_x \frac{\partial}{\partial x}\left(\frac{1}{R}\right) + \mathbf{a}_y \frac{\partial}{\partial y}\left(\frac{1}{R}\right) + \mathbf{a}_z \frac{\partial}{\partial z}\left(\frac{1}{R}\right) \\ &= -\frac{\mathbf{a}_x(x - x') + \mathbf{a}_y(y - y') + \mathbf{a}_z(z - z')}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} \\ &= -\frac{\mathbf{R}}{R^3} = -\mathbf{a}_R \frac{1}{R^2}, \end{aligned} \tag{6–31}$$

where $\mathbf{a}_R$ is the unit vector directed *from the source point to the field point*. Substituting Eq. (6–31) in Eq. (6–30), we get

$$\boxed{\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\boldsymbol{\ell}' \times \mathbf{a}_R}{R^2} \quad \text{(T)}.} \tag{6–32}$$

Equation (6–32) is known as ***Biot-Savart law***. It is a formula for determining **B** caused by a current I in a closed path C' and is obtained by taking the curl of **A** in Eq. (6–27). Sometimes it is convenient to write Eq. (6–32) in two steps:

$$\boxed{\mathbf{B} = \oint_{C'} d\mathbf{B} \quad \text{(T)},} \tag{6–33a}$$

with

$$\boxed{d\mathbf{B} = \frac{\mu_0 I}{4\pi} \left(\frac{d\boldsymbol{\ell}' \times \mathbf{a}_R}{R^2} \right) \quad \text{(T)},} \tag{6–33b}$$

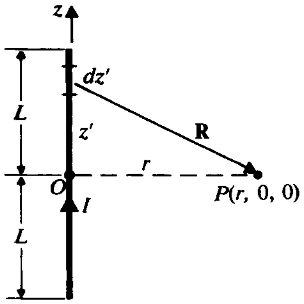


FIGURE 6–5
A current-carrying straight wire (Example 6–4).

which is the magnetic flux density due to a current element $I\,d\boldsymbol{\ell}'$. An alternative and sometimes more convenient form for Eq. (6–33b) is

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi}\left(\frac{d\boldsymbol{\ell}' \times \mathbf{R}}{R^3}\right) \quad \text{(T)}. \tag{6–33c}$$

Comparison of Eq. (6–32) with Eq. (6–10) will reveal that Biot-Savart law is, in general, more difficult to apply than Ampère's circuital law. However, Ampère's circuital law is not useful for determining **B** from I in a circuit if a closed path cannot be found over which **B** has a constant magnitude.

EXAMPLE 6–4 A direct current I flows in a straight wire of length $2L$. Find the magnetic flux density **B** at a point located at a distance r from the wire in the bisecting plane: (a) by determining the vector magnetic potential **A** first, and (b) by applying Biot-Savart law.

Solution Currents exist only in closed circuits. Hence the wire in the present problem must be a part of a current-carrying loop with several straight sides. Since we do not know the rest of the circuit, Ampère's circuital law cannot be used to advantage. Refer to Fig. 6–5. The current-carrying line segment is aligned with the z-axis. A typical element on the wire is

$$d\boldsymbol{\ell}' = \mathbf{a}_z\,dz'.$$

The cylindrical coordinates of the field point P are $(r, 0, 0)$.

a) *By finding* **B** *from* $\nabla \times \mathbf{A}$. Substituting $R = \sqrt{z'^2 + r^2}$ into Eq. (6–27), we have

$$\begin{aligned} \mathbf{A} &= \mathbf{a}_z \frac{\mu_0 I}{4\pi} \int_{-L}^{L} \frac{dz'}{\sqrt{z'^2 + r^2}} \\ &= \mathbf{a}_z \frac{\mu_0 I}{4\pi} \left[\ln\left(z' + \sqrt{z'^2 + r^2}\right)\right]\Big|_{-L}^{L} \\ &= \mathbf{a}_z \frac{\mu_0 I}{4\pi} \ln \frac{\sqrt{L^2 + r^2} + L}{\sqrt{L^2 + r^2} - L}. \end{aligned} \tag{6–34}$$

Therefore,

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times (\mathbf{a}_z A_z) = \mathbf{a}_r \frac{1}{r}\frac{\partial A_z}{\partial \phi} - \mathbf{a}_\phi \frac{\partial A_z}{\partial r}.$$

Cylindrical symmetry around the wire assures that $\partial A_z/\partial\phi = 0$. Thus,

$$\mathbf{B} = -\mathbf{a}_\phi \frac{\partial}{\partial r}\left[\frac{\mu_0 I}{4\pi}\ln\frac{\sqrt{L^2+r^2}+L}{\sqrt{L^2+r^2}-L}\right] = \mathbf{a}_\phi \frac{\mu_0 I L}{2\pi r\sqrt{L^2+r^2}}. \tag{6–35}$$

When $r \ll L$, Eq. (6–35) reduces to

$$\mathbf{B}_\phi = \mathbf{a}_\phi \frac{\mu_0 I}{2\pi r}, \tag{6–36}$$

which is the expression for **B** at a point located at a distance r from an infinitely long, straight wire carrying current I, as given in Eq. (6–11b).

b) *By applying Biot-Savart law.* From Fig. 6–5 we see that the distance vector *from* the source element dz' *to* the field point P is

$$\mathbf{R} = \mathbf{a}_r r - \mathbf{a}_z z'$$

$$d\boldsymbol{\ell}' \times \mathbf{R} = \mathbf{a}_z\, dz' \times (\mathbf{a}_r r - \mathbf{a}_z z') = \mathbf{a}_\phi r\, dz'.$$

Substitution in Eq. (6–33c) gives

$$\mathbf{B} = \int d\mathbf{B} = \mathbf{a}_\phi \frac{\mu_0 I}{4\pi}\int_{-L}^{L}\frac{r\,dz'}{(z'^2+r^2)^{3/2}} = \mathbf{a}_\phi \frac{\mu_0 I L}{2\pi r\sqrt{L^2+r^2}},$$

which is the same as Eq. (6–35).

EXAMPLE 6–5 Find the magnetic flux density at the center of a square loop, with side w carrying a direct current I.

Solution Assume that the loop lies in the xy-plane, as shown in Fig. 6–6. The magnetic flux density at the center of the square loop is equal to four times that caused

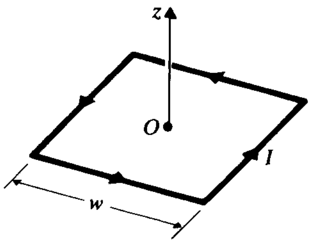


FIGURE 6–6
A square loop carrying current I (Example 6–5).

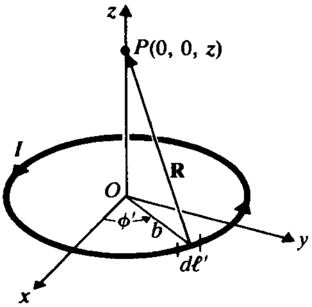


FIGURE 6–7
A circular loop carrying current I (Example 6–6).

by a single side of length w. We have, by setting $L = r = w/2$ in Eq. (6–35),

$$\mathbf{B} = \mathbf{a}_z \frac{\mu_0 I}{\sqrt{2}\pi w} \times 4 = \mathbf{a}_z \frac{2\sqrt{2}\mu_0 I}{\pi w}, \tag{6–37}$$

where the direction of **B** and that of the current in the loop follow the right-hand rule.

EXAMPLE 6–6 Find the magnetic flux density at a point on the axis of a circular loop of radius b that carries a direct current I.

Solution We apply Biot-Savart law to the circular loop shown in Fig. 6–7:

$$d\boldsymbol{\ell}' = \mathbf{a}_\phi b\, d\phi',$$
$$\mathbf{R} = \mathbf{a}_z z - \mathbf{a}_r b,$$
$$R = (z^2 + b^2)^{1/2}.$$

Again it is important to remember that **R** is the vector *from* the source element $d\boldsymbol{\ell}'$ *to* the field point P. We have

$$d\boldsymbol{\ell}' \times \mathbf{R} = \mathbf{a}_\phi b\, d\phi' \times (\mathbf{a}_z z - \mathbf{a}_r b)$$
$$= \mathbf{a}_r bz\, d\phi' + \mathbf{a}_z b^2\, d\phi'.$$

Because of cylindrical symmetry, it is easy to see that the $\mathbf{a}_r$-component is canceled by the contribution of the element located diametrically opposite to $d\boldsymbol{\ell}'$, so we need only consider the $\mathbf{a}_z$-component of this cross product.

We write, from Eqs. (6–33a) and (6–33c),

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \mathbf{a}_z \frac{b^2\, d\phi'}{(z^2 + b^2)^{3/2}}$$

or

$$\boxed{\mathbf{B} = \mathbf{a}_z \frac{\mu_0 I b^2}{2(z^2 + b^2)^{3/2}} \quad \text{(T).}} \tag{6–38}$$

6–5 The Magnetic Dipole

We begin this section with an example.

EXAMPLE 6–7 Find the magnetic flux density at a distant point of a small circular loop of radius b that carries current I (a ***magnetic dipole***).

Solution It is apparent from the statement of the problem that we are interested in determining **B** at a point whose distance, R, from the center of the loop satisfies the relation $R \gg b$; that being the case, we may make certain simplifying approximations.

We select the center of the loop to be the origin of spherical coordinates, as shown in Fig. 6–8. The source coordinates are primed. We first find the vector magnetic potential **A** and then determine **B** by $\nabla \times \mathbf{A}$:

$$\mathbf{A} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\boldsymbol{\ell}'}{R_1}. \tag{6–39}$$

Equation (6–39) is the same as Eq. (6–27), except for one important point: R in Eq. (6–27) denotes the distance between the source element $d\boldsymbol{\ell}'$ at P' and the field point P; but it must be replaced by R_1 in accordance with the notation in Fig. 6–8. Because of symmetry, the magnetic field is obviously independent of the angle ϕ of the field point. We pick $P(R, \theta, \pi/2)$ in the yz-plane for convenience.

Another point of importance is that $\mathbf{a}_{\phi'}$ at $d\boldsymbol{\ell}'$ is *not* the same as $\mathbf{a}_\phi$ at point P. In fact, $\mathbf{a}_\phi$ at P, shown in Fig. 6–8 is $-\mathbf{a}_x$, and

$$d\boldsymbol{\ell}' = (-\mathbf{a}_x \sin \phi' + \mathbf{a}_y \cos \phi')b\, d\phi'. \tag{6–40}$$

For every $I\, d\boldsymbol{\ell}'$ there is another symmetrically located differential current element on the other side of the y-axis that will contribute an equal amount to **A** in the $-\mathbf{a}_x$ direction but will cancel the contribution of $I\, d\boldsymbol{\ell}'$ in the $\mathbf{a}_y$ direction. Equation (6–39)

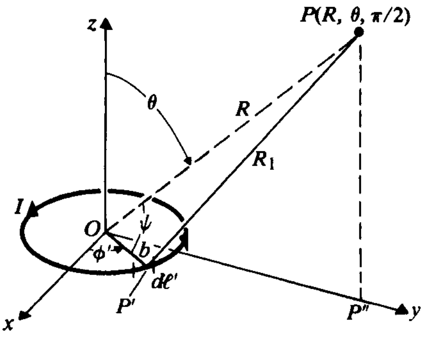


FIGURE 6–8
A small circular loop carrying current I (Example 6–7).

can be written as

$$\mathbf{A} = -\mathbf{a}_x \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{b \sin \phi'}{R_1} \, d\phi'$$

or

$$\mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 I b}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{\sin \phi'}{R_1} \, d\phi'. \tag{6-41}$$

The law of cosines applied to the triangle OPP' gives

$$R_1^2 = R^2 + b^2 - 2bR \cos \psi,$$

where $R \cos \psi$ is the projection of R on the radius OP', which is the same as the projection of OP'' ($OP'' = R \sin \theta$) on OP'. Hence,

$$R_1^2 = R^2 + b^2 - 2bR \sin \theta \sin \phi',$$

and

$$\frac{1}{R_1} = \frac{1}{R}\left(1 + \frac{b^2}{R^2} - \frac{2b}{R} \sin \theta \sin \phi'\right)^{-1/2}.$$

When $R^2 \gg b^2$, b^2/R^2 can be neglected in comparison with 1:

$$\begin{aligned}\frac{1}{R_1} &\cong \frac{1}{R}\left(1 - \frac{2b}{R} \sin \theta \sin \phi'\right)^{-1/2} \\ &\cong \frac{1}{R}\left(1 + \frac{b}{R} \sin \theta \sin \phi'\right).\end{aligned} \tag{6-42}$$

Substitution of Eq. (6–42) in Eq. (6–41) yields

$$\mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 I b}{2\pi R} \int_{-\pi/2}^{\pi/2} \left(1 + \frac{b}{R} \sin \theta \sin \phi'\right) \sin \phi' \, d\phi'$$

or

$$\mathbf{A} = \mathbf{a}_\phi \frac{\mu_0 I b^2}{4R^2} \sin \theta. \tag{6-43}$$

The magnetic flux density is $\mathbf{B} = \nabla \times \mathbf{A}$. Equation (2–139) can be used to find

$$\mathbf{B} = \frac{\mu_0 I b^2}{4R^3} (\mathbf{a}_R \, 2 \cos \theta + \mathbf{a}_\theta \sin \theta), \tag{6-44}$$

which is our answer. ■

At this point we recognize the similarity between Eq. (6–44) and the expression for the electric field intensity *in the far field* of an electrostatic dipole as given in Eq. (3–54). Hence, at distant points the magnetic flux lines of a magnetic dipole (placed in the xy-plane) such as that in Fig. 6–8 will have the same form as the dashed electric field lines of an electric dipole (lying in the z-direction) given in Fig. 3–15. In the vicinity of the dipoles, however, the flux lines of a magnetic dipole are continuous, whereas the field lines of an electric dipole terminate on the charges, always going from the positive to the negative charge. This is illustrated in Fig. 6–9.

Let us now rearrange the expression of the vector magnetic potential in Eq. (6–43) as

$$\mathbf{A} = \mathbf{a}_\phi \frac{\mu_0(I\pi b^2)}{4\pi R^2} \sin\theta$$

or

$$\boxed{\mathbf{A} = \frac{\mu_0 \mathbf{m} \times \mathbf{a}_R}{4\pi R^2} \qquad \text{(Wb/m)},} \qquad (6\text{–}45)$$

where

$$\mathbf{m} = \mathbf{a}_z I\pi b^2 = \mathbf{a}_z IS = \mathbf{a}_z m \qquad (\text{A}\cdot\text{m}^2) \qquad (6\text{–}46)$$

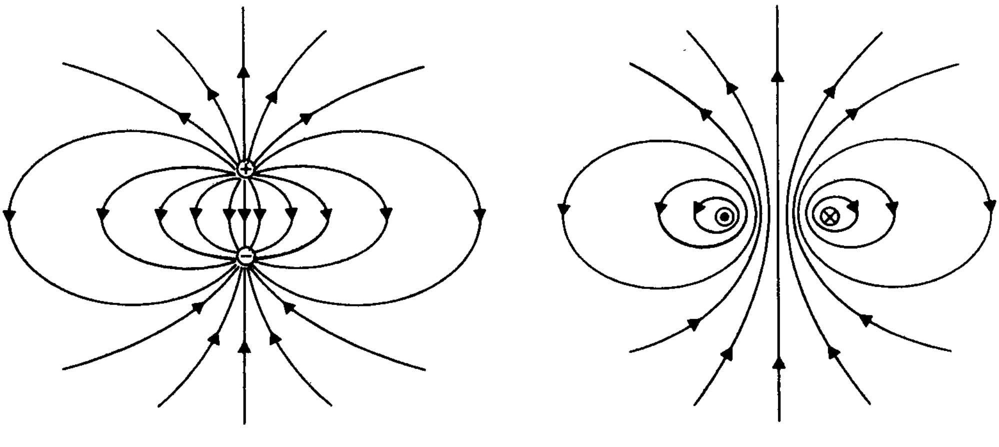
is defined as the ***magnetic dipole moment***, which is a vector whose magnitude is the product of the current in and the area of the loop and whose direction is the direction of the thumb as the fingers of the right hand follow the direction of the current. Comparison of Eq. (6–45) with the expression for the scalar electric potential of an electric dipole in Eq. (3–53b),

$$V = \frac{\mathbf{p}\cdot\mathbf{a}_R}{4\pi\epsilon_0 R^2} \qquad \text{(V)}, \qquad (6\text{–}47)$$

reveals that, for the two cases, **A** is analogous to V. We call a small current-carrying loop a ***magnetic dipole.***

In a similar manner we can also rewrite Eq. (6–44) as

$$\boxed{\mathbf{B} = \frac{\mu_0 m}{4\pi R^3}(\mathbf{a}_R\, 2\cos\theta + \mathbf{a}_\theta \sin\theta) \qquad \text{(T)}.} \qquad (6\text{–}48)$$



(a) Electric dipole. (b) Magnetic dipole.

FIGURE 6–9
Electric field lines of an electric dipole and magnetic flux lines of a magnetic dipole.

Except for the change of p to m and ϵ_0 to $1/\mu_0$, Eq. (6–48) has the same form as Eq. (3–54) does for the expression for $\mathbf{E}$ at a distant point of an electric dipole. Hence the magnetic flux lines of a magnetic dipole lying in the xy-plane will have the same form as that of the electric field lines of an electric dipole positioned along the z-axis, as noted before.

Although the magnetic dipole in Example 6–7 was taken to be a circular loop, it can be shown (Problem P.6–19) that the same expressions—Eqs. (6–45) and (6–48)—are obtained when the loop has a rectangular shape, with $m = IS$, as given in Eq. (6–46).

6–5.1 SCALAR MAGNETIC POTENTIAL

In a current-free region $\mathbf{J} = 0$, Eq. (6–7) becomes

$$\nabla \times \mathbf{B} = 0. \tag{6–49}$$

The magnetic flux density $\mathbf{B}$ is then curl-free and can be expressed as the gradient of a scalar field. Let

$$\mathbf{B} = -\mu_0 \nabla V_m, \tag{6–50}$$

where V_m is called the ***scalar magnetic potential*** (expressed in amperes). The negative sign in Eq. (6–50) is conventional (see the definition of the scalar electric potential V in Eq. 3–43), and the permeability of free space μ_0 is simply a proportionality constant. Analogous to Eq. (3–45), we can write the scalar magnetic potential difference between two points, P_2 and P_1, in free space as

$$V_{m2} - V_{m1} = -\int_{P_1}^{P_2} \frac{1}{\mu_0} \mathbf{B} \cdot d\boldsymbol{\ell}. \tag{6–51}$$

If there *were* magnetic charges with a volume density ρ_m (A/m^2) in a volume V', we *would* be able to find V_m from

$$V_m = \frac{1}{4\pi} \int_{V'} \frac{\rho_m}{R}\, dv' \qquad \text{(A)}. \tag{6–52}$$

The magnetic flux density $\mathbf{B}$ could then be determined from Eq. (6–50). However, isolated magnetic charges have never been observed experimentally; they must be considered fictitious. Nevertheless, the consideration of fictitious magnetic charges in a mathematical (not physical) model is expedient both to the discussion of some magnetostatic relations in terms of our knowledge of electrostatics and to the establishment of a bridge between the traditional magnetic-pole viewpoint of magnetism and the concept of microscopic circulating currents as sources of magnetism.

The magnetic field of a small bar magnet is the same as that of a magnetic dipole. This can be verified experimentally by observing the contours of iron filings around a magnet. The traditional understanding is that the ends (the north and south poles) of a permanent magnet are the location of positive and negative magnetic charges, respectively. For a bar magnet the fictitious magnetic charges $+q_m$ and $-q_m$ are assumed to be separated by a distance d and to form an equivalent magnetic dipole of moment

$$\mathbf{m} = q_m\mathbf{d} = \mathbf{a}_n IS. \tag{6–53}$$

The scalar magnetic potential V_m caused by this magnetic dipole can then be found by following the procedure used in Subsection 3–5.1 for finding the scalar electric potential that is caused by an electric dipole. We obtain, as in Eq. (3–53b),

$$V_m = \frac{\mathbf{m}\cdot\mathbf{a}_R}{4\pi R^2} \quad \text{(A)}. \tag{6–54}$$

Substitution of Eq. (6–54) in Eq. (6–50) yields the same **B** as is given in Eq. (6–48).

We note that the expression of the scalar magnetic potential V_m in Eq. (6–54) for a magnetic dipole is exactly analogous to that of the scalar electric potential V in Eq. (6–47) for an electric dipole. The likeness between the vector magnetic potential **A** in Eq. (6–45) and V in Eq. (6–47) is, however, not as exact. It is noted that the curl-free nature of **B** indicated in Eq. (6–49), from which the scalar magnetic potential V_m is defined, holds only at points with no currents. In a region where currents exist, the magnetic field is *not conservative*, and the scalar magnetic potential is not a single-valued function; hence the magnetic potential difference evaluated by Eq. (6–51) depends on the path of integration. For these reasons we will use the circulating-current-and-vector-potential approach, instead of the fictitious magnetic-charge-and-scalar-potential approach, for the study of magnetic fields in magnetic materials. We ascribe the macroscopic properties of a bar magnet to circulating atomic currents (Ampèrian currents) caused by orbiting and spinning electrons. Some aspects of equivalent (fictitious) magnetic charge densities will be discussed in Subsection 6–6.1.

6–6 Magnetization and Equivalent Current Densities

According to the elementary atomic model of matter, all materials are composed of atoms, each with a positively charged nucleus and a number of orbiting negatively charged electrons. The orbiting electrons cause circulating currents and form microscopic magnetic dipoles. In addition, both the electrons and the nucleus of an atom rotate (spin) on their own axes with certain magnetic dipole moments. The magnetic dipole moment of a spinning nucleus is usually negligible in comparison to that of an orbiting or spinning electron because of the much larger mass and lower angular velocity of the nucleus. A complete understanding of the magnetic effects of materials requires a knowledge of quantum mechanics. (We give a qualitative description of the behavior of different kinds of magnetic materials in Section 6–9.)

In the absence of an external magnetic field the magnetic dipoles of the atoms of most materials (except permanent magnets) have random orientations, resulting in no net magnetic moment. The application of an external magnetic field causes both an alignment of the magnetic moments of the spinning electrons and an induced magnetic moment due to a change in the orbital motion of electrons. To obtain a formula for determining the quantitative change in the magnetic flux density caused by the presence of a magnetic material, we let $\mathbf{m}_k$ be the magnetic dipole moment of an atom. If there are n atoms per unit volume, we define a ***magnetization vector***, $\mathbf{M}$, as

$$\mathbf{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\,\Delta v} \mathbf{m}_k}{\Delta v} \qquad \text{(A/m)}, \tag{6-55}$$

which is the volume density of magnetic dipole moment. The magnetic dipole moment $d\mathbf{m}$ of an elemental volume dv' is $d\mathbf{m} = \mathbf{M}\,dv'$ that, according to Eq. (6–45), will produce a vector magnetic potential

$$d\mathbf{A} = \frac{\mu_0 \mathbf{M} \times \mathbf{a}_R}{4\pi R^2}\,dv'. \tag{6-56}$$

Using Eq. (3–83), we can write Eq. (6–56) as

$$d\mathbf{A} = \frac{\mu_0}{4\pi}\mathbf{M} \times \nabla'\left(\frac{1}{R}\right) dv'.$$

Thus,

$$\mathbf{A} = \int_{V'} d\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \mathbf{M} \times \nabla'\left(\frac{1}{R}\right) dv', \tag{6-57}$$

where V' is the volume of the magnetized material.

We now use the vector identity in Eq. (6–29) to write

$$\mathbf{M} \times \nabla'\left(\frac{1}{R}\right) = \frac{1}{R}\nabla' \times \mathbf{M} - \nabla' \times \left(\frac{\mathbf{M}}{R}\right) \tag{6-58}$$

and expand the right side of Eq. (6–57) into two terms:

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\nabla' \times \mathbf{M}}{R}\,dv' - \frac{\mu_0}{4\pi}\int_{V'} \nabla' \times \left(\frac{\mathbf{M}}{R}\right) dv'. \tag{6-59}$$

The following vector identity (see Problem P.6–20) enables us to change the volume integral of the curl of a vector into a surface integral:

$$\int_{V'} \nabla' \times \mathbf{F}\,dv' = -\oint_{S'} \mathbf{F} \times d\mathbf{s}', \tag{6-60}$$

where $\mathbf{F}$ is any vector with continuous first derivatives. We have, from Eq. (6–59),

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\nabla' \times \mathbf{M}}{R}\,dv' + \frac{\mu_0}{4\pi}\oint_{S'} \frac{\mathbf{M} \times \mathbf{a}'_n}{R}\,ds', \tag{6-61}$$

where $\mathbf{a}'_n$ is the unit outward normal vector from ds' and S' is the surface bounding the volume V'.

A comparison of the expressions on the right side of Eq. (6-61) with the form of **A** in Eq. (6-23) expressed in terms of volume current density **J** suggests that the effect of the magnetization vector is equivalent to both a volume current density

$$\boxed{\mathbf{J}_m = \nabla \times \mathbf{M} \qquad (\text{A/m}^2)} \tag{6-62}$$

and a surface current density

$$\boxed{\mathbf{J}_{ms} = \mathbf{M} \times \mathbf{a}_n \qquad (\text{A/m}).} \tag{6-63}$$

In Eqs. (6-62) and (6-63) we have omitted the primes on ∇ and $\mathbf{a}_n$ for simplicity, since it is clear that both refer to the coordinates of the source point where the magnetization vector **M** exists. However, the primes should be retained when there is a possibility of confusing the coordinates of the source and field points.

The problem of finding the magnetic flux density **B** caused by a given volume density of magnetic dipole moment **M** is then reduced to finding the equivalent ***magnetization current densities*** $\mathbf{J}_m$ and $\mathbf{J}_{ms}$ by using Eqs. (6-62) and (6-63), determining **A** from Eq. (6-61), and then obtaining **B** from the curl of **A**. The externally applied magnetic field, if it also exists, must be accounted for separately.

The mathematical derivation of Eqs. (6-62) and (6-63) is straightforward. The equivalence of a volume density of magnetic dipole moment to a volume current density and a surface current density can be appreciated qualitatively by referring to Fig. 6-10, in which a cross section of a magnetized material is shown. It is assumed that an externally applied magnetic field has caused the atomic circulating currents to align with it, thereby magnetizing the material. The strength of this magnetizing effect is measured by the magnetization vector **M**. On the surface of the material there will be a surface current density $\mathbf{J}_{ms}$, whose direction is correctly given

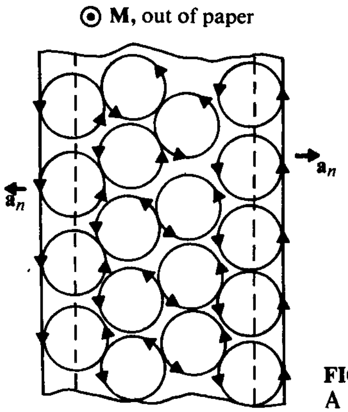


FIGURE 6-10
A cross section of a magnetized material.

by that of the cross product $\mathbf{M} \times \mathbf{a}_n$. If $\mathbf{M}$ is uniform inside the material, the currents of the neighboring atomic dipoles that flow in opposite directions will cancel everywhere, leaving no net currents in the interior. This is predicted by Eq. (6–62), since the space derivatives (and therefore the curl) of a constant $\mathbf{M}$ vanish. However, if $\mathbf{M}$ has space variations, the internal atomic currents do not completely cancel, resulting in a net volume current density $\mathbf{J}_m$. It is possible to justify the quantitative relationships between $\mathbf{M}$ and the current densities by deriving the atomic currents on the surface *and* in the interior. But since this additional derivation is really not necessary and tends to be tedious, we will not attempt it here.

EXAMPLE 6–8 Determine the magnetic flux density on the axis of a uniformly magnetized circular cylinder of a magnetic material. The cylinder has a radius b, length L, and axial magnetization $\mathbf{M} = \mathbf{a}_z M_0$.

Solution In this problem concerning a cylindrical bar magnet, let the axis of the magnetized cylinder coincide with the z-axis of a cylindrical coordinate system, as shown in Fig. 6–11. Since the magnetization $\mathbf{M}$ is a constant within the magnet, $\mathbf{J}_m = \nabla' \times \mathbf{M} = 0$, and there is no equivalent volume current density. The equivalent magnetization surface current density on the side wall is

$$\begin{aligned} \mathbf{J}_{ms} &= \mathbf{M} \times \mathbf{a}'_n = (\mathbf{a}_z M_0) \times \mathbf{a}_r \\ &= \mathbf{a}_\phi M_0. \end{aligned} \quad (6\text{–}64)$$

The magnet is then like a cylindrical sheet with a lineal current density of M_0 (A/m). There is no surface current on the top and bottom faces. To find $\mathbf{B}$ at $P(0, 0, z)$, we consider a differential length dz' with a current $\mathbf{a}_\phi M_0\, dz'$ and use Eq. (6–38) to obtain

$$d\mathbf{B} = \mathbf{a}_z \frac{\mu_0 M_0 b^2\, dz'}{2[(z - z')^2 + b^2]^{3/2}}$$

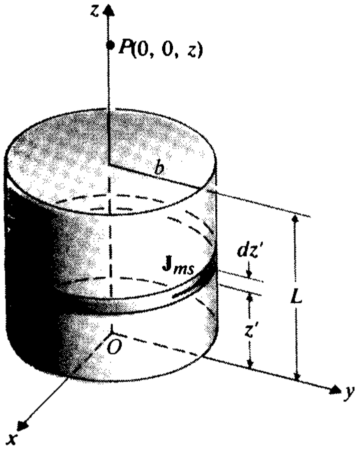


FIGURE 6–11
A uniformly magnetized circular cylinder (Example 6–8).

and

$$\mathbf{B} = \int d\mathbf{B} = \mathbf{a}_z \int_0^L \frac{\mu_0 M_0 b^2\, dz'}{2[(z-z')^2 + b^2]^{3/2}}$$
$$= \mathbf{a}_z \frac{\mu_0 M_0}{2}\left[\frac{z}{\sqrt{z^2+b^2}} - \frac{z-L}{\sqrt{(z-L)^2+b^2}}\right]. \tag{6–65}$$

6–6.1 EQUIVALENT MAGNETIZATION CHARGE DENSITIES

In subsection 6–5.1 we noted that in a current-free region we may define a scalar magnetic potential V_m, from which the magnetic flux density **B** can be found by differentiation, as in Eq. (6–50). In terms of magnetization vector **M** (volume density of magnetic dipole moment) we may write, in lieu of Eq. (6–54),

$$dV_m = \frac{\mathbf{M}\cdot\mathbf{a}_R}{4\pi R^2}. \tag{6–66}$$

Integrating Eq. (6–66) over a magnetized body (a magnet) carrying no current, we have

$$V_m = \frac{1}{4\pi}\int_{V'} \frac{\mathbf{M}\cdot\mathbf{a}_R}{R^2}\, dv'. \tag{6–67}$$

Equation (6–67) is of exactly the same form as Eq. (3–81) for the scalar electric potential of a polarized dielectric. Following the steps leading to Eq. (3–87), we obtain

$$V_m = \frac{1}{4\pi}\oint_{S'} \frac{\mathbf{M}\cdot\mathbf{a}_n'}{R}\, ds' + \frac{1}{4\pi}\int_{V'} \frac{-(\nabla'\cdot\mathbf{M})}{R}\, dv', \tag{6–68}$$

where $\mathbf{a}_n'$ is the outward normal to the surface element ds' of the magnetized body. We saw in Section 3–7 that, for field calculations, a polarized dielectric may be replaced by an equivalent polarization surface charge density, given in Eq. (3–88), and an equivalent polarization volume charge density, given in Eq. (3–89). Similarly, we can conclude that, for field calculations, a magnetized body may be replaced by an equivalent (fictitious) magnetization surface charge density ρ_{ms} and an equivalent (fictitious) magnetization volume charge density ρ_m such that

$$\boxed{\rho_{ms} = \mathbf{M}\cdot\mathbf{a}_n \quad (\text{A/m})} \tag{6–69}$$

and

$$\boxed{\rho_m = -\nabla\cdot\mathbf{M} \quad (\text{A/m}^2).} \tag{6–70}$$

The use of the equivalent magnetization charge density concept for determining the magnetic flux density of a magnetized body will be illustrated in the following example.

EXAMPLE 6–9 A cylindrical bar magnet of radius b and length L has a uniform magnetization $\mathbf{M} = \mathbf{a}_z M_0$ along its axis. Use the equivalent magnetization charge density concept to determine the magnetic flux density at an arbitrary distant point.

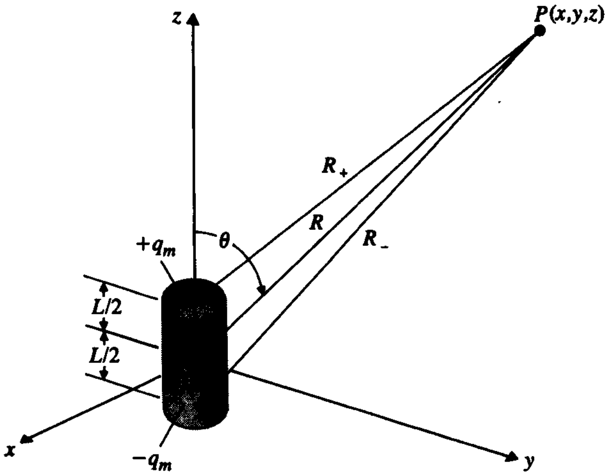


FIGURE 6–12
A cylindrical bar magnet (Example 6–9).

Solution Refer to Fig. 6–12. The equivalent magnetization charge densities for $\mathbf{M} = \mathbf{a}_z M_0$ are, according to Eqs. (6–69) and (6–70):

$$\rho_{ms} = \begin{cases} M_0 & \text{on top face,} \\ -M_0 & \text{on bottom face,} \\ 0 & \text{on side wall;} \end{cases}$$

$$\rho_m = 0 \qquad \text{in the interior.}$$

At a distant point the total equivalent magnetic charges on the top and bottom faces appear as point charges: $q_m = \pi b^2 \rho_{ms} = \pi b^2 M_0$. We have at $P(x, y, z)$

$$V_m = \frac{q_m}{4\pi}\left(\frac{1}{R_+} - \frac{1}{R_-}\right) \qquad \text{(A)}, \tag{6–71}$$

which is similar to Eq. (3–50) for an electric dipole. If $R \gg b$, Eq. (6–71) can be reduced to (see Eq. 3–53a)

$$V_m = \frac{q_m L \cos\theta}{4\pi R^2} = \frac{(\pi b^2 M_0) L \cos\theta}{4\pi R^2}$$
$$= \frac{M_T \cos\theta}{4\pi R^2}, \tag{6–72}$$

where $M_T = \pi b^2 L M_0$ is the total dipole moment of the cylindrical magnet. The magnetic flux density **B** can then be found by applying Eq. (6–50):

$$\mathbf{B} = -\mu_0 \nabla V_m = \frac{\mu_0 M_T}{4\pi R^3}(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta) \qquad \text{(T)}, \tag{6–73}$$

which is of the same form as the expression in Eq. (6–44) for **B** at a distant point due to a single magnetic dipole having a moment $I\pi b^2$. ■

This problem can be solved just as easily by using the equivalent magnetization current density concept. (See Problem P.6–25.)

6–7 Magnetic Field Intensity and Relative Permeability

Because the application of an external magnetic field causes both an alignment of the internal dipole moments and an induced magnetic moment in a magnetic material, we expect that the resultant magnetic flux density in the presence of a magnetic material will be different from its value in free space. The macroscopic effect of magnetization can be studied by incorporating the equivalent volume current density, $\mathbf{J}_m$ in Eq. (6–62), into the basic curl equation, Eq. (6–7). We have

$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{J} + \mathbf{J}_m = \mathbf{J} + \nabla \times \mathbf{M}$$

or

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M}\right) = \mathbf{J}. \tag{6–74}$$

We now define a new fundamental field quantity, the ***magnetic field intensity*** **H**, such that

$$\boxed{\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \qquad \text{(A/m)}.} \tag{6–75}$$

The use of the vector **H** enables us to write a curl equation relating the magnetic field and the distribution of free currents in any medium. There is no need to deal explicitly with the magnetization vector **M** or the equivalent volume current density $\mathbf{J}_m$. Combining Eqs. (6–74) and (6–75), we obtain the new equation

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} \qquad (\text{A/m}^2),} \tag{6–76}$$

where **J** (A/M^2) is the volume density of *free current*. Equations (6–6) and (6–76) are the two fundamental governing differential equations for magnetostatics. The permeability of the medium does not appear explicitly in these two equations.

The corresponding integral form of Eq. (6–76) is obtained by taking the scalar surface integral of both sides:

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = \int_S \mathbf{J} \cdot d\mathbf{s} \tag{6–77}$$

or, according to Stokes's theorem,

$$\boxed{\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I \qquad \text{(A)},} \tag{6-78}$$

where C is the contour (closed path) bounding the surface S and I is the total free current passing through S. The relative directions of C and current flow I follow the right-hand rule. Equation (6–78) is another form of ***Ampère's circuital law***: It states that ***the circulation of the magnetic field intensity around any closed path is equal to the free current flowing through the surface bounded by the path.*** As we indicated in Section 6–2, Ampère's circuital law is most useful in determining the magnetic field caused by a current when cylindrical symmetry exists—that is, when there is a closed path around the current over which the magnetic field is constant.

When the magnetic properties of the medium are *linear* and *isotropic*, the magnetization is directly proportional to the magnetic field intensity:

$$\mathbf{M} = \chi_m \mathbf{H}, \tag{6-79}$$

where χ_m is a dimensionless quantity called ***magnetic susceptibility***. Substitution of Eq. (6–79) in Eq. (6–75) yields

$$\boxed{\begin{aligned} \mathbf{B} &= \mu_0(1 + \chi_m)\mathbf{H} \\ &= \mu_0\mu_r\mathbf{H} = \mu\mathbf{H} \qquad \text{(Wb/m}^2\text{)} \end{aligned}} \tag{6-80a}$$

or

$$\boxed{\mathbf{H} = \frac{1}{\mu}\mathbf{B} \qquad \text{(A/m)},} \tag{6-80b}$$

where

$$\boxed{\mu_r = 1 + \chi_m = \frac{\mu}{\mu_0}} \tag{6-81}$$

is another dimensionless quantity known as the ***relative permeability*** of the medium. The parameter $\mu = \mu_0\mu_r$ is the ***absolute permeability*** (or sometimes just ***permeability***) of the medium and is measured in H/m; χ_m, and therefore μ_r, can be a function of space coordinates. For a simple medium—linear, isotropic, and homogeneous—χ_m and μ_r are constants.

The permeability of most materials is very close to that of free space (μ_0). For ferromagnetic materials such as iron, nickel, and cobalt, μ_r could be very large (50–5000 and up to 10^6 or more for special alloys); the permeability depends not only on the magnitude of **H** but also on the previous history of the material. Section 6–9 contains some qualitative discussions of the macroscopic behavior of magnetic materials.

At this point we note a number of analogous relations between the quantities in electrostatics and those in magnetostatics as follows:

Electrostatics	Magnetostatics
$\mathbf{E}$	$\mathbf{B}$
$\mathbf{D}$	$\mathbf{H}$
ϵ	$\frac{1}{\mu}$
$\mathbf{P}$	$-\mathbf{M}$
ρ	$\mathbf{J}$
V	$\mathbf{A}$
$\cdot$	$\times$
$\times$	$\cdot$

With the above table, most of the equations relating the basic quantities in electrostatics can be converted into corresponding analogous ones in magnetostatics.

6–8 Magnetic Circuits

In electric-circuit problems we are required to find the voltages across and the currents in various branches and elements of an electric network that are excited by voltage and/or current sources. There is an analogous class of problems dealing with magnetic circuits. In a magnetic circuit we are generally concerned with the determination of the magnetic fluxes and magnetic field intensities in various parts of a circuit caused by windings carrying currents around ferromagnetic cores. Magnetic circuit problems arise in transformers, generators, motors, relays, magnetic recording devices, and so on.

Analysis of magnetic circuits is based on the two basic equations for magnetostatics, (6–6) and (6–76), which are repeated below for convenience:

$$\nabla \cdot \mathbf{B} = 0, \tag{6–82}$$

$$\nabla \times \mathbf{H} = \mathbf{J}. \tag{6–83}$$

We have seen in Eq. (6–78) that Eq. (6–83) converts to Ampère's circuital law. If the closed path C is chosen to enclose N turns of a winding carrying a current I that excites a magnetic circuit, we have

$$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = NI = \mathscr{V}_m. \tag{6–84}$$

The quantity $\mathscr{V}_m$ $(=NI)$ here plays a role that is analogous to electromotive force (emf) in an electric circuit and is therefore called a ***magnetomotive force*** (mmf). Its SI unit is ampere (A); but, because of Eq. (6–84), mmf is frequently measured in ampere-turns (A·t). An mmf is *not* a force measured in newtons.

EXAMPLE 6–10 Assume that N turns of wire are wound around a toroidal core of a ferromagnetic material with permeability μ. The core has a mean radius r_o, a circular cross section of radius a ($a \ll r_o$), and a narrow air gap of length ℓ_g, as shown in Fig. 6–13. A steady current I_o flows in the wire. Determine (a) the magnetic flux density, $\mathbf{B}_f$, in the ferromagnetic core; (b) the magnetic field intensity, $\mathbf{H}_f$, in the core; and (c) the magnetic field intensity, $\mathbf{H}_g$, in the air gap.

Solution

a) Applying Ampère's circuital law, Eq. (6–84), around the circular contour C in Fig. 6–13, which has a mean radius r_o, we have

$$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = NI_o. \tag{6–85}$$

If flux leakage is neglected, the same total flux will flow in both the ferromagnetic core and in the air gap. If the fringing effect of the flux in the air gap is also neglected, the magnetic flux density $\mathbf{B}$ in both the core and the air gap will also be the same. However, because of the different permeabilities, the magnetic field intensities in both parts will be different. We have

$$\mathbf{B}_f = \mathbf{B}_g = \mathbf{a}_\phi B_f, \tag{6–86}$$

where the subscripts f and g denote ferromagnetic and gap, respectively. In the ferromagnetic core,

$$\mathbf{H}_f = \mathbf{a}_\phi \frac{B_f}{\mu}; \tag{6–87}$$

and, in the air gap,

$$\mathbf{H}_g = \mathbf{a}_\phi \frac{B_f}{\mu_0}. \tag{6–88}$$

Substituting Eqs. (6–87) and (6–88) in Eq. (6–85), we obtain

$$\frac{B_f}{\mu}(2\pi r_o - \ell_g) + \frac{B_f}{\mu_0}\ell_g = NI_o$$

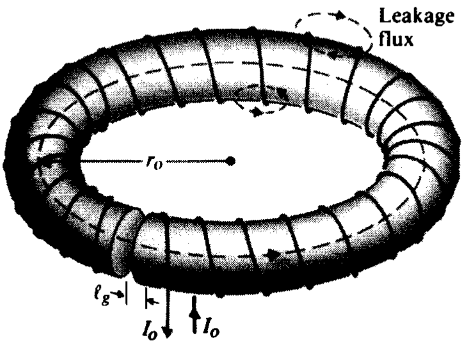


FIGURE 6–13
Coil on ferromagnetic toroid with air gap (Example 6–10).

and

$$\mathbf{B}_f = \mathbf{a}_\phi \frac{\mu_0 \mu N I_o}{\mu_0(2\pi r_o - \ell_g) + \mu \ell_g}. \tag{6-89}$$

b) From Eqs. (6–87) and (6–89) we get

$$\mathbf{H}_f = \mathbf{a}_\phi \frac{\mu_0 N I_o}{\mu_0(2\pi r_o - \ell_g) + \mu \ell_g}. \tag{6-90}$$

c) Similarly, from Eqs. (6–88) and (6–89) we have

$$\mathbf{H}_g = \mathbf{a}_\phi \frac{\mu N I_o}{\mu_0(2\pi r_o - \ell_g) + \mu \ell_g}. \tag{6-91}$$

Since $H_g/H_f = \mu/\mu_0$, the magnetic field intensity in the air gap is much stronger than that in the ferromagnetic core. ■

If the radius of the cross section of the core is much smaller than the mean radius of the toroid, the magnetic flux density **B** in the core is approximately constant, and the magnetic flux in the circuit is

$$\Phi \cong BS, \tag{6-92}$$

where S is the cross-sectional area of the core. Combination of Eqs. (6–92) and (6–89) yields

$$\Phi = \frac{N I_o}{(2\pi r_o - \ell_g)/\mu S + \ell_g/\mu_0 S}. \tag{6-93}$$

Equation (6–93) can be rewritten as

$$\Phi = \frac{\mathscr{V}_m}{\mathscr{R}_f + \mathscr{R}_g}, \tag{6-94}$$

with

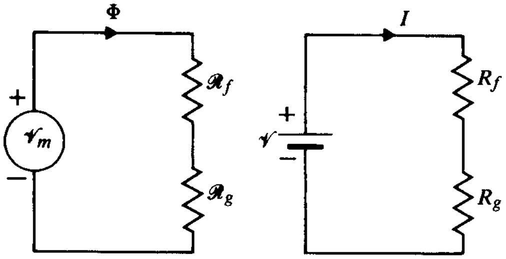
$$\mathscr{R}_f = \frac{2\pi r_o - \ell_g}{\mu S} = \frac{\ell_f}{\mu S}, \tag{6-95}$$

where $\ell_f = 2\pi r_o - \ell_g$ is the length of the ferromagnetic core, and

$$\mathscr{R}_g = \frac{\ell_g}{\mu_0 S}. \tag{6-96}$$

Both $\mathscr{R}_f$ and $\mathscr{R}_g$ have the same form as the formula, Eq. (5–27), for the d-c resistance of a straight piece of homogeneous material with a uniform cross section S. Both are called ***reluctance***: $\mathscr{R}_f$, of the ferromagnetic core; and $\mathscr{R}_g$, of the air gap. The SI unit for reluctance is reciprocal henry (H^{-1}). The fact that Eqs. (6–95) and (6–96) are as they are, even though the core is not straight, is a consequence of assuming that **B** is approximately constant over the core cross section.

Equation (6–94) is analogous to the expression for the current I in an electric circuit, in which an ideal voltage source of emf $\mathscr{V}$ is connected in series with two



(a) Magnetic circuit. (b) Electric circuit.

FIGURE 6–14
Equivalent magnetic circuit and analogous electric circuit for toroidal coil with air gap in Fig. 6–13.

resistances R_f and R_g:

$$I = \frac{\mathscr{V}}{R_f + R_g}. \tag{6–97}$$

The analogous magnetic and electric circuits are shown in Figs. 6–14(a) and 6–14(b), respectively. Magnetic circuits can, by analogy, be analyzed by the same techniques we have used in analyzing electric circuits. The analogous quantities are as follows:

Magnetic Circuits	Electric Circuits
mmf, $\mathscr{V}_m$ $(=NI)$	emf, $\mathscr{V}$
magnetic flux, Φ	electric current, I
reluctance, $\mathscr{R}$	resistance, R
permeability, μ	conductivity, σ

In spite of this convenient likeness an exact analysis of magnetic circuits is inherently very difficult to achieve.

First, it is very difficult to account for leakage fluxes, fluxes that stray or leak from the main flux paths of a magnetic circuit. For the toroidal coil in Fig. 6–13, leakage flux paths encircle every turn of the winding; they partially transverse the space around the core, as illustrated, because the permeability of air is not zero. (There is little need for considering leakage currents outside the conducting paths of electric circuits that carry direct currents. The reason is that the conductivity of air is practically zero compared to that of a good conductor.)

A second difficulty is the fringing effect that causes the magnetic flux lines at the air gap to spread and bulge.[†] (The purpose of specifying the "narrow air gap" in Example 6–10 was to minimize this fringing effect.)

[†] To obtain a more accurate numerical result, it is customary to consider the effective area of the air gap as slightly larger than the cross-sectional area of the ferromagnetic core, with each of the lineal dimensions of the core cross section increased by the length of the air gap. If we were to make a correction like this in Eq. (6–86), B_g would become

$$B_g = \frac{a^2 B_f}{(a + \ell_g)^2} < B_f.$$

A third difficulty is that the permeability of ferromagnetic materials depends on the magnetic field intensity; that is, **B** and **H** have a nonlinear relationship. (They might not even be in the same direction). The problem of Example 6–10, which assumes a given μ before either $\mathbf{B}_c$ or $\mathbf{H}_c$ is known, is therefore not a realistic one.

In a practical problem the B–H curve of the ferromagnetic material, such as that shown later in Fig. 6–17, should be given. The ratio of B to H is obviously not a constant, and B_f can be known only when H_f is known. So how does one solve the problem? Two conditions must be satisfied. First, the sum of $H_g\ell_g$ and $H_f\ell_f$ must equal the total mmf NI_o:

$$H_g\ell_g + H_f\ell_f = NI_o. \tag{6–98}$$

Second, if we assume no leakage flux, the total flux Φ in the ferromagnetic core and in the air gap must be the same, or $B_f = B_g$:†

$$B_f = \mu_0 H_g. \tag{6–99}$$

Substitution of Eq. (6–99) in Eq. (6–98) yields an equation relating B_f and H_f in the core:

$$B_f + \mu_0 \frac{\ell_f}{\ell_g} H_f = \frac{\mu_0}{\ell_g} NI_o. \tag{6–100}$$

This is an equation for a straight line in the B–H plane with a negative slope $(-\mu_0\ell_f/\ell_g)$. The intersection of this line and the given B–H curve determines the operating point. Once the operating point has been found, μ and H_f and all other quantities can be obtained.

The similarity between Eqs. (6–94) and (6–97) can be extended to the writing of two basic equations for magnetic circuits that correspond to Kirchhoff's voltage and current laws for electric circuits. Similar to Kirchhoff's voltage law in Eq. (5–41), we may write, for any closed path in a magnetic circuit,

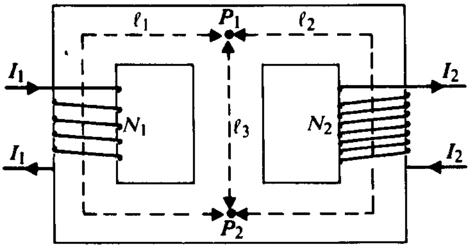
$$\boxed{\sum_j N_j I_j = \sum_k \mathscr{R}_k \Phi_k.} \tag{6–101}$$

Equation (6–101) states that ***around a closed path in a magnetic circuit the algebraic sum of ampere-turns is equal to the algebraic sum of the products of the reluctances and fluxes.***

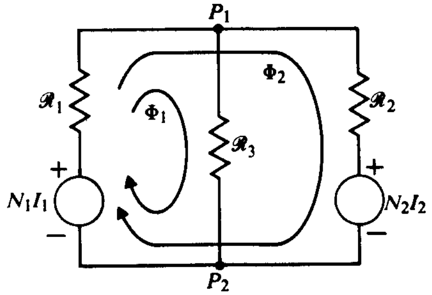
Kirchhoff's current law for a junction in an electric circuit, Eq. (5–47), is a consequence of $\nabla \cdot \mathbf{J} = 0$. Similarly, the fundamental postulate $\nabla \cdot \mathbf{B} = 0$ in Eq. (6–82) leads to Eq. (6–9). Thus, we have

$$\boxed{\sum_j \Phi_j = 0,} \tag{6–102}$$

† This assumes an equal cross-sectional area for the core and the gap. If the core were to be constructed of insulated laminations of ferromagnetic material, the effective area for flux passage in the core would be smaller than the geometrical cross-sectional area, and B_c would be larger than B_g by a factor. This factor can be determined from the data on the insulated laminations.



(a) Magnetic core with current-carrying windings.



(b) Magnetic circuit for loop analysis.

FIGURE 6-15
A magnetic circuit (Example 6-11).

which states that ***the algebraic sum of all the magnetic fluxes flowing out of a junction in a magnetic circuit is zero.*** Equations (6-101) and (6-102) form the bases for the loop and node analysis, respectively, of magnetic circuits.

EXAMPLE 6-11 Consider the magnetic circuit in Fig. 6-15(a). Steady currents I_1 and I_2 flow in windings of N_1 and N_2 turns, respectively, on the outside legs of the ferromagnetic core. The core has a cross-sectional area S_c and a permeability μ. Determine the magnetic flux in the center leg.

Solution The equivalent magnetic circuit for loop analysis is shown in Fig. 6-15(b). Two sources of mmf's, N_1I_1 and N_2I_2, are shown with proper polarities in series with reluctances $\mathscr{R}_1$ and $\mathscr{R}_2$, respectively. This is obviously a two-loop network. Since we are determining magnetic flux in the center leg P_1P_2, it is expedient to choose the two loops in such a way that only one loop flux (Φ_1) flows through the center leg. The reluctances are computed on the basis of average path lengths. These are, of course, approximations. We have

$$\mathscr{R}_1 = \frac{\ell_1}{\mu S_c}, \tag{6-103a}$$

$$\mathscr{R}_2 = \frac{\ell_2}{\mu S_c}, \tag{6-103b}$$

$$\mathscr{R}_3 = \frac{\ell_3}{\mu S_c}. \tag{6-103c}$$

The two loop equations are, from Eq. (6-101),

$$\textit{Loop 1:} \quad N_1I_1 = (\mathscr{R}_1 + \mathscr{R}_3)\Phi_1 + \mathscr{R}_1\Phi_2; \tag{6-104}$$

$$\textit{Loop 2:} \quad N_1I_1 - N_2I_2 = \mathscr{R}_1\Phi_1 + (\mathscr{R}_1 + \mathscr{R}_2)\Phi_2. \tag{6-105}$$

Solving these simultaneous equations, we obtain

$$\Phi_1 = \frac{\mathscr{R}_2 N_1 I_1 + \mathscr{R}_1 N_2 I_2}{\mathscr{R}_1\mathscr{R}_2 + \mathscr{R}_1\mathscr{R}_3 + \mathscr{R}_2\mathscr{R}_3}, \tag{6–106}$$

which is the desired answer. ■

Actually, since the magnetic fluxes and therefore the magnetic flux densities in the three legs are different, different permeabilities should be used in computing the reluctances in Eqs. (6–103a), (6–103b), and (6–103c). But the value of permeability, in turn, depends on the magnetic flux density. The only way to improve the accuracy of the solution, provided that the B–H curve of the core material is given, is to use a procedure of successive approximation. For instance, Φ_1, Φ_2, and Φ_3 (and therefore B_1, B_2, and B_3) are first solved with an assumed μ and reluctances computed from the three parts of Eq. (6–103). From B_1, B_2, and B_3 the corresponding μ_1, μ_2, and μ_3 can be found from the B–H curve. These will modify the reluctances. A second approximation for B_1, B_2, and B_3 is then obtained with the modified reluctances. From the new flux densities, new permeabilities and new reluctances are determined. This procedure is repeated until further iterations bring little change in the computed values.

We remark here that the currents in the windings in Fig. 6–15(a) are independent of time and that Example 6–11 is strictly a d-c magnetic circuit problem. If the currents vary with time, we must deal with the effects of electromagnetic induction, and we will have a transformer problem. Other fundamental laws are involved, which we shall discuss in Chapter 7.

6–9 Behavior of Magnetic Materials

In Eq. (6–79), Section 6–7, we described the macroscopic magnetic property of a linear, isotropic medium by defining the magnetic susceptibility χ_m, a dimensionless coefficient of proportionality between magnetization **M** and magnetic field intensity **H**. The relative permeability μ_r is simply $1 + \chi_m$. Magnetic materials can be roughly classified into three main groups in accordance with their μ_r values. A material is said to be

Diamagnetic, if $\mu_r \lesssim 1$ (χ_m is a very small negative number).

Paramagnetic, if $\mu_r \gtrsim 1$ (χ_m is a very small positive number).

Ferromagnetic, if $\mu_r \gg 1$ (χ_m is a large positive number).

As mentioned before, a thorough understanding of microscopic magnetic phenomena requires a knowledge of quantum mechanics. In the following we give a qualitative description of the behavior of the various types of magnetic materials based on the classical atomic model.

In a *diamagnetic* material the net magnetic moment due to the orbital and spinning motions of the electrons in any particular atom is zero in the absence of an externally applied magnetic field. As predicted by Eq. (6–4), the application of an external magnetic field to this material produces a force on the orbiting electrons, causing a perturbation in the angular velocities. As a consequence, a net magnetic moment is created. This is a process of induced magnetization. According to ***Lenz's law*** of electromagnetic induction (Section 7–2), the induced magnetic moment always *opposes* the applied field, thus reducing the magnetic flux density. The macroscopic effect of this process is equivalent to that of a negative magnetization that can be described by a negative magnetic susceptibility. This effect is usually very small, and χ_m for most known diamagnetic materials (bismuth, copper, lead, mercury, germanium, silver, gold, diamond) is of the order of -10^{-5}.

Diamagnetism arises mainly from the orbital motion of the electrons within an atom and is present in all materials. In most materials it is too weak to be of any practical importance. The diamagnetic effect is masked in paramagnetic and ferromagnetic materials. Diamagnetic materials exhibit no permanent magnetism, and the induced magnetic moment disappears when the applied field is withdrawn.

In some materials the magnetic moments due to the orbiting and spinning electrons do not cancel completely, and the atoms and molecules have a net average magnetic moment. An externally applied magnetic field, in addition to causing a very weak diamagnetic effect, tends to align the molecular magnetic moments *in the direction of* the applied field, thus increasing the magnetic flux density. The macroscopic effect is, then, equivalent to that of a positive magnetization that is described by a positive magnetic susceptibility. The alignment process is, however, impeded by the forces of random thermal vibrations. There is little coherent interaction, and the increase in magnetic flux density is quite small. Materials with this behavior are said to be *paramagnetic*. Paramagnetic materials generally have very small positive values of magnetic susceptibility, of the order of 10^{-5} for aluminum, magnesium, titanium, and tungsten.

Paramagnetism arises mainly from the magnetic dipole moments of the spinning electrons. The alignment forces, acting upon molecular dipoles by the applied field, are counteracted by the deranging effects of thermal agitation. Unlike diamagnetism, which is essentially independent of temperature, the paramagnetic effect is temperature dependent, being stronger at lower temperatures where there is less thermal collision.

The magnetization of *ferromagnetic* materials can be many orders of magnitude larger than that of paramagnetic substances. (See Appendix B–5 for typical values of relative permeability.) ***Ferromagnetism*** can be explained in terms of magnetized ***domains***. According to this model, which has been experimentally confirmed, a ferromagnetic material (such as cobalt, nickel, and iron) is composed of many small domains, their linear dimensions ranging from a few microns to about 1 mm. These domains, each containing about 10^{15} or 10^{16} atoms, are fully magnetized in the sense that they contain aligned magnetic dipoles resulting from spinning electrons even in the absence of an applied magnetic field. Quantum theory asserts that strong coupling forces exist between the magnetic dipole moments of the atoms in a domain, holding the dipole moments in parallel. Between adjacent domains there is a transition region

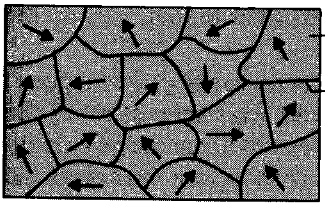


FIGURE 6–16
Domain structure of a polycrystalline ferromagnetic specimen.

about 100 atoms thick called a ***domain wall.*** In an unmagnetized state the magnetic moments of the adjacent domains in a ferromagnetic material have different directions, as exemplified in Fig. 6–16 by the polycrystalline specimen shown. Viewed as a whole, the random nature of the orientations in the various domains results in no net magnetization.

When an external magnetic field is applied to a ferromagnetic material, the walls of those domains having magnetic moments aligned with the applied field move in such a way as to make the volumes of those domains grow at the expense of other domains. As a result, magnetic flux density is increased. For weak applied fields, say up to point P_1 in Fig. 6–17, domain-wall movements are reversible. But when an applied field becomes stronger (past P_1), domain-wall movements are no longer reversible, and domain rotation toward the direction of the applied field will also occur. For example, if an applied field is reduced to zero at point P_2, the B–H relationship will not follow the solid curve P_2P_1O, but will go down from P_2 to P_2', along the lines of the broken curve in the figure. This phenomenon of magnetization lagging behind the field producing it is called ***hysteresis***, which is derived from a Greek word meaning "to lag." As the applied field becomes even much stronger (past P_2 to P_3), domain-wall motion and domain rotation will cause essentially a total alignment of the microscopic magnetic moments with the applied field, at which point

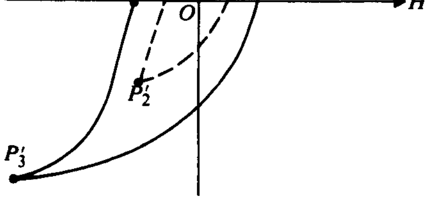


FIGURE 6–17
Hysteresis loops in the B–H plane for ferromagnetic material.

the magnetic material is said to have reached *saturation*. The curve $OP_1P_2P_3$ on the B–H plane is called the ***normal magnetization curve***.

If the applied magnetic field is reduced to zero from the value at P_3, the magnetic flux density does not go to zero but assumes the value at B_r. This value is called the ***residual*** or ***remanent flux density*** (in Wb/m^2) and is dependent on the maximum applied field intensity. The existence of a remanent flux density in a ferromagnetic material makes permanent magnets possible.

To make the magnetic flux density of a specimen zero, it is necessary to apply a magnetic field intensity H_c in the opposite direction. This required H_c is called *coercive force*, but a more appropriate name is ***coercive field intensity*** (in A/m). Like B_r, H_c also depends on the maximum value of the applied magnetic field intensity.

It is evident from Fig. 6–17 that the B–H relationship for a ferromagnetic material is nonlinear. Hence, if we write $\mathbf{B} = \mu\mathbf{H}$ as in Eq. (6–80a), the permeability μ itself is a function of the magnitude of $\mathbf{H}$. Permeability μ also depends on the history of the material's magnetization, since—even for the same $\mathbf{H}$—we must know the location of the operating point on a particular branch of a particular hysteresis loop in order to determine the value of μ exactly. In some applications a small alternating current may be superimposed on a large steady magnetizing current. The steady magnetizing field intensity locates the operating point, and the local slope of the hysteresis curve at the operating point determines the ***incremental permeability***.

Ferromagnetic materials for use in electric generators, motors, and transformers should have a large magnetization for a very small applied field; they should have tall, narrow hysteresis loops. As the applied magnetic field intensity varies periodically between $\pm H_{max}$, the hysteresis loop is traced once per cycle. The area of the hysteresis loop corresponds to energy loss (***hysteresis loss***) per unit volume per cycle (Problem P.6–29). Hysteresis loss is the energy lost in the form of heat in overcoming the friction encountered during domain-wall motion and domain rotation. Ferromagnetic materials, which have tall, narrow hysteresis loops with small loop areas, are referred to as "soft" materials; they are usually well-annealed materials with very few dislocations and impurities so that the domain walls can move easily.

Good permanent magnets, on the other hand, should show a high resistance to demagnetization. This requires that they be made with materials that have large coercive field intensities H_c and hence fat hysteresis loops. These materials are referred to as "hard" ferromagnetic materials. The coercive field intensity of hard ferromagnetic materials (such as Alnico alloys) can be 10^5 (A/m) or more, whereas that for soft materials is usually 50 (A/m) or less.

As indicated before, ferromagnetism is the result of strong coupling effects between the magnetic dipole moments of the atoms in a domain. Figure 6–18(a) depicts the atomic spin structure of a ferromagnetic material. When the temperature of a ferromagnetic material is raised to such an extent that the thermal energy exceeds the coupling energy, the magnetized domains become disorganized. Above this critical temperature, known as the ***curie temperature***, a ferromagnetic material behaves like a paramagnetic substance. Hence, when a permanent magnet is heated above its curie temperature it loses its magnetization. The curie temperature of most ferromagnetic materials lies between a few hundred to a thousand degrees Celsius, that of iron being 770°C.

Some elements, such as chromium and manganese, which are close to ferromagnetic elements in atomic number and are neighbors of iron in the periodic table, also have strong coupling forces between the atomic magnetic dipole moments; but their coupling forces produce antiparallel alignments of electron spins, as illustrated in Fig. 6–18(b). The spins alternate in direction from atom to atom and result in no net magnetic moment. A material possessing this property is said to be ***antiferromagnetic***. Antiferromagnetism is also temperature dependent. When an antiferromagnetic material is heated above its curie temperature, the spin directions suddenly become random, and the material becomes paramagnetic.

There is another class of magnetic materials that exhibit a behavior between ferromagnetism and antiferromagnetism. Here quantum mechanical effects make the directions of the magnetic moments in the ordered spin structure alternate and the magnitudes unequal, resulting in a net nonzero magnetic moment, as depicted in Fig. 6–18(c). These materials are said to be ***ferrimagnetic***. Because of the partial cancellation, the maximum magnetic flux density attained in a ferrimagnetic substance is substantially lower than that in a ferromagnetic specimen. Typically, it is about 0.3 Wb/m^2, approximately one-tenth that for ferromagnetic substances.

Ferrites are a subgroup of ferrimagnetic material. One type of ferrites, called ***magnetic spinels***, crystallize in a complicated spinel structure and have the formula $XO \cdot Fe_2O_3$, where X denotes a divalent metallic ion such as Fe, Co, Ni, Mn, Mg, Zn, Cd, etc. These are ceramiclike compounds with very low conductivities (for instance, 10^{-4} to 1 (S/m) compared with 10^7 (S/m) for iron). Low conductivity limits eddy-current losses at high frequencies. Hence ferrites find extensive uses in such high-frequency and microwave applications as cores for FM antennas, high-frequency transformers, and phase shifters. Ferrite material also has broad applications in

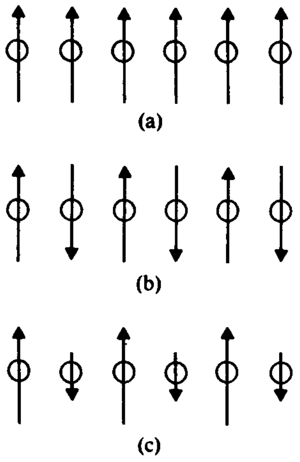


FIGURE 6–18
Schematic atomic spin structures for (a) ferromagnetic, (b) antiferromagnetic, and (c) ferrimagnetic materials.

computer magnetic-core and magnetic-disk memory devices. Other ferrites include magnetic-oxide garnets, of which yttrium-iron-garnet ("YIG," $Y_3Fe_5O_{12}$) is typical. Garnets are used in microwave multiport junctions.

Ferrites are anisotropic in the presence of a magnetic field. This means that **H** and **B** vectors in ferrites generally have different directions, and permeability is a tensor. The relation between the components of **H** and **B** can be represented in a matrix form similar to that between the components of **D** and **E** in an anisotropic dielectric medium, as given in Eq. (3–104) or Eq. (3–105). Analysis of problems containing anisotropic and/or nonlinear media is beyond the scope of this book.

6–10 Boundary Conditions for Magnetostatic Fields

In order to solve problems concerning magnetic fields in regions having media with different physical properties, it is necessary to study the conditions (boundary conditions) that **B** and **H** vectors must satisfy at the interfaces of different media. Using techniques similar to those employed in Section 3–9 to obtain the boundary conditions for electrostatic fields, we derive magnetostatic boundary conditions by applying the two fundamental governing equations, Eqs. (6–82) and (6–83), to a small pillbox and a small closed path, respectively, which include the interface. From the divergenceless nature of the **B** field in Eq. (6–82) we may conclude directly, in light of past experience, that ***the normal component of* B *is continuous across an interface***; that is,

$$\boxed{B_{1n} = B_{2n} \qquad \text{(T)}.} \tag{6–107}$$

For linear media, $\mathbf{B}_1 = \mu_1 \mathbf{H}_1$ and $\mathbf{B}_2 = \mu_2 \mathbf{H}_2$, Eq. (6–107) becomes

$$\boxed{\mu_1 H_{1n} = \mu_2 H_{2n}.} \tag{6–108}$$

The boundary condition for the tangential components of magnetostatic field is obtained from the integral form of the curl equation for **H**, Eq. (6–78), which is repeated here for convenience:

$$\oint_C \mathbf{H} \cdot d\ell = I. \tag{6–109}$$

We now choose the closed path *abcda* in Fig. 6–19 as the contour *C*. Applying Eq. (6–109) and letting $bc = da = \Delta h$ approach zero, we have†

$$\oint_{abcda} \mathbf{H} \cdot d\ell = \mathbf{H}_1 \cdot \Delta\mathbf{w} + \mathbf{H}_2 \cdot (-\Delta\mathbf{w}) = J_{sn} \Delta w$$

or

$$H_{1t} - H_{2t} = J_{sn} \qquad \text{(A/m)}, \tag{6–110}$$

† Equation (6–109) is assumed to be valid for regions containing discontinuous media.

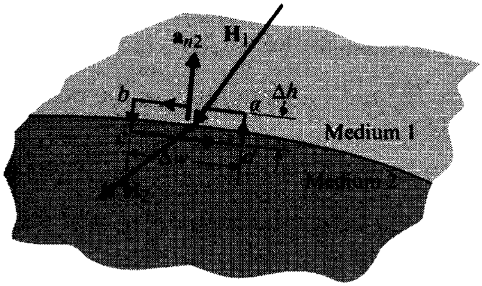


FIGURE 6–19
Closed path about the interface of two media for determining the boundary condition of H_t.

where J_{sn} is the surface current density on the interface normal to the contour C. The direction of J_{sn} is that of the thumb when the fingers of the right hand follow the direction of the path. In Fig. 6–19 the positive direction of J_{sn} for the chosen path is out of the paper. The following is a more concise expression of the boundary condition for the tangential components of H, which includes both magnitude and direction relations (Problem P.6–30).

$$\boxed{\mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \qquad \text{(A/m)},} \tag{6–111}$$

where $\mathbf{a}_{n2}$ is the *outward unit normal from medium* 2 at the interface. Thus, ***the tangential component of the*** **H** ***field is discontinuous across an interface where a free surface current exists***, the amount of discontinuity being determined by Eq. (6–111).

When the conductivities of both media are finite, currents are defined by volume current densities and free surface currents do not exist on the interface. Hence $\mathbf{J}_s$ equals zero, and ***the tangential component of*** **H** ***is continuous across the boundary of almost all physical media; it is discontinuous only when an interface with an ideal perfect conductor or a superconductor is assumed.***

EXAMPLE 6–12 Two magnetic media with permeabilities μ_1 and μ_2 have a common boundary, as shown in Fig. 6–20. The magnetic field intensity in medium 1 at the

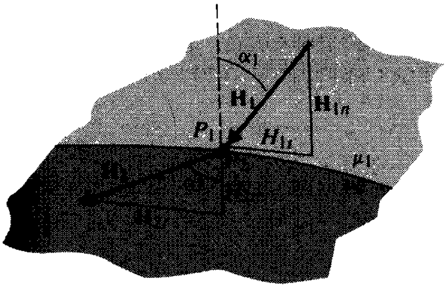


FIGURE 6–20
Boundary conditions for magnetostatic field at an interface (Example 6–12).

point P_1 has a magnitude H_1 and makes an angle α_1 with the normal. Determine the magnitude and the direction of the magnetic field intensity at point P_2 in medium 2.

Solution The desired unknown quantities are H_2 and α_2. Continuity of the normal component of **B** field requires, from Eq. (6–108),

$$\mu_2 H_2 \cos\alpha_2 = \mu_1 H_1 \cos\alpha_1. \tag{6–112}$$

Since neither of the media is a perfect conductor, the tangential component of **H** field is continuous. We have

$$H_2 \sin\alpha_2 = H_1 \sin\alpha_1. \tag{6–113}$$

Division of Eq. (6–113) by Eq. (6–112) gives

$$\boxed{\frac{\tan\alpha_2}{\tan\alpha_1} = \frac{\mu_2}{\mu_1}} \tag{6–114}$$

or

$$\alpha_2 = \tan^{-1}\left(\frac{\mu_2}{\mu_1}\tan\alpha_1\right), \tag{6–115}$$

which describes the refraction property of the magnetic field. The magnitude of $\mathbf{H}_2$ is

$$H_2 = \sqrt{H_{2t}^2 + H_{2n}^2} = \sqrt{(H_2 \sin\alpha_2)^2 + (\mathrm{H}_2 \cos\alpha_2)^2}.$$

From Eqs. (6–112) and (6–113) we obtain

$$\boxed{H_2 = H_1\left[\sin^2\alpha_1 + \left(\frac{\mu_1}{\mu_2}\cos\alpha_1\right)^2\right]^{1/2}.} \tag{6–116}$$

We make three remarks here. First, Eqs. (6–114) and (6–116) are entirely similar to Eqs. (3–129) and (3–130), respectively, for the electric fields in dielectric media—except for the use of permeabilities (instead of permittivities) in the case of magnetic fields. Second, if medium 1 is nonmagnetic (like air) and medium 2 is ferromagnetic (like iron), then $\mu_2 \gg \mu_1$, and, from Eq. (6–114), α_2 will be nearly 90°. This means that for any arbitrary angle α_1 that is not close to zero, the magnetic field in a ferromagnetic medium runs almost parallel to the interface. Third, if medium 1 is ferromagnetic and medium 2 is air ($\mu_1 \gg \mu_2$), then α_2 will be nearly zero; that is, if a magnetic field originates in a ferromagnetic medium, the flux lines will emerge into air in a direction almost normal to the interface.

EXAMPLE 6–13 Sketch the magnetic flux lines both inside and outside a cylindrical bar magnet having a uniform axial magnetization $\mathbf{M} = \mathbf{a}_z M_0$.

Solution In Example 6–8 we noted that the problem of a cylindrical bar magnet could be replaced by that of a magnetization current sheet having a surface current density $\mathbf{J}_{ms} = \mathbf{a}_\phi M_0$ (the equivalent volume current density being zero). The determination of $\mathbf{B}$ at an arbitrary point inside and outside the magnet involves integrals that are difficult to evaluate. We shall use the result in Example 6–8 for a point on the magnet axis to obtain a rough sketch of the $\mathbf{B}$ lines.

A cross section of a cylindrical bar magnet having a radius b and length L is shown in Fig. 6–21. From Eq. (6–65) we get

$$\mathbf{B}_{P_0} = \mathbf{a}_z \frac{\mu_0 M_0}{2}\left[\frac{L}{\sqrt{(L/2)^2 + b^2}}\right] \tag{6–117}$$

$$\mathbf{B}_{P_1} = \mathbf{a}_z \frac{\mu_0 M_0}{2}\left[\frac{L}{\sqrt{L^2 + b^2}}\right] = \mathbf{B}_{P_1'}. \tag{6–118}$$

It is obvious from Eqs. (6–117) and (6–118) that $\mathbf{B}_{P_1} = \mathbf{B}_{P_1'} < \mathbf{B}_{P_0}$; that is, the magnetic flux density along the axis at the end faces of the magnet is less than that at the center. This is because the flux lines tend to flare out at the end faces. We know that, at points off the axis, $\mathbf{B}$ has a radial component. We also know that $\mathbf{B}$ lines are not refracted at the end faces and that they close upon themselves.

On the side of the magnet there is a surface current given by Eq. (6–64):

$$\mathbf{J}_{ms} = \mathbf{a}_\phi M_0. \tag{6–119}$$

Hence according to Eq. (6–111), the axial component of $\mathbf{B}$ changes by an amount equal to $\mu_0 M_0$. From Eqs. (6–117) and (6–118) we see that B_z inside the magnet is less

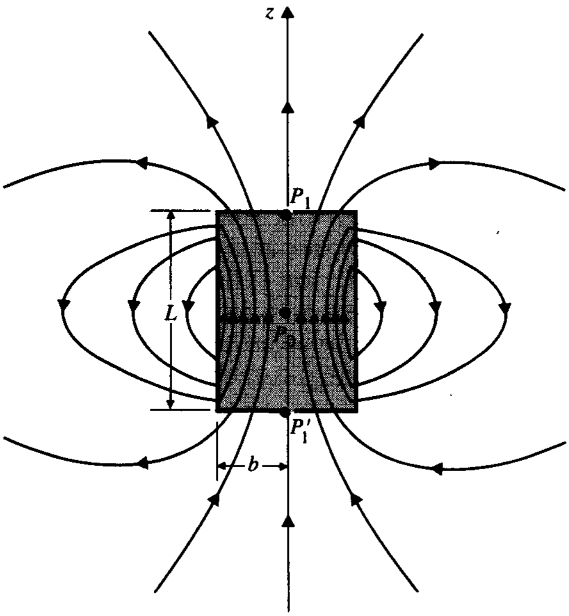


FIGURE 6–21
Magnetic flux lines around a cylindrical bar magnet (Example 6–13).

than $\mu_0 M_0$. Consequently, there is a change in both the magnitude and the direction for B_z as it crosses the side wall. The magnetic flux lines will then assume the form sketched in Fig. 6–21.

It must be remarked here that while $\mathbf{H} = \mathbf{B}/\mu_0$ outside the magnet, **H** and **B** inside the magnet are far from being proportional vectors in the same direction. From Eq. (6–75),

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}, \tag{6–120}$$

and the fact that B/μ_0 along the axis inside is less than M_0, we observe that **H** and **B** are in opposite directions along the axis inside. For a long, thin magnet, $L \gg b$, Eq. (6–117) gives approximately $B_{P_0} = \mu_0 M_0$. From Eq. (6–120) we obtain $H_{P_0} \cong 0$. Hence **H** nearly vanishes at the center of a long, thin magnet, where **B** is maximum. By hypothesis the magnetization vector **M** is zero outside and is a constant vector everywhere inside the magnet. ▬

In current-free regions the magnetic flux density **B** is irrotational and can be expressed as the gradient of a scalar magnetic potential V_m, as indicated in Section 6–5.1.

$$\mathbf{B} = -\mu \nabla V_m. \tag{6–121}$$

Assuming a constant μ, substitution of Eq. (6–121) in $\nabla \cdot \mathbf{B} = 0$ (Eq. 6–6) yields a Laplace's equation in V_m:

$$\nabla^2 V_m = 0. \tag{6–122}$$

Equation (6–122) is entirely similar to the Laplace's equation, Eq. (4–10), for the scalar electric potential V in a charge-free region. That the solution for Eq. (6–122) satisfying given boundary conditions is unique can be proved in the same way as for Eq. (4–10)—see Section 4–3. Thus the techniques (method of images and method of separation of variables) discussed in Chapter 4 for solving electrostatic boundary-value problems can be adapted to solving analogous magnetostatic boundary-value problems. However, although electrostatic problems with conducting boundaries maintained at fixed potentials occur quite often in practice, analogous magnetostatic problems with constant magnetic-potential boundaries are of little practical importance. (We recall that isolated magnetic charges do not exist and that magnetic flux lines always form closed paths.) The nonlinearity in the relationship between **B** and **H** in ferromagnetic materials also complicates the analytical solution of boundary-value problems in magnetostatics.

6–11 Inductances and Inductors

Consider two neighboring closed loops, C_1 and C_2 bounding surfaces S_1 and S_2, respectively, as shown in Fig. 6–22. If a current I_1 flows in C_1, a magnetic field $\mathbf{B}_1$ will be created. Some of the magnetic flux due to $\mathbf{B}_1$ will link with C_2—that is, will pass through the surface S_2 bounded by C_2. Let us designate this mutual flux Φ_{12}. We have

$$\Phi_{12} = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}_2 \qquad \text{(Wb)}. \tag{6-123}$$

From physics we know that a time-varying I_1 (and therefore a time-varying Φ_{12}) will produce an induced electromotive force or voltage in C_2 as a result of Faraday's law of electromagnetic induction. (We defer the discussion of Faraday's law until the next chapter.) However, Φ_{12} exists even if I_1 is a steady d-c current.

From the Biot-Savart law, Eq. (6–32), we see that B_1 is directly proportional to I_1; hence Φ_{12} is also proportional to I_1. We write

$$\Phi_{12} = L_{12}I_1, \tag{6-124}$$

where the proportionality constant L_{12} is called the ***mutual inductance*** between loops C_1 and C_2, with SI unit henry (H). In case C_2 has N_2 turns, the ***flux linkage*** Λ_{12} due to Φ_{12} is

$$\Lambda_{12} = N_2\Phi_{12} \qquad \text{(Wb)}, \tag{6-125}$$

and Eq. (6–124) generalizes to

$$\Lambda_{12} = L_{12}I_1 \qquad \text{(Wb)} \tag{6-126}$$

or

$$\boxed{L_{12} = \frac{\Lambda_{12}}{I_1} \qquad \text{(H)}.} \tag{6-127}$$

The ***mutual inductance between two circuits*** *is then the magnetic flux linkage with one circuit per unit current in the other.* In Eq. (6–124) it is implied that the permeability of the medium does not change with I_1. In other words, Eq. (6–124) and hence Eq.

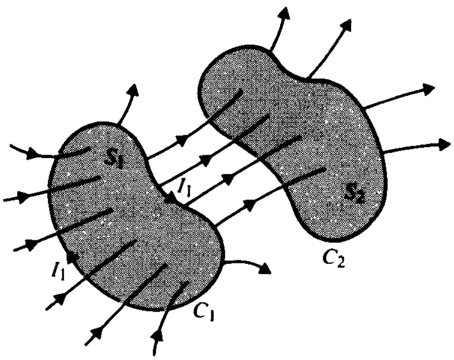


FIGURE 6–22
Two magnetically coupled loops.

(6–127) apply only to *linear* media. A more general definition for L_{12} is

$$\boxed{L_{12} = \frac{d\Lambda_{12}}{dI_1} \quad \text{(H).}} \tag{6–128}$$

Some of the magnetic flux produced by I_1 links only with C_1 itself, and not with C_2. The total flux linkage with C_1 caused by I_1 is

$$\Lambda_{11} = N_1\Phi_{11} > N_1\Phi_{12}. \tag{6–129}$$

The ***self-inductance*** *of loop C_1 is defined as the magnetic flux linkage per unit current in the loop itself*; that is,

$$\boxed{L_{11} = \frac{\Lambda_{11}}{I_1} \quad \text{(H),}} \tag{6–130}$$

for a linear medium. In general,

$$\boxed{L_{11} = \frac{d\Lambda_{11}}{dI_1} \quad \text{(H).}} \tag{6–131}$$

The self-inductance of a loop or circuit depends on the geometrical shape and the physical arrangement of the conductor constituting the loop or circuit, as well as on the permeability of the medium. With a linear medium, self-inductance does not depend on the current in the loop or circuit. As a matter of fact, it exists regardless of whether the loop or circuit is open or closed, or whether it is near another loop or circuit.

A conductor arranged in an appropriate shape (such as a conducting wire wound as a coil) to supply a certain amount of self-inductance is called an ***inductor***. Just as a capacitor can store electric energy, an inductor can storage magnetic energy, as we shall see in Section 6–12. When we deal with only one loop or coil, there is no need to carry the subscripts in Eq. (6–130) or Eq. (6–131), and ***inductance*** without an adjective will be taken to mean self-inductance. The procedure for determining the self-inductance of an inductor is as follows:

1. Choose an appropriate coordinate system for the given geometry.
2. Assume a current I in the conducting wire.
3. Find $\mathbf{B}$ from I by Ampère's circuital law, Eq. (6–10), if symmetry exists; if not, Biot-Savart law, Eq. (6–32), must be used.
4. Find the flux linking with each turn, Φ, from $\mathbf{B}$ by integration:

   $$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s},$$

   where S is the area over which $\mathbf{B}$ exists and links with the assumed current.

5. Find the flux linkage Λ by multiplying Φ by the number of turns.
6. Find L by taking the ratio $L = \Lambda/I$.

Only a slight modification of this procedure is needed to determine the mutual inductance L_{12} between two circuits. After choosing an appropriate coordinate system, proceed as follows: Assume $I_1 \rightarrow$ Find $\mathbf{B}_1 \rightarrow$ Find Φ_{12} by integrating $\mathbf{B}_1$ over surface $S_2 \rightarrow$ Find flux linkage $\Lambda_{12} = N_2\Phi_{12} \rightarrow$ Find $L_{12} = \Lambda_{12}/I_1$.

EXAMPLE 6–14 Assume that N turns of wire are tightly wound on a toroidal frame of a rectangular cross section with dimensions as shown in Fig. 6–23. Then, assuming the permeability of the medium to be μ_0, find the self-inductance of the toroidal coil.

Solution It is clear that the cylindrical coordinate system is appropriate for this problem because the toroid is symmetrical about its axis. Assuming a current I in the conducting wire, we find, by applying Eq. (6–10) to a circular path with radius r $(a < r < b)$:

$$\mathbf{B} = \mathbf{a}_\phi B_\phi,$$
$$d\ell = \mathbf{a}_\phi r d\phi,$$
$$\oint_C \mathbf{B} \cdot d\ell = \int_0^{2\pi} B_\phi r\, d\phi = 2\pi r B_\phi.$$

This result is obtained because both B_ϕ and r are constant around the circular path C. Since the path encircles a total current NI, we have

$$2\pi r B_\phi = \mu_0 NI$$

and

$$B_\phi = \frac{\mu_0 NI}{2\pi r}.$$

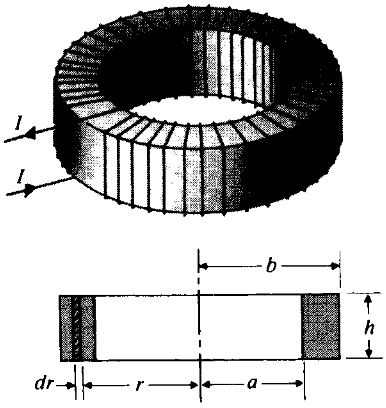


FIGURE 6–23
A closely wound toroidal coil (Example 6–14).

Next we find

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \int_S \left(\mathbf{a}_\phi \frac{\mu_0 NI}{2\pi r}\right) \cdot (\mathbf{a}_\phi h\, dr)$$

$$= \frac{\mu_0 NIh}{2\pi} \int_a^b \frac{dr}{r} = \frac{\mu_0 NIh}{2\pi} \ln \frac{b}{a}.$$

The flux linkage Λ is $N\Phi$ or

$$\Lambda = \frac{\mu_0 N^2 Ih}{2\pi} \ln \frac{b}{a}.$$

Finally, we obtain

$$L = \frac{\Lambda}{I} = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{b}{a} \qquad \text{(H)}. \tag{6-132}$$

We note that the self-inductance is not a function of I (for a constant medium permeability). The qualification that the coil be closely wound on the toroid is to minimize the linkage flux around the individual turns of the wire. ▬

EXAMPLE 6–15 Find the inductance per unit length of a very long solenoid with air core having n turns per unit length.

Solution The magnetic flux density inside an infinitely long solenoid has been found in Example 6–3. For current I we have, from Eq. (6–14),

$$B = \mu_0 nI,$$

which is constant inside the solenoid. Hence,

$$\Phi = BS = \mu_0 nSI, \tag{6-133}$$

where S is the cross-sectional area of the solenoid. The flux linkage per unit length is

$$\Lambda' = n\Phi = \mu_0 n^2 SI. \tag{6-134}$$

Therefore the inductance per unit length is

$$L' = \mu_0 n^2 S \qquad \text{(H/m)}. \tag{6-135}$$

Equation (6–135) is an approximate formula, based on the assumption that the length of the solenoid is very much greater than the linear dimensions of its cross section. A more accurate derivation for the magnetic flux density and flux linkage per unit length near the ends of a finite solenoid will show that they are less than the values given, respectively, by Eqs. (6–14) and (6–134). Hence the total inductance of a finite solenoid is somewhat less than the values of L', as given in Eq. (6–135), multiplied by the length. ▬

The following is a significant observation about the results of the previous two examples: The self-inductance of wire-wound inductors is proportional to the *square* of the number of turns.

EXAMPLE 6–16 An air coaxial transmission line has a solid inner conductor of radius a and a very thin outer conductor of inner radius b. Determine the inductance per unit length of the line.

Solution Refer to Fig. 6–24. Assume that a current I flows in the inner conductor and returns via the outer conductor in the other direction. Because of the cylindrical symmetry, $\mathbf{B}$ has only a ϕ-component with different expressions in the two regions: (a) inside the inner conductor, and (b) between the inner and outer conductors. Also assume that the current I is uniformly distributed over the cross section of the inner conductor.

a) *Inside the inner conductor,*

$$0 \leq r \leq a.$$

From Eq. (6–11a),

$$\mathbf{B}_1 = \mathbf{a}_\phi B_{\phi 1} = \mathbf{a}_\phi \frac{\mu_0 r I}{2\pi a^2}. \tag{6–136}$$

b) *Between the inner and outer conductors,*

$$a \leq r \leq b.$$

From Eq. (6–11b),

$$\mathbf{B}_2 = \mathbf{a}_\phi B_{\phi 2} = \mathbf{a}_\phi \frac{\mu_0 I}{2\pi r}. \tag{6–137}$$

Now consider an annular ring in the inner conductor between radii r and $r + dr$. The current in a unit length of this annular ring is linked by the flux that can be obtained by integrating Eqs. (6–136) and (6–137). We have

$$\begin{aligned} d\Phi' &= \int_r^a B_{\phi 1}\, dr + \int_a^b B_{\phi 2}\, dr \\ &= \frac{\mu_0 I}{2\pi a^2} \int_r^a r\, dr + \frac{\mu_0 I}{2\pi} \int_a^b \frac{dr}{r} \\ &= \frac{\mu_0 I}{4\pi a^2}(a^2 - r^2) + \frac{\mu_0 I}{2\pi} \ln \frac{b}{a}. \end{aligned} \tag{6–138}$$

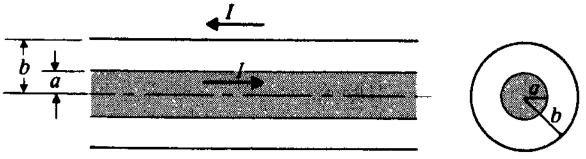


FIGURE 6–24
Two views of a coaxial transmission line (Example 6–16).

But the current in the annular ring is only a fraction $(2\pi r\,dr/\pi a^2 = 2r\,dr/a^2)$ of the total current I.† Hence the flux linkage for this annular ring is

$$d\Lambda' = \frac{2r\,dr}{a^2}\,d\Phi'. \tag{6-139}$$

The total flux linkage per unit length is

$$\begin{aligned}\Lambda' &= \int_{r=0}^{r=a} d\Lambda' \\ &= \frac{\mu_0 I}{\pi a^2}\left[\frac{1}{2a^2}\int_0^a (a^2 - r^2)r\,dr + \left(\ln\frac{b}{a}\right)\int_0^a r\,dr\right] \\ &= \frac{\mu_0 I}{2\pi}\left(\frac{1}{4} + \ln\frac{b}{a}\right).\end{aligned}$$

The inductance of a unit length of the coaxial transmission line is therefore

$$\boxed{L' = \frac{\Lambda'}{I} = \frac{\mu_0}{8\pi} + \frac{\mu_0}{2\pi}\ln\frac{b}{a} \qquad \text{(H/m)}.} \tag{6-140}$$

The first term $\mu_0/8\pi$ arises from the flux linkage internal to the solid inner conductor; it is known as the ***internal inductance*** per unit length of the inner conductor. The second term comes from the linkage of the flux that exists between the inner and the outer conductors; this term is known as the ***external inductance*** per unit length of the coaxial line. ■

In high-frequency applications the current in a good conductor tends to shift to the surface of the conductor (due to ***skin effect***, as we shall see in Chapter 8), resulting in an uneven current distribution in the inner conductor and thereby changing the value of the internal inductances. In the extreme case the current may essentially concentrate in the "skin" of the inner conductor as a surface current, and the internal self-inductance is reduced to zero.

EXAMPLE 6–17 Calculate the internal and external inductances per unit length of a transmission line consisting of two long parallel conducting wires of radius a that carry currents in opposite directions. The axes of the wires are separated by a distance d, which is much larger than a.

Solution The internal self-inductance per unit length of each wire is, from Eq. (6–140), $\mu_0/8\pi$. So for two wires we have

$$L_i' = 2 \times \frac{\mu_0}{8\pi} = \frac{\mu_0}{4\pi} \qquad \text{(H/m)}. \tag{6-141}$$

† It is assumed that the current is distributed uniformly in the inner conductor. This assumption does not hold for high-frequency a-c currents.

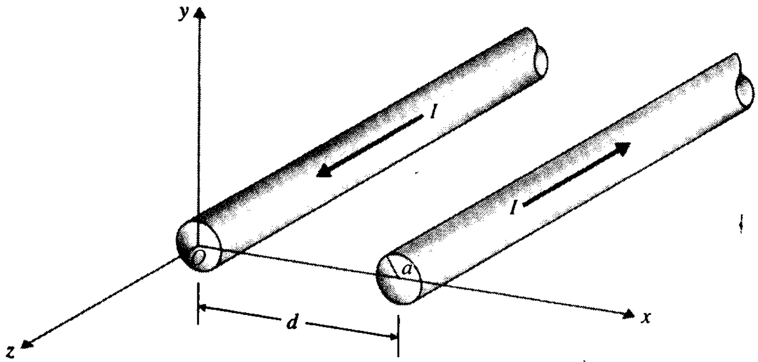


FIGURE 6–25
A two-wire transmission line (Example 6–17).

To find the external self-inductance per unit length, we first calculate the magnetic flux linking with a unit length of the transmission line for an assumed current I in the wires. In the xz-plane where the two wires lie, as in Fig. 6–25, the contributing **B** vectors due to the equal and opposite currents in the two wires have only a y-component:

$$B_{y1} = \frac{\mu_0 I}{2\pi x} \tag{6–142}$$

$$B_{y2} = \frac{\mu_0 I}{2\pi(d - x)}. \tag{6–143}$$

The flux linkage per unit length is then

$$\begin{aligned}\Phi' &= \int_a^{d-a} (B_{y1} + B_{y2})\,dx \\ &= \int_a^{d-a} \frac{\mu_0 I}{2\pi}\left[\frac{1}{x} + \frac{1}{d - x}\right] dx \\ &= \frac{\mu_0 I}{\pi} \ln\left(\frac{d-a}{a}\right) \cong \frac{\mu_0 I}{\pi} \ln\frac{d}{a} \qquad \text{(Wb/m)}.\end{aligned}$$

Therefore,

$$L'_e = \frac{\Phi'}{I} = \frac{\mu_0}{\pi} \ln\frac{d}{a} \qquad \text{(H/m)}, \tag{6–144}$$

and the total self-inductance per unit length of the two-wire line is

$$L' = L'_i + L'_e = \frac{\mu_0}{\pi}\left(\frac{1}{4} + \ln\frac{d}{a}\right) \qquad \text{(H/m)}. \tag{6–145}$$

▬

Before we present some examples showing how to determine the mutual inductance between two circuits, we pose the following question about Fig. 6–22 and Eq. (6–127): Is the flux linkage with loop C_2 caused by a unit current in loop C_1 equal to the flux linkage with C_1 caused by a unit current in C_2? That is, is it true that

$$L_{12} = L_{21}? \tag{6-146}$$

We may vaguely and intuitively expect that the answer is in the affirmative "because of reciprocity." But how do we prove it? We may proceed as follows. Combining Eqs. (6–123), (6–125) and (6–127), we obtain

$$L_{12} = \frac{N_2}{I_1}\int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}_2. \tag{6-147}$$

But in view of Eq. (6–15), $\mathbf{B}_1$ can be written as the curl of a vector magnetic potential $\mathbf{A}_1$, $\mathbf{B}_1 = \nabla \times \mathbf{A}_1$. We have

$$\begin{aligned} L_{12} &= \frac{N_2}{I_1}\int_{S_2} (\nabla \times \mathbf{A}_1) \cdot d\mathbf{s}_2 \\ &= \frac{N_2}{I_1}\oint_{C_2} \mathbf{A}_1 \cdot d\boldsymbol{\ell}_2. \end{aligned} \tag{6-148}$$

Now, from Eq. (6–27),

$$\mathbf{A}_1 = \frac{\mu_0 N_1 I_1}{4\pi}\oint_{C_1} \frac{d\boldsymbol{\ell}_1}{R}. \tag{6-149}$$

In Eqs. (6–148) and (6–149) the contour integrals are evaluated only *once* over the periphery of the *loops* C_2 and C_1, respectively—the effects of multiple turns having been taken care of separately by the factors N_2 and N_1. Substitution of Eq. (6–149) in Eq. (6–148) yields

$$L_{12} = \frac{\mu_0 N_1 N_2}{4\pi}\oint_{C_1}\oint_{C_2} \frac{d\boldsymbol{\ell}_1 \cdot d\boldsymbol{\ell}_2}{R}, \tag{6-150a}$$

where R is the distance between the differential lengths $d\boldsymbol{\ell}_1$ and $d\boldsymbol{\ell}_2$. It is customary to write Eq. (6–150a) as

$$\boxed{L_{12} = \frac{\mu_0}{4\pi}\oint_{C_1}\oint_{C_2} \frac{d\boldsymbol{\ell}_1 \cdot d\boldsymbol{\ell}_2}{R} \quad \text{(H)},} \tag{6-150b}$$

where N_1 and N_2 have been absorbed in the contour integrals over the *circuits* C_1 and C_2 from one end to the other. Equation (6–150b) is the ***Neumann formula*** for mutual inductance. It is a general formula requiring the evaluation of a double line integral. For any given problem we always first look for symmetry conditions that may simplify the determination of flux linkage and mutual inductance without resorting to Eq. (6–150b) directly.

It is clear from Eq. (6–150b) that mutual inductance is a property of the geometrical shape and the physical arrangement of coupled circuits. For a *linear medium*, mutual inductance is proportional to the medium's permeability and is independent of the currents in the circuits. It is obvious that interchanging the subscripts 1 and

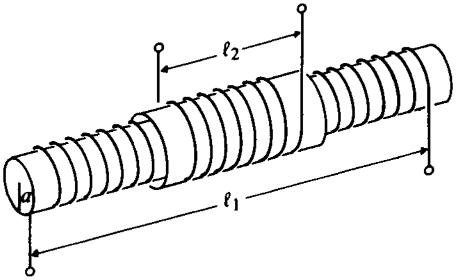


FIGURE 6–26
A solenoid with two windings (Example 6–18).

2 does not change the value of the double integral; hence an affirmative answer to the question posed in Eq. (6–146) follows. This is an important conclusion because it allows us to use the simpler of the two ways (finding L_{12} or L_{21}) to determine the mutual inductance.†

EXAMPLE 6–18 Two coils of N_1 and N_2 turns are wound concentrically on a straight cylindrical core of radius a and permeability μ. The windings have lengths ℓ_1 and ℓ_2, respectively. Find the mutual inductance between the coils.

Solution Figure 6–26 shows such a solenoïd with two concentric windings. Assume that current I_1 flows in the inner coil. From Eq. (6–133) we find that the flux Φ_{12} in the solenoid core that links with the outer coil is

$$\Phi_{12} = \mu\left(\frac{N_1}{\ell_1}\right)(\pi a^2)I_1.$$

Since the outer coil has N_2 turns, we have

$$\Lambda_{12} = N_2\Phi_{12} = \frac{\mu}{\ell_1} N_1 N_2 \pi a^2 I_1.$$

Hence the mutual inductance is

$$L_{12} = \frac{\Lambda_{12}}{I_1} = \frac{\mu}{\ell_1} N_1 N_2 \pi a^2 \qquad \text{(H)}. \tag{6–151}$$

Leakage flux has been neglected.

EXAMPLE 6–19 Determine the mutual inductance between a conducting triangular loop and a very long straight wire as shown in Fig. 6–27.

Solution Let us designate the triangular loop as circuit 1 and the long wire as circuit 2. If we assume a current I_1 in the triangular loop, it is difficult to find the magnetic flux density $\mathbf{B}_1$ everywhere. Consequently, it is difficult to determine the mutual

† In circuit theory books the symbol M is frequently used to denote mutual inductance.

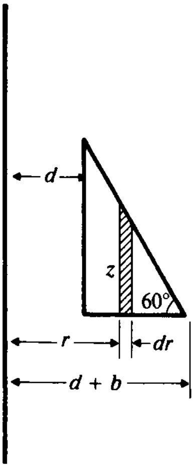


FIGURE 6–27
A conducting triangular loop and a long straight wire (Example 6–19).

inductance L_{12} from Λ_{12}/I_1 in Eq. (6–127). We can, however, apply Ampère's circuital law and readily write the expression for $\mathbf{B}_2$ that is caused by a current I_2 in the long straight wire:

$$\mathbf{B}_2 = \mathbf{a}_\phi \frac{\mu_0 I_2}{2\pi r}. \tag{6–152}$$

The flux linkage $\Lambda_{21} = \Phi_{21}$ is

$$\Lambda_{21} = \int_{S_1} \mathbf{B}_2 \cdot d\mathbf{s}_1, \tag{6–153}$$

where

$$d\mathbf{s}_1 = \mathbf{a}_\phi z\, dr. \tag{6–154}$$

The relation between z and r is given by the equation of the hypotenuse of the triangle:

$$\begin{aligned} z &= -[r - (d + b)] \tan 60^\circ \\ &= -\sqrt{3}[r - (d + b)]. \end{aligned} \tag{6–155}$$

Substituting Eqs. (6–152), (6–154), and (6–155) in Eq. (6–153), we have

$$\begin{aligned} \Lambda_{21} &= -\frac{\sqrt{3}\mu_0 I_2}{2\pi} \int_d^{d+b} \frac{1}{r} [r - (d + b)]\, dr \\ &= \frac{\sqrt{3}\mu_0 I_2}{2\pi} \left[(d + b) \ln\left(1 + \frac{b}{d}\right) - b \right]. \end{aligned}$$

Therefore, the mutual inductance is

$$L_{21} = \frac{\Lambda_{21}}{I_2} = \frac{\sqrt{3}\mu_0}{2\pi} \left[(d + b) \ln\left(1 + \frac{b}{d}\right) - b \right] \quad \text{(H)}. \tag{6–156}$$

■

6–12 Magnetic Energy

So far we have discussed self- and mutual inductances in static terms. Because inductances depend on the geometrical shape and the physical arrangement of the conductors constituting the circuits, and, for a linear medium, are independent of the currents, we were not concerned with nonsteady currents in the defining of inductances. However, we know that resistanceless inductors appear as short-circuits to steady (d-c) currents; it is obviously necessary that we consider alternating currents when the effects of inductances on circuits and magnetic fields are of interest. A general consideration of time-varying electromagnetic fields (electrodynamics) will be deferred until the next chapter. For now we assume ***quasi-static conditions***, which imply that the currents vary very slowly in time (are low of frequency) and that the dimensions of the circuits are very small in comparison to the wavelength. These conditions are tantamount to ignoring retardation and radiation effects, as we shall see when electromagnetic waves are discussed in Chapter 8.

In Section 3–11 we discussed the fact that work is required to assemble a group of charges and that the work is stored as electric energy. We certainly expect that work also needs to be expended in sending currents into conducting loops and that it will be stored as magnetic energy. Consider a single closed loop with a self-inductance L_1 in which the current is initially zero. A current generator is connected to the loop, which increases the current i_1 from zero to I_1. From physics we know that an electromotive force (emf) will be induced in the loop that opposes the current change.† An amount of work must be done to overcome this induced emf. Let $v_1 = L_1\, di_1/dt$ be the voltage across the inductance. The work required is

$$W_1 = \int v_1 i_1\, dt = L_1 \int_0^{I_1} i_1\, di_1 = \tfrac{1}{2} L_1 I_1^2. \tag{6–157}$$

Since $L_1 = \Phi_1/I_1$ for *linear* media, Eq. (6–157) can be written alternatively in terms of flux linkage as

$$W_1 = \tfrac{1}{2} I_1 \Phi_1, \tag{6–158}$$

which is stored as *magnetic energy*.

Now consider two closed loops C_1 and C_2 carrying currents i_1 and i_2, respectively. The currents are initially zero and are to be increased to I_1 and I_2, respectively. To find the amount of work required, we first keep $i_2 = 0$ and increase i_1 from zero to I_1. This requires a work W_1 in loop C_1, as given in Eq. (6–157) or (6–158); no work is done in loop C_2, since $i_2 = 0$. Next we keep i_1 at I_1 and increase i_2 from zero to I_2. Because of mutual coupling, some of the magnetic flux due to i_2 will link with loop C_1, giving rise to an induced emf that must be overcome by a voltage $v_{21} = L_{21}\, di_2/dt$ in order to keep i_1 constant at its value I_1. The work involved is

$$W_{21} = \int v_{21} I_1\, dt = L_{21} I_1 \int_0^{I_2} di_2 = L_{21} I_1 I_2. \tag{6–159}$$

† The subject of electromagnetic induction will be discussed in Chapter 7.

At the same time a work W_{22} must be done in loop C_2 in order to counteract the induced emf and increase i_2 to I_2.

$$W_{22} = \tfrac{1}{2}L_2I_2^2. \tag{6-160}$$

The total amount of work done in raising the currents in loops C_1 and C_2 from zero to I_1 and I_2, respectively, is then the sum of W_1, W_{21}, and W_{22}:

$$\begin{aligned} W_2 &= \tfrac{1}{2}L_1I_1^2 + L_{21}I_1I_2 + \tfrac{1}{2}L_2I_2^2 \\ &= \frac{1}{2}\sum_{j=1}^{2}\sum_{k=1}^{2} L_{jk}I_jI_k. \end{aligned} \tag{6-161}$$

Generalizing this result to a system of N loops carrying currents $I_1, I_2, \ldots, I_n$, we obtain

$$\boxed{W_m = \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} L_{jk}I_jI_k \qquad \text{(J)},} \tag{6-162}$$

which is the energy stored in the magnetic field. For a current I flowing in a single inductor with inductance L, the stored magnetic energy is

$$\boxed{W_m = \tfrac{1}{2}LI^2 \qquad \text{(J)}.} \tag{6-163}$$

It is instructive to derive Eq. (6–162) in an alternative way. Consider a typical kth loop of N magnetically coupled loops. Let v_k and i_k be the voltage across and the current in the loop, respectively. The work done to the kth loop in time dt is

$$dW_k = v_ki_k\,dt = i_k\,d\phi_k, \tag{6-164}$$

where we have used the relation $v_k = d\phi_k/dt$. Note that the change, $d\phi_k$, in the flux ϕ_k linking with the kth loop is the result of the changes of the currents in all the coupled loops. The differential work done to, or the differential magnetic energy stored in, the system is then

$$dW_m = \sum_{k=1}^{N} dW_k = \sum_{k=1}^{N} i_k\,d\phi_k. \tag{6-165}$$

The total stored energy is the integration of dW_m and is independent of the manner in which the final values of the currents and fluxes are reached. Let us assume that all the currents and fluxes are brought to their final values in concert by an equal fraction α that increases from 0 to 1; that is, $i_k = \alpha I_k$, and $\phi_k = \alpha\Phi_k$ at any instant of time. We obtain the total ***magnetic energy***:

$$W_m = \int dW_m = \sum_{k=1}^{N} I_k\Phi_k \int_0^1 \alpha\,d\alpha$$

or

$$W_m = \frac{1}{2}\sum_{k=1}^{N} I_k\Phi_k \qquad \text{(J)}, \tag{6–166}$$

which simplifies to Eq. (6–158) for $N = 1$, as expected. Noting that, for *linear* media,

$$\Phi_k = \sum_{j=1}^{N} L_{jk}I_j,$$

we obtain Eq. (6–162) immediately.

6–12.1 MAGNETIC ENERGY IN TERMS OF FIELD QUANTITIES

Equation (6–166) can be generalized to determine the magnetic energy of a continuous distribution of current within a volume. A single current-carrying loop can be considered as consisting of a large number, N, of contiguous filamentary current elements of closed paths C_k, each with a current ΔI_k flowing in an infinitesimal cross-sectional area $\Delta a'_k$ and linking with magnetic flux Φ_k.

$$\Phi_k = \int_{S_k} \mathbf{B}\cdot\mathbf{a}_n\, ds'_k = \oint_{C_k} \mathbf{A}\cdot d\boldsymbol{\ell}'_k, \tag{6–167}$$

where S_k is the surface bounded by C_k. Substituting Eq. (6–167) in Eq. (6–166), we have

$$W_m = \frac{1}{2}\sum_{k=1}^{N} \Delta I_k \oint_{C_k} \mathbf{A}\cdot d\boldsymbol{\ell}'_k. \tag{6–168}$$

Now,

$$\Delta I_k\, d\boldsymbol{\ell}'_k = J(\Delta a'_k)\, d\boldsymbol{\ell}'_k = \mathbf{J}\,\Delta v'_k.$$

As $N \to \infty$, $\Delta v'_k$ becomes dv', and the summation in Eq. (6–168) can be written as an integral. We have

$$W_m = \tfrac{1}{2}\int_{V'} \mathbf{A}\cdot\mathbf{J}\, dv' \qquad \text{(J)}, \tag{6–169}$$

where V' is the volume of the loop or the *linear medium* in which $\mathbf{J}$ exists. This volume can be extended to include all space, since the inclusion of a region where $\mathbf{J} = 0$ does not change W_m. Equation (6–169) should be compared with the expression for the electric energy W_e in Eq. (3–170).

It is often desirable to express the magnetic energy in terms of field quantities $\mathbf{B}$ and $\mathbf{H}$ instead of current density $\mathbf{J}$ and vector potential $\mathbf{A}$. Making use of the vector identity,

$$\nabla\cdot(\mathbf{A}\times\mathbf{H}) = \mathbf{H}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{H}),$$

(see Problem P.2–33 or the list at the end of book), we have

$$\mathbf{A}\cdot(\nabla\times\mathbf{H}) = \mathbf{H}\cdot(\nabla\times\mathbf{A}) - \nabla\cdot(\mathbf{A}\times\mathbf{H})$$

or

$$\mathbf{A} \cdot \mathbf{J} = \mathbf{H} \cdot \mathbf{B} - \nabla \cdot (\mathbf{A} \times \mathbf{H}). \tag{6–170}$$

Substituting Eq. (6–170) in Eq. (6–169), we obtain

$$W_m = \frac{1}{2} \int_{V'} \mathbf{H} \cdot \mathbf{B} \, dv' - \frac{1}{2} \oint_{S'} (\mathbf{A} \times \mathbf{H}) \cdot \mathbf{a}_n \, ds'. \tag{6–171}$$

In Eq. (6–171) we have applied the divergence theorem, and S' is the surface bounding V'. If V' is taken to be sufficiently large, the points on its surface S' will be very far from the currents. At those far-away points the contribution of the surface integral in Eq. (6–171) tends to zero because $|\mathbf{A}|$ falls off as $1/R$ and $|\mathbf{H}|$ falls off as $1/R^2$, as can be seen from Eqs. (6–23) and (6–32), respectively. Thus, the magnitude of $(\mathbf{A} \times \mathbf{H})$ decreases as $1/R^3$, whereas at the same time the surface S' increases only as R^2. When R approaches infinity, the surface integral in Eq. (6–171) vanishes. We have then

$$\boxed{W_m = \frac{1}{2} \int_{V'} \mathbf{H} \cdot \mathbf{B} \, dv' \qquad \text{(J)}.} \tag{6–172a}$$

Noting that $\mathbf{H} = \mathbf{B}/\mu$, we can write Eq. (6–172a) in two alternative forms:

$$\boxed{W_m = \frac{1}{2} \int_{V'} \frac{B^2}{\mu} \, dv' \qquad \text{(J)}} \tag{6–172b}$$

and

$$\boxed{W_m = \frac{1}{2} \int_{V'} \mu H^2 \, dv' \qquad \text{(J)}.} \tag{6–172c}$$

The expressions in Eqs. (6–172a), (6–172b), and (6–172c) for the magnetic energy W_m in a linear medium are analogous to those of electrostatic energy W_e in Eqs. (3–176a), (3–176b), and (3–176c), respectively.

If we define a ***magnetic energy density***, w_m, such that its volume integral equals the total magnetic energy

$$W_m = \int_{V'} w_m \, dv', \tag{6–173}$$

we can write w_m in three different forms:

$$w_m = \frac{1}{2} \mathbf{H} \cdot \mathbf{B} \qquad (\text{J/m}^3) \tag{6–174a}$$

or

$$w_m = \frac{B^2}{2\mu} \qquad (\text{J/m}^3) \tag{6–174b}$$

or

$$w_m = \frac{1}{2} \mu H^2 \qquad (\text{J/m}^3). \tag{6–174c}$$

By using Eq. (6–163) we can often determine self-inductance more easily from stored magnetic energy calculated in terms of $\mathbf{B}$ and/or $\mathbf{H}$, than from flux linkage.

We have

$$L = \frac{2W_m}{I^2} \qquad \text{(H)}. \tag{6-175}$$

EXAMPLE 6–20 By using stored magnetic energy, determine the inductance per unit length of an air coaxial transmission line that has a solid inner conductor of radius a and a very thin outer conductor of inner radius b.

Solution This is the same problem as that in Example 6–16, in which the self-inductance was determined through a consideration of flux linkages. Refer again to Fig. 6–24. Assume that a uniform current I flows in the inner conductor and returns in the outer conductor. The magnetic energy per unit length stored in the inner conductor is, from Eqs. (6–136) and (6–172b),

$$\begin{aligned} W'_{m1} &= \frac{1}{2\mu_0}\int_0^a B_{\phi 1}^2 2\pi r\,dr \\ &= \frac{\mu_0 I^2}{4\pi a^4}\int_0^a r^3\,dr = \frac{\mu_0 I^2}{16\pi} \qquad \text{(J/m)}. \end{aligned} \tag{6-176a}$$

The magnetic energy per unit length stored in the region between the inner and outer conductors is, from Eq. (6–137) and (6–172b),

$$\begin{aligned} W'_{m2} &= \frac{1}{2\mu_0}\int_a^b B_{\phi 2}^2 2\pi r\,dr \\ &= \frac{\mu_0 I^2}{4\pi}\int_a^b \frac{1}{r}\,dr = \frac{\mu_0 I^2}{4\pi}\ln\frac{b}{a} \qquad \text{(J/m)}. \end{aligned} \tag{6-176b}$$

Therefore, from Eq. (6–175) we have

$$\begin{aligned} L' &= \frac{2}{I^2}(W'_{m1} + W'_{m2}) \\ &= \frac{\mu_0}{8\pi} + \frac{\mu_0}{2\pi}\ln\frac{b}{a} \qquad \text{(H/m)}, \end{aligned}$$

which is the same as Eq. (6–140). The procedure used in this solution is comparatively simpler than that used in Example 6–16, especially the part leading to the internal inductance $\mu_0/8\pi$.

6–13 Magnetic Forces and Torques

In Section 6–1 we noted that a charge q moving with a velocity $\mathbf{u}$ in a magnetic field with flux density $\mathbf{B}$ experiences a magnetic force $\mathbf{F}_m$ given by Eq. (6–4), which is repeated below:

$$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B} \qquad \text{(N)}. \tag{6-177}$$

In this section we will discuss various aspects of forces and torques in static magnetic fields.

6–13.1 HALL EFFECT

Consider a conducting material of a $d \times b$ rectangular cross section in a uniform magnetic field $\mathbf{B} = \mathbf{a}_z B_0$, as shown in Fig. 6–28. A uniform direct current flows in the y-direction:

$$\mathbf{J} = \mathbf{a}_y J_0 = Nq\mathbf{u}, \tag{6–178}$$

where N is the number of charge carriers per unit volume, moving with a velocity $\mathbf{u}$, and q is the charge on each charge carrier. Because of Eq. (6–177), the charge carriers experience a force perpendicular to both $\mathbf{B}$ and $\mathbf{u}$. If the material is a conductor or an n-type semiconductor, the charge carriers are electrons, and q is negative. The magnetic force tends to move the electrons in the positive x-direction, creating a transverse electric field. This will continue until the transverse field is sufficient to stop the drift of the charge carriers. In the steady state the net force on the charge carriers is zero:

$$\mathbf{E}_h + \mathbf{u} \times \mathbf{B} = 0 \tag{6–179a}$$

or

$$\mathbf{E}_h = -\mathbf{u} \times \mathbf{B}. \tag{6–179b}$$

This is known as the ***Hall effect***, and $\mathbf{E}_h$ is called the ***Hall field***. For conductors and n-type semiconductors and a positive J_0, $\mathbf{u} = -\mathbf{a}_y u_0$, and

$$\begin{aligned} \mathbf{E}_h &= -(-\mathbf{a}_y u_0) \times \mathbf{a}_z B_0 \\ &= \mathbf{a}_x u_0 B_0. \end{aligned} \tag{6–180}$$

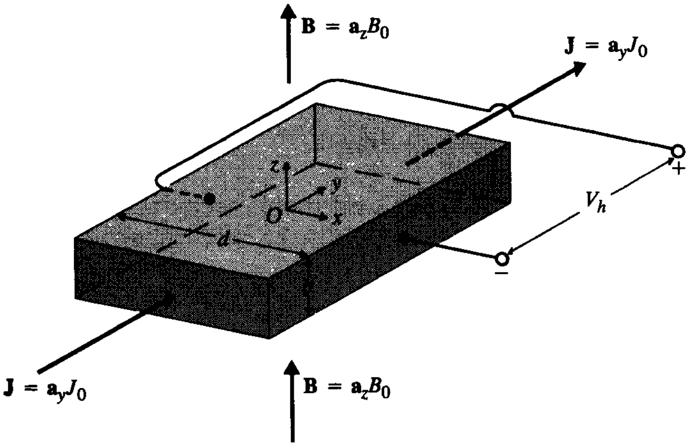


FIGURE 6–28
Illustrating the Hall effect.

A transverse potential appears across the sides of the material. Thus, we have

$$V_h = -\int_0^d E_h\,dx = u_0 B_0 d, \tag{6–181}$$

for electron carriers. In Eq. (6–181), V_h is called the ***Hall voltage***. The ratio $E_x/J_yB_z = 1/Nq$ is a characteristic of the material and is known as the ***Hall coefficient***.

If the charge carriers are holes, such as in a *p*-type semiconductor, the Hall field will be reversed, and the Hall voltage in Eq. (6–181) will be negative with the reference polarities shown in Fig. 6–28.

The Hall effect can be used for measuring the magnetic field and determining the sign of the predominant charge carriers (distinguishing an *n*-type from a *p*-type semiconductor). We have given here a simplified version of the Hall effect. In actuality it is a complex affair involving quantum theory concepts.

6–13.2 FORCES AND TORQUES ON CURRENT-CARRYING CONDUCTORS

Let us consider an element of conductor $d\ell$ with a cross-sectional area S. If there are N charge carriers (electrons) per unit volume moving with a velocity $\mathbf{u}$ in the direction of $d\ell$, then the magnetic force on the differential element is

$$\begin{aligned} d\mathbf{F}_m &= -NeS|d\ell|\mathbf{u} \times \mathbf{B} \\ &= -NeS|\mathbf{u}|\,d\ell \times \mathbf{B}, \end{aligned} \tag{6–182}$$

where e is the electronic charge. The two expressions in Eq. (6–182) are equivalent, since $\mathbf{u}$ and $d\ell$ have the same direction. Now, since $-NeS|\mathbf{u}|$ equals the current in the conductor, we can write Eq. (6–182) as

$$\boxed{d\mathbf{F}_m = I\,d\ell \times \mathbf{B} \quad \text{(N).}} \tag{6–183}$$

The magnetic force on a complete (closed) circuit of contour C that carries a current I in a magnetic field $\mathbf{B}$ is then

$$\boxed{\mathbf{F}_m = I \oint_C d\ell \times \mathbf{B} \quad \text{(N).}} \tag{6–184}$$

When we have two circuits carrying currents I_1 and I_2, respectively, the situation is that of one current-carrying circuit in the magnetic field of the other. In the presence of the magnetic field $\mathbf{B}_{21}$, which was caused by the current I_2 in C_2, the force $\mathbf{F}_{21}$ on circuit C_1 can be written as

$$\mathbf{F}_{21} = I_1 \oint_{C_1} d\ell_1 \times \mathbf{B}_{21}, \tag{6–185a}$$

where $\mathbf{B}_{21}$ is, from the Biot-Savart law in Eq. (6–32),

$$\mathbf{B}_{21} = \frac{\mu_0 I_2}{4\pi} \oint_{C_2} \frac{d\ell_2 \times \mathbf{a}_{R_{21}}}{R_{21}^2}. \tag{6–185b}$$

Combining Eqs. (6–185a) and (6–185b), we obtain

$$\boxed{\mathbf{F}_{21} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1} \oint_{C_2} \frac{d\boldsymbol{\ell}_1 \times (d\boldsymbol{\ell}_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2} \quad \text{(N)},} \tag{6–186a}$$

which is ***Ampere's law of force*** between two current-carrying circuits. It is an inverse-square relationship and should be compared with Coulomb's law of force in Eq. (3–17) between two stationary charges, the latter being much the simpler.

The force $\mathbf{F}_{12}$ on circuit C_2, in the presence of the magnetic field set up by the current I_1 in circuit C_1, can be written from Eq. (6–186a) by interchanging the subscripts 1 and 2:

$$\mathbf{F}_{12} = \frac{\mu_0}{4\pi} I_2 I_1 \oint_{C_2} \oint_{C_1} \frac{d\boldsymbol{\ell}_2 \times (d\boldsymbol{\ell}_1 \times \mathbf{a}_{R_{12}})}{R_{12}^2}. \tag{6–186b}$$

However, since $d\boldsymbol{\ell}_2 \times (d\boldsymbol{\ell}_1 \times \mathbf{a}_{R_{12}}) \neq -d\boldsymbol{\ell}_1 \times (d\boldsymbol{\ell}_2 \times \mathbf{a}_{R_{21}})$, we inquire whether this means $\mathbf{F}_{21} \neq -\mathbf{F}_{12}$—that is, whether Newton's third law governing the forces of action and reaction fails here. Let us expand the vector triple product in the integrand of Eq. (6–186a) by the back-cab rule, Eq. (2–20):

$$\frac{d\boldsymbol{\ell}_1 \times (d\boldsymbol{\ell}_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2} = \frac{d\boldsymbol{\ell}_2(d\boldsymbol{\ell}_1 \cdot \mathbf{a}_{R_{21}})}{R_{21}^2} - \frac{\mathbf{a}_{R_{21}}(d\boldsymbol{\ell}_1 \cdot d\boldsymbol{\ell}_2)}{R_{21}^2}. \tag{6–187}$$

Now the double closed line integral of the first term on the right side of Eq. (6–187) is

$$\begin{aligned} \oint_{C_1} \oint_{C_2} \frac{d\boldsymbol{\ell}_2(d\boldsymbol{\ell}_1 \cdot \mathbf{a}_{R_{21}})}{R_{21}^2} &= \oint_{C_2} d\boldsymbol{\ell}_2 \oint_{C_1} \frac{d\boldsymbol{\ell}_1 \cdot \mathbf{a}_{R_{21}}}{R_{21}^2} \\ &= \oint_{C_2} d\boldsymbol{\ell}_2 \oint_{C_1} d\boldsymbol{\ell}_1 \cdot \left(-\nabla_1 \frac{1}{R_{21}}\right) \\ &= -\oint_{C_2} d\boldsymbol{\ell}_2 \oint_{C_1} d\left(\frac{1}{R_{21}}\right) = 0. \end{aligned} \tag{6–188}$$

In Eq. (6–188) we have made use of Eq. (2–88) and the relation $\nabla_1(1/R_{21}) = -\mathbf{a}_{R_{21}}/R_{21}^2$. The closed line integral (with identical upper and lower limits) of $d(1/R_{21})$ around circuit C_1 vanishes. Substituting Eq. (6–187) in Eq. (6–186a) and using Eq. (6–188), we get

$$\mathbf{F}_{21} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1} \oint_{C_2} \frac{\mathbf{a}_{R_{21}}(d\boldsymbol{\ell}_1 \cdot d\boldsymbol{\ell}_2)}{R_{21}^2}, \tag{6–189}$$

which obviously equals $-\mathbf{F}_{12}$, inasmuch as $\mathbf{a}_{R_{12}} = -\mathbf{a}_{R_{21}}$. It follows that Newton's third law holds here, as expected.

EXAMPLE 6–21 Determine the force per unit length between two infinitely long parallel conducting wires carrying currents I_1 and I_2 in the same direction. The wires are separated by a distance d.

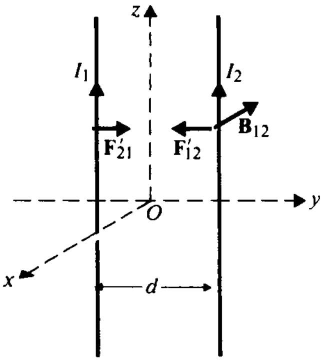


FIGURE 6–29
Force between two parallel current-carrying wires (Example 6–21).

Solution Let the wires lie in the *yz*-plane and be parallel to the *z*-axis, as shown in Fig. 6–29. This problem is a straightforward application of Eq. (6–185a). Using $\mathbf{F}'_{12}$ to denote the force per unit length on wire 2, we have

$$\mathbf{F}'_{12} = I_2(\mathbf{a}_z \times \mathbf{B}_{12}), \tag{6–190}$$

where $\mathbf{B}_{12}$, the magnetic flux density at wire 2, set up by the current I_1 in wire 1, is constant over wire 2. Because the wires are assumed to be infinitely long and cylindrical symmetry exists, it is not necessary to use Eq. (6–185b) for the determination of $\mathbf{B}_{12}$. We apply Ampère's circuital law and write, from Eq. (6–11b),

$$\mathbf{B}_{12} = -\mathbf{a}_x \frac{\mu_0 I_1}{2\pi d}. \tag{6–191}$$

Substitution of Eq. (6–191) in Eq. (6–190) yields

$$\mathbf{F}'_{12} = -\mathbf{a}_y \frac{\mu_0 I_1 I_2}{2\pi d} \quad \text{(N/m)}. \tag{6–192}$$

We see that the force on wire 2 pulls it toward wire 1. Hence the force between two wires carrying *currents in the same direction* is one of *attraction* (unlike the force between two charges of the same polarity, which is one of repulsion). It is trivial to prove that $\mathbf{F}'_{21} = -\mathbf{F}'_{12} = \mathbf{a}_y(\mu_0 I_1 I_2/2\pi d)$ and that the force between two wires carrying *currents in opposite directions* is one of *repulsion.* ■

Let us now consider a small circular loop of radius b and carrying a current I in a uniform magnetic field of flux density $\mathbf{B}$. It is convenient to resolve $\mathbf{B}$ into two components, $\mathbf{B} = \mathbf{B}_\perp + \mathbf{B}_\parallel$, where $\mathbf{B}_\perp$ and $\mathbf{B}_\parallel$ are perpendicular and parallel, respectively, to the plane of the loop. As illustrated in Fig. 6–30(a), the perpendicular component $\mathbf{B}_\perp$ tends to expand the loop (or contract it if the direction of I is reversed), but $\mathbf{B}_\perp$ exerts no net force to move the loop. The parallel component $\mathbf{B}_\parallel$ produces an upward force $d\mathbf{F}_1$ (out from the paper) on element $d\ell_1$ and a downward force (into the paper) $d\mathbf{F}_2 = -d\mathbf{F}_1$ on the symmetrically located element $d\ell_2$, as shown in

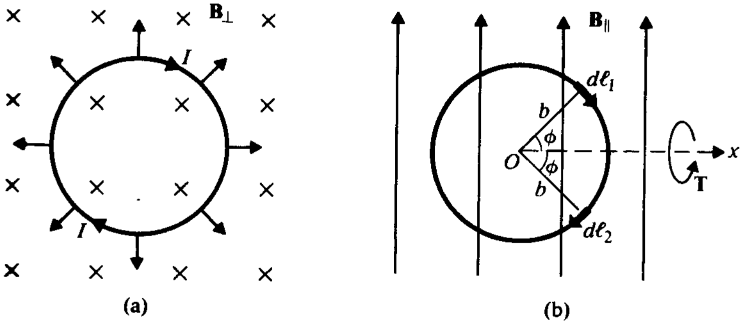


FIGURE 6–30
A circular loop in a uniform magnetic field $\mathbf{B} = \mathbf{B}_\perp + \mathbf{B}_{||}$.

Fig. 6–30(b). Although the net force on the entire loop caused by $\mathbf{B}_{||}$ is also zero, a torque exists that tends to rotate the loop about the x-axis in such a way as to *align* the magnetic field (due to I) with the external $\mathbf{B}_{||}$ field. The differential torque produced by $d\mathbf{F}_1$ and $d\mathbf{F}_2$ is

$$\begin{aligned} d\mathbf{T} &= \mathbf{a}_x(dF)2b\sin\phi \\ &= \mathbf{a}_x(I\,d\ell\,B_{||}\sin\phi)2b\sin\phi \\ &= \mathbf{a}_x 2Ib^2B_{||}\sin^2\phi\,d\phi, \end{aligned} \tag{6–193}$$

where $dF = |d\mathbf{F}_1| = |d\mathbf{F}_2|$ and $d\ell = |d\ell_1| = |d\ell_2| = b\,d\phi$. The total torque acting on the loop is then

$$\begin{aligned} \mathbf{T} &= \int d\mathbf{T} = \mathbf{a}_x 2Ib^2B_{||}\int_0^\pi \sin^2\phi\,d\phi \\ &= \mathbf{a}_x I(\pi b^2)B_{||}. \end{aligned} \tag{6–194}$$

If the definition of the magnetic dipole moment in Eq. (6–46) is used,

$$\mathbf{m} = \mathbf{a}_n I(\pi b^2) = \mathbf{a}_n IS,$$

where $\mathbf{a}_n$ is a unit vector in the direction of the right thumb (normal to the plane of the loop) as the fingers of the right hand follow the direction of the current, we can write Eq. (6–194) as

$$\boxed{\mathbf{T} = \mathbf{m}\times\mathbf{B} \qquad (\text{N}\cdot\text{m}).} \tag{6–195}$$

The vector $\mathbf{B}$ (instead of $\mathbf{B}_{||}$) is used in Eq. (6–195) because $\mathbf{m}\times(\mathbf{B}_\perp + \mathbf{B}_{||}) = \mathbf{m}\times\mathbf{B}_{||}$. This is the torque that aligns the microscopic magnetic dipoles in magnetic materials and causes the materials to be magnetized by an applied magnetic field. It should be remembered that Eq. (6–195) does not hold if $\mathbf{B}$ is not uniform over the current-carrying loop.

EXAMPLE 6–22 A rectangular loop in the xy-plane with sides b_1 and b_2 carrying a current I lies in a *uniform* magnetic field $\mathbf{B} = \mathbf{a}_x B_x + \mathbf{a}_y B_y + \mathbf{a}_z B_z$. Determine the force and torque on the loop.

Solution Resolving $\mathbf{B}$ into perpendicular and parallel components $\mathbf{B}_\perp$ and $\mathbf{B}_{||}$, we have

$$\mathbf{B}_\perp = \mathbf{a}_z B_z \tag{6–196a}$$

$$\mathbf{B}_{||} = \mathbf{a}_x B_x + \mathbf{a}_y B_y. \tag{6–196b}$$

Assuming that the current flows in a clockwise direction, as shown in Fig. 6–31, we find that the perpendicular component $\mathbf{a}_z B_z$ results in forces Ib_1B_z on sides (1) and (3) and forces Ib_2B_z on sides (2) and (4), all directed toward the center of the loop. The vector sum of these four contracting forces is zero, and no torque is produced.

The parallel component of the magnetic flux density, $\mathbf{B}_{||}$, produces the following forces on the four sides:

$$\begin{aligned}\mathbf{F}_1 &= Ib_1\mathbf{a}_x \times (\mathbf{a}_x B_x + \mathbf{a}_y B_y)\\ &= \mathbf{a}_z Ib_1 B_y = -\mathbf{F}_3;\end{aligned} \tag{6–197a}$$

$$\begin{aligned}\mathbf{F}_2 &= Ib_2(-\mathbf{a}_y) \times (\mathbf{a}_x B_x + \mathbf{a}_y B_y)\\ &= \mathbf{a}_z Ib_2 B_x = -\mathbf{F}_4.\end{aligned} \tag{6–197b}$$

Again, the net force on the loop, $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4$, is zero. However, these forces result in a net torque that can be computed as follows. The torque $\mathbf{T}_{13}$, due to forces $\mathbf{F}_1$ and $\mathbf{F}_3$ on sides (1) and (3), is

$$\mathbf{T}_{13} = \mathbf{a}_x Ib_1b_2B_y; \tag{6–198a}$$

the torque $\mathbf{T}_{24}$, due to forces $\mathbf{F}_2$ and $\mathbf{F}_4$ on sides (2) and (4), is

$$\mathbf{T}_{24} = -\mathbf{a}_y Ib_1b_2B_x. \tag{6–198b}$$

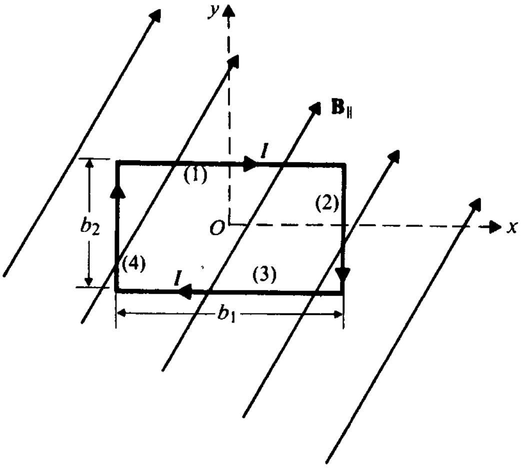


FIGURE 6–31
A rectangular loop in a uniform magnetic field (Example 6–22).

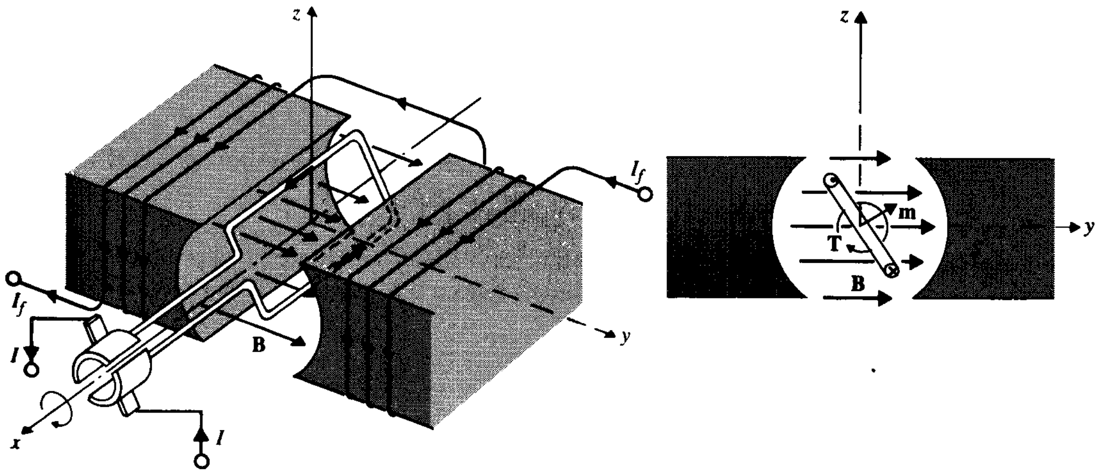
The total torque on the rectangular loop is then

$$\mathbf{T} = \mathbf{T}_{13} + \mathbf{T}_{24} = Ib_1b_2(\mathbf{a}_xB_y - \mathbf{a}_yB_x) \quad (\text{N}\cdot\text{m}). \tag{6-199}$$

Since the magnetic moment of the loop is $\mathbf{m} = -\mathbf{a}_zIb_1b_2$, the result in Eq. (6–199) is exactly $\mathbf{T} = \mathbf{m} \times (\mathbf{a}_xB_x + \mathbf{a}_yB_y) = \mathbf{m} \times \mathbf{B}$. Hence in spite of the fact that Eq. (6–195) was derived for a circular loop, the torque formula holds also for a rectangular loop. As a matter of fact, it can be proved that Eq. (6–195) holds for a planar loop of any shape as long as it is located in a uniform magnetic field. Can you suggest a proof for the last statement?

The principle of operation of direct-current (d-c) motor is based on Eq. (6–195). Figure 6–32(a) shows a schematic diagram of such a motor. The magnetic field **B** is produced by a field current I_f in a winding around the pole pieces. When a current I is sent through the rectangular loop, a torque results that makes the loop rotate in a clockwise direction as viewed from the $+x$-direction. This is illustrated in Fig. 6–32(b). A split ring with brushes is necessary so that the currents in the two legs of the coil reverse their directions every half of a turn in order to maintain the torque **T** always in the same direction; the magnetic moment **m** of the loop must have a positive z-component.

To obtain a smooth operation, an actual d-c motor has many such rectangular loops wound and distributed around an armature. The ends of each loop are attached to a pair of conducting bars arranged on a small cylindrical drum called a ***commutator***.



(a) Perspective view. (b) Schematic view from $+x$ direction.

FIGURE 6–32
Illustrating the principle of operation of d-c motor.

The commutator has twice as many parallel conducting bars insulated from one another as there are loops.

6–13.3 FORCES AND TORQUES IN TERMS OF STORED MAGNETIC ENERGY

All current-carrying conductors and circuits experience magnetic forces when situated in a magnetic field. They are held in place only if mechanical forces, equal and opposite to the magnetic forces, exist. Except for special symmetrical cases (such as the case of the two infinitely long, current-carrying, parallel conducting wires in Example 6–21), determining the magnetic forces between current-carrying circuits by Ampère's law of force is a tedious task. We now examine an alternative method of finding magnetic forces and torques based on the ***principle of virtual displacement***. This principle was used in Section 3–11.2 to determine electrostatic forces between charged conductors. We consider two cases: first, a system of circuits with constant magnetic flux linkages, and second, a system of circuits with constant currents.

System of Circuits with Constant Flux Linkages If we assume that no changes in flux linkages result from a virtual differential displacement $d\ell$ of one of the current-carrying circuits, there will be no induced emf's, and the sources will supply no energy to the system. The mechanical work, $\mathbf{F}_\Phi \cdot d\ell$, done *by the system* is at the expense of a decrease in the stored magnetic energy, where $\mathbf{F}_\Phi$ denotes the force under the constant-flux condition. Thus,

$$\mathbf{F}_\Phi \cdot d\ell = -dW_m = -(\nabla W_m)\cdot d\ell, \tag{6–200}$$

from which we obtain

$$\boxed{\mathbf{F}_\Phi = -\nabla W_m \qquad \text{(N).}} \tag{6–201}$$

In Cartesian coordinates the component forces are

$$(F_\Phi)_x = -\frac{\partial W_m}{\partial x}, \tag{6–202a}$$

$$(F_\Phi)_y = -\frac{\partial W_m}{\partial y}, \tag{6–202b}$$

$$(F_\Phi)_z = -\frac{\partial W_m}{\partial z}. \tag{6–202c}$$

If the circuit is constrained to rotate about an axis, say the z-axis, the mechanical work done by the system will be $(T_\Phi)_z\, d\phi$, and

$$\boxed{(T_\Phi)_z = -\frac{\partial W_m}{\partial \phi} \qquad \text{(N·m),}} \tag{6–203}$$

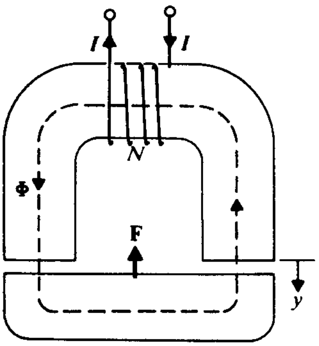


FIGURE 6–33
An electromagnet (Example 6–23).

which is the z-component of the torque acting on the circuit under the condition of constant flux linkages.

EXAMPLE 6–23 Consider the electromagnet in Fig. 6–33, in which a current I in an N-turn coil produces a flux Φ in the magnetic circuit. The cross-sectional area of the core is S. Determine the lifting force on the armature.

Solution Let the armature take a virtual displacement dy (a differential increase in y) and the source be adjusted to keep the flux Φ constant. A displacement of the armature changes only the length of the air gaps; consequently, the displacement changes only the magnetic energy stored in the two air gaps. We have, from Eq. (6–172b),

$$dW_m = d(W_m)_{\substack{\text{air}\\\text{gap}}} = 2\left(\frac{B^2}{2\mu_0}S\,dy\right) = \frac{\Phi^2}{\mu_0 S}\,dy. \tag{6–204}$$

An increase in the air-gap length (a positive dy) increases the stored magnetic energy if Φ is constant. Using Eq. (6–202b), we obtain

$$\mathbf{F}_\Phi = \mathbf{a}_y(F_\Phi)_y = -\mathbf{a}_y\frac{\Phi^2}{\mu_0 S} \quad \text{(N)}. \tag{6–205}$$

Here the negative sign indicates that the force tends to reduce the air-gap length; that is, it is a force of attraction.

System of Circuits with Constant Currents In this case the circuits are connected to current sources that counteract the induced emf's resulting from changes in flux linkages that are caused by a virtual displacement $d\ell$. The work done or energy supplied by the sources is (see Eq. 6–165)

$$dW_s = \sum_k I_k \, d\Phi_k. \tag{6–206}$$

This energy must be equal to the sum of the mechanical work done by the system dW ($dW = \mathbf{F}_I \cdot d\boldsymbol{\ell}$, where $\mathbf{F}_I$ denotes the force on the displaced circuit under the constant-current condition) and the increase in the stored magnetic energy, dW_m. That is,

$$dW_s = dW + dW_m. \tag{6–207}$$

From Eq. (6–166) we have

$$dW_m = \frac{1}{2} \sum_k I_k \, d\Phi_k = \frac{1}{2} \, dW_s. \tag{6–208}$$

Equations (6–207) and (6–208) combine to give

$$\begin{aligned} dW = \mathbf{F}_I \cdot d\boldsymbol{\ell} &= dW_m \\ &= (\nabla W_m) \cdot d\boldsymbol{\ell} \end{aligned}$$

or

$$\boxed{\mathbf{F}_I = \nabla W_m \qquad \text{(N)},} \tag{6–209}$$

which differs from the expression for $\mathbf{F}_\Phi$ in Eq. (6–201) only by a sign change. If the circuit is constrained to rotate about the z-axis, the z-component of the torque acting on the circuit is

$$\boxed{(T_I)_z = \frac{\partial W_m}{\partial \phi} \qquad (\text{N}\cdot\text{m}).} \tag{6–210}$$

The difference between the expression above and $(T_\Phi)_z$ in Eq. (6–203) is, again, only in the sign. It must be understood that, despite the difference in the sign, Eqs. (6–201) and (6–203) should yield the same answers to a given problem as do Eqs. (6–209) and (6–210), respectively. The formulations using the method of virtual displacement under constant-flux-linkage and constant-current conditions are simply two means of solving the same problem.

Let us solve the electromagnet problem in Example 6–23 assuming a virtual displacement under the constant-current condition. For this purpose we express W_m in terms of the current I:

$$W_m = \tfrac{1}{2} L I^2, \tag{6–211}$$

where L is the self-inductance of the coil. The flux, Φ, in the electromagnet is obtained by dividing the applied magnetomotive force (NI) by the sum of the reluctance of the core ($\mathscr{R}_c$) and that of the two air gaps ($2y/\mu_0 S$). Thus,

$$\Phi = \frac{NI}{\mathscr{R}_c + 2y/\mu_0 S}. \tag{6–212}$$

Inductance L is equal to flux linkage per unit current:

$$L = \frac{N\Phi}{I} = \frac{N^2}{\mathscr{R}_c + 2y/\mu_0 S}. \tag{6-213}$$

Combining Eqs. (6–209) and (6–211) and using Eq. (6–213), we obtain

$$\begin{aligned}\mathbf{F}_I &= \mathbf{a}_y \frac{I^2}{2}\frac{dL}{dy} = -\mathbf{a}_y \frac{1}{\mu_0 S}\left(\frac{NI}{\mathscr{R}_c + 2y/\mu_0 S}\right)^2 \\ &= -\mathbf{a}_y \frac{\Phi^2}{\mu_0 S} \qquad \text{(N)},\end{aligned} \tag{6-214}$$

which is exactly the same as the $\mathbf{F}_\Phi$ in Eq. (6–205).

6–13.4 FORCES AND TORQUES IN TERMS OF MUTUAL INDUCTANCE

The method of virtual displacement for constant currents provides a powerful technique for determining the forces and torques between rigid current-carrying circuits. For two circuits with currents I_1 and I_2, self-inductances L_1 and L_2, and mutual inductance L_{12}, the magnetic energy is, from Eq. (6–161),

$$W_m = \tfrac{1}{2}L_1 I_1^2 + L_{12}I_1 I_2 + \tfrac{1}{2}L_2 I_2^2. \tag{6-215}$$

If one of the circuits is given a virtual displacement under the condition of constant currents, there would be a change in W_m, and Eq. (6–209) applies. Substitution of Eq. (6–215) in Eq. (6–209) yields

$$\boxed{\mathbf{F}_I = I_1 I_2(\nabla L_{12}) \qquad \text{(N)}.} \tag{6-216}$$

Similarly, we obtain, from Eq. (6–210),

$$\boxed{(T_I)_z = I_1 I_2 \frac{\partial L_{12}}{\partial \phi} \qquad \text{(N·m)}.} \tag{6-217}$$

EXAMPLE 6–24 Determine the force between two coaxial circular coils of radii b_1 and b_2 separated by a distance d that is much larger than the radii ($d \gg b_1, b_2$). The coils consist of N_1 and N_2 closely wound turns and carry currents I_1 and I_2, respectively.

Solution This problem is rather a difficult one if we try to solve it with Ampère's law of force, as expressed in Eq. (6–185a). Therefore we will base our solution on Eq. (6–216). First, we determine the mutual inductance between the two coils. In Example 6–7 we found, in Eq. (6–43), the vector potential at a distant point, which was caused by a single-turn circular loop carrying a current I. Referring to Fig. 6–34 for this problem, at the point P on coil 2 we have $\mathbf{A}_{12}$ due to current I_1 in coil 1 with N_1 turns as follows:

$$\begin{aligned}\mathbf{A}_{12} &= \mathbf{a}_\phi \frac{\mu_0 N_1 I_1 b_1^2}{4R^2} \sin\theta \\ &= \mathbf{a}_\phi \frac{\mu_0 N_1 I_1 b_1^2}{4R^2}\left(\frac{b_2}{R}\right) \\ &= \mathbf{a}_\phi \frac{\mu_0 N_1 I_1 b_1^2 b_2}{4(z^2+b_2^2)^{3/2}}.\end{aligned} \tag{6–218}$$

In Eq. (6–218), z, instead of d, is used because we anticipate a virtual displacement, and z is to be kept as a variable for the time being. Using Eq. (6–218) in Eq. (6–25), we find the mutual flux.

$$\begin{aligned}\Phi_{12} &= \oint_{C_2} \mathbf{A}_{12}\cdot d\boldsymbol{\ell}_2 = \int_0^{2\pi} A_{12} b_2\, d\phi \\ &= \frac{\mu_0 N_1 I_1 b_1^2 b_2^2 \pi}{2(z^2+b_2^2)^{3/2}}.\end{aligned} \tag{6–219}$$

The mutual inductance is then, from Eq. (6–127),

$$L_{12} = \frac{N_2\Phi_{12}}{I_1} = \frac{\mu_0 N_1 N_2 \pi b_1^2 b_2^2}{2(z^2+b_2^2)^{3/2}} \quad \text{(H)}. \tag{6–220}$$

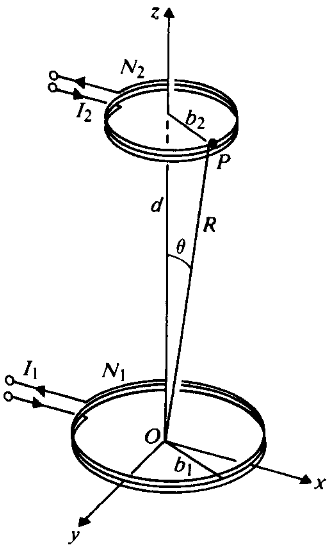


FIGURE 6–34
Coaxial current-carrying circular loops (Example 6–24).

On coil 2 the force due to the magnetic field of coil 1 can now be obtained directly by substituting Eq. (6–220) in Eq. (6–216):

$$\mathbf{F}_{12} = \mathbf{a}_z I_1 I_2 \left.\frac{dL_{12}}{dz}\right|_{z=d} = -\mathbf{a}_z I_1 I_2 \frac{3\mu_0 N_1 N_2 \pi b_1^2 b_2^2 d}{2(d^2 + b_2^2)^{5/2}},$$

which can be written as

$$F_{12} \cong -\mathbf{a}_z \frac{3\mu_0 m_1 m_2}{2\pi d^4} \qquad \text{(N)}, \tag{6–221}$$

where $(d^2 + b_2^2)$ has been replaced approximately by d^2, and m_1 and m_2 are the magnitudes of the magnetic moments of coils 1 and 2, respectively:

$$m_1 = N_1 I_1 \pi b_1^2, \qquad m_2 = N_2 I_2 \pi b_2^2.$$

The negative sign in Eq. (6–221) indicates that $\mathbf{F}_{12}$ is a force of attraction for currents flowing in the same direction. This force diminishes very rapidly as the inverse fourth power of the distance of separation. ■

Review Questions

R.6–1 What is the expression for the force on a test charge q that moves with velocity $\mathbf{u}$ in a magnetic field of flux density $\mathbf{B}$?

R.6–2 Verify that tesla (T), the unit for magnetic flux density, is the same as volt-second per square meter ($\text{V}\cdot\text{s/m}^2$).

R.6–3 Write Lorentz's force equation.

R.6–4 Which postulate of magnetostatics denies the existence of isolated magnetic charges?

R.6–5 State the law of conservation of magnetic flux.

R.6–6 State Ampère's circuital law.

R.6–7 In applying Ampère's circuital law, must the path of integration be circular? Explain.

R.6–8 Why cannot the **B**-field of an infinitely long, straight, current-carrying conductor have a component in the direction of the current?

R.6–9 Do the formulas for **B**, as derived in Eqs. (6–11) and (6–12) for a round conductor, apply to a conductor having a square cross section of the same area and carrying the same current? Explain.

R.6–10 In what manner does the **B**-field of an infinitely long straight filament carrying a direct current I vary with distance?

R.6–11 Can a static magnetic field exist in a good conductor? Explain.

R.6–12 Define in words *vector magnetic potential* **A**. What is its SI unit?

R.6–13 What is the relation between magnetic flux density **B** and vector magnetic potential **A**? Give an example of a situation in which **B** is zero and **A** is not.

R.6–14 What is the relation between vector magnetic potential **A** and the magnetic flux through a given area?

R.6–15 State Biot-Savart law.

R.6–16 Compare the usefulness of Ampère's circuital law and Biot-Savart law in determining **B** of a current-carrying circuit.

R.6–17 What is a magnetic dipole? Define *magnetic dipole moment.* What is its SI unit?

R.6–18 Define *scalar magnetic potential* V_m. What is its SI unit?

R.6–19 Discuss the relative merits of using the vector and scalar magnetic potentials in magnetostatics.

R.6–20 Define *magnetization vector.* What is its SI unit?

R.6–21 What is meant by "equivalent magnetization current densities"? What are the SI units for $\nabla \times \mathbf{M}$ and $\mathbf{M} \times \mathbf{a}_n$?

R.6–22 Define *magnetic field intensity vector.* What is its SI unit?

R.6–23 What are *magnetization charge densities*? What are the SI units for $\mathbf{M} \cdot \mathbf{a}_n$ and $-\nabla \cdot \mathbf{M}$?

R.6–24 Describe a procedure for finding the external magnetic field of a bar magnet having a known volume density of dipole moment.

R.6–25 Define *magnetic susceptibility* and *relative permeability.* What are their SI units?

R.6–26 Does the magnetic field intensity due to a current distribution depend on the properties of the medium? Does the magnetic flux density?

R.6–27 Define *magnetomotive force.* What is its SI unit?

R.6–28 What is the reluctance of a piece of magnetic material of permeability μ, length ℓ, and a constant cross section S? What is its SI unit?

R.6–29 An air gap is cut in a ferromagnetic toroidal core. The core is excited with an mmf of NI ampere-turns. Is the magnetic field intensity in the air gap higher or lower than that in the core?

R.6–30 Define *diamagnetic, paramagnetic,* and *ferromagnetic* materials.

R.6–31 What is a magnetic domain?

R.6–32 Define *remanent flux density* and *coercive field intensity.*

R.6–33 Discuss the difference between soft and hard ferromagnetic materials.

R.6–34 What is *curie temperature*?

R.6–35 What are the characteristics of ferrites?

R.6–36 What are the boundary conditions for magnetostatic fields at an interface between two different magnetic media?

R.6–37 Explain why magnetic flux lines leave the surface of a ferromagnetic medium perpendicularly.

R.6–38 Explain qualitatively the statement that **H** and **B** along the axis of a cylindrical bar magnet are in opposite directions.

R.6–39 Define (a) the mutual inductance between two circuits, and (b) the self-inductance of a single coil.

R.6–40 Explain how the self-inductance of a wire-wound inductor depends on its number of turns.

R.6–41 In Example 6–16, would the answer be the same if the outer conductor were not "very thin"? Explain.

R.6–42 What is implied by "quasi-static conditions" in electromagnetics?

R.6–43 Give an expression of magnetic energy in terms of **B** and/or **H**.

R.6–44 Explain the *Hall effect*.

R.6–45 Give the integral expression for the force on a closed circuit that carries a current I in a magnetic field **B**.

R.6–46 Discuss first the net force and then the net torque acting on a current-carrying circuit situated in a uniform magnetic field.

R.6–47 Explain the principle of operation of d-c motors.

R.6–48 What is the relation between the force and the stored magnetic energy in a system of current-carrying circuits under the condition of constant flux linkages? Under the condition of constant currents?

Problems

P.6–1 A positive point charge q of mass m is injected with a velocity $\mathbf{u}_0 = \mathbf{a}_y u_0$ into the $y > 0$ region where a uniform magnetic field $\mathbf{B} = \mathbf{a}_x B_0$ exists. Obtain the equation of motion of the charge, and describe the path that the charge follows.

P.6–2 An electron is injected with a velocity $\mathbf{u}_0 = \mathbf{a}_y u_0$ into a region where both an electric field **E** and a magnetic field **B** exist. Describe the motion of the electron if

a) $\mathbf{E} = \mathbf{a}_z E_0$ and $\mathbf{B} = \mathbf{a}_x B_0$,
b) $\mathbf{E} = -\mathbf{a}_z E_0$ and $\mathbf{B} = -\mathbf{a}_z B_0$.

Discuss the effect of the relative magnitudes of E_0 and B_0 on the electron paths in parts (a) and (b).

P.6–3 A current I flows in the inner conductor of an infinitely long coaxial line and returns via the outer conductor. The radius of the inner conductor is a, and the inner and outer radii of the outer conductor are b and c, respectively. Find the magnetic flux density **B** for all regions and plot $|\mathbf{B}|$ versus r.

P.6–4 A current I flows lengthwise in a very long, thin conducting sheet of width w, as shown in Fig. 6–35.

a) Assuming that the current flows into the paper, determine the magnetic flux density $\mathbf{B}_1$ at point $P_1(0, d)$.
b) Use the result in part (a) to find the magnetic flux density $\mathbf{B}_2$ at point $P_2(2w/3, d)$.

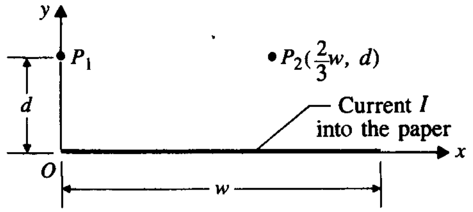


FIGURE 6–35
A thin conducting sheet carrying a current I (Problem P.6–4).

P.6–5 A current I flows in a $w \times w$ square loop as in Fig. 6–36. Find the magnetic flux density at the off-center point $P(w/4, w/2)$.

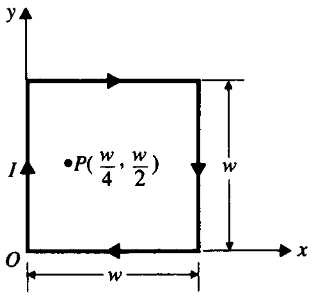


FIGURE 6–36
A square loop carrying a current I (Problem P.6–5).

P.6–6 Figure 6–37 shows an infinitely long solenoid with air core having a radius b and n closely wound turns per unit length. The windings are slanted at an angle α and carry a current I. Determine the magnetic flux density both inside and outside the solenoid.

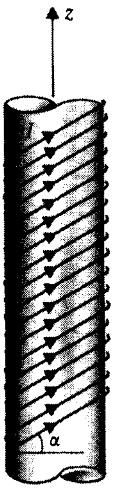


FIGURE 6–37
A long solenoid with closely wound windings carrying a current I (Problem P.6–6).

P.6–7 Determine the magnetic flux density at a point on the axis of a solenoid with radius b and length L, and with a current I in its N turns of closely wound coil. Show that the result reduces to that given in Eq. (6–14) when L approaches infinity.

P.6–8 Starting from the expression for vector magnetic potential **A** in Eq. (6–23), prove that

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\mathbf{J} \times \mathbf{a}_R}{R^2}\, dv'. \qquad (6\text{–}222)$$

Furthermore, prove that **B** in Eq. (6–222) satisfies the fundamental postulates of magnetostatics in free space, Eqs. (6–6) and (6–7).

P.6–9 Combine Eqs. (6–4) and (6–33) to obtain a formula for the magnetic force $\mathbf{F}_{12}$ exerted by a charge q_1 moving with a velocity $\mathbf{u}_1$ on a charge q_2 moving with a velocity $\mathbf{u}_2$.

P.6–10 A very long, thin conducting strip of width w lies in the xz-plane between $x = \pm w/2$. A surface current $\mathbf{J}_s = \mathbf{a}_z J_{s0}$ flows in the strip. Find the magnetic flux density at an arbitrary point outside the strip.

P.6–11 A long wire carrying a current I folds back with a semicircular bend of radius b as in Fig. 6–38. Determine the magnetic flux density at the center point P of the bend.

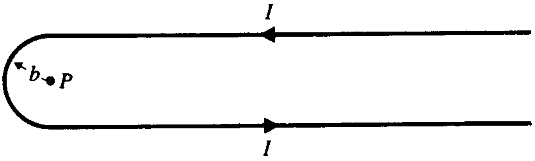


FIGURE 6–38
A very long wire with a semicircular bend (Problem P.6–11).

P.6–12 Two identical coaxial coils, each of N turns and radius b, are separated by a distance d, as depicted in Fig. 6–39. A current I flows in each coil in the same direction.

a) Find the magnetic flux density $\mathbf{B} = \mathbf{a}_x B_x$ at a point midway between the coils.

b) Show that dB_x/dx vanishes at the midpoint.

c) Find the relation between b and d such that d^2B_x/dx^2 also vanishes at the midpoint. Such a pair of coils are used to obtain an approximately uniform magnetic field in the midpoint region. They are known as ***Helmholtz coils.***

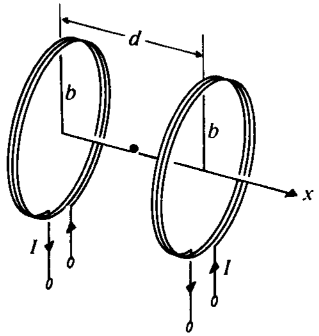


FIGURE 6–39
Helmholtz coils (Problems P.6–12).

P.6–13 A thin conducting wire is bent into the shape of a regular polygon of N sides. A current I flows in the wire. Show that the magnetic flux density at the center is

$$\mathbf{B} = \mathbf{a}_n \frac{\mu_0 NI}{2\pi b} \tan\frac{\pi}{N},$$

where b is the radius of the circle circumscribing the polygon and $\mathbf{a}_n$ is a unit vector normal to the plane of the polygon. Show also that, as N becomes very large, this result reduces to that given in Eq. (6–38) with $z = 0$.

P.6–14 Find the total magnetic flux through a circular toroid with a rectangular cross section of height h. The inner and outer radii of the toroid are a and b, respectively. A current I flows in N turns of closely wound wire around the toroid. Determine the percentage of error if the flux is found by multiplying the cross-sectional area by the flux density at the mean radius.

P.6–15 In certain experiments it is desirable to have a region of constant magnetic flux density. This can be created in an off-center cylindrical cavity that is cut in a very long cylindrical conductor carrying a uniform current density. Refer to the cross section in

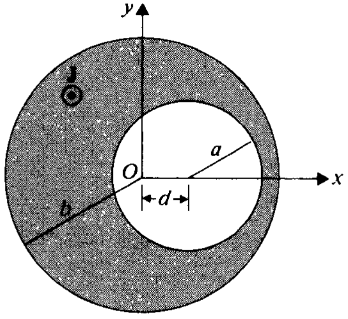


FIGURE 6–40
Cross section of a long cylindrical conductor with cavity (Problem P.6–15).

Fig. 6–40. The uniform axial current density is $\mathbf{J} = \mathbf{a}_z J$. Find the magnitude and direction of **B** in the cylindrical cavity whose axis is displaced from that of the conducting part by a distance d. (*Hint:* Use principle of superposition and consider **B** in the cavity as that due to two long cylindrical conductors with radii b and a and current densities **J** and $-\mathbf{J}$, respectively.)

P.6–16 Prove the following:

a) If Cartesian coordinates are used, Eq. (6–18) for the Laplacian of a vector field holds.

b) If cylindrical coordinates are used, $\nabla^2 \mathbf{A} \neq \mathbf{a}_r \nabla^2 A_r + \mathbf{a}_\phi \nabla^2 A_\phi + \mathbf{a}_z \nabla^2 A_z$.

P.6–17 The magnetic flux density **B** for an infinitely long cylindrical conductor has been found in Example 6–1. Determine the vector magnetic potential **A** both inside and outside the conductor from the relation $\mathbf{B} = \nabla \times \mathbf{A}$.

P.6–18 Starting from the expression of **A** in Eq. (6–34) for the vector magnetic potential at a point in the bisecting plane of a straight wire of length $2L$ that carries a current I:

a) Find **A** at point $P(x, y, 0)$ in the bisecting plane of two parallel wires each of length $2L$, located at $y = \pm d/2$ and carrying equal and opposite currents, as shown in Fig. 6–41.

b) Find **A** due to equal and opposite currents in a very long two-wire transmission line.

c) Find **B** from **A** in part (b), and check your answer against the result obtained by applying Ampère's circuital law.

d) Find the equation for the magnetic flux lines in the xy-plane.

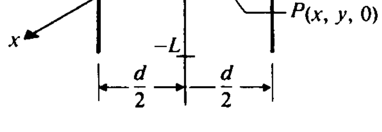


FIGURE 6–41
Parallel wires carrying equal and opposite currents (Problem P.6–18).

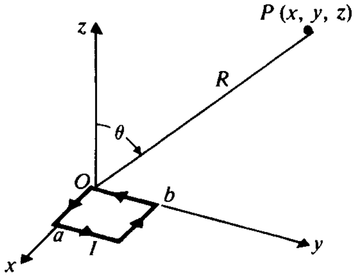


FIGURE 6–42
A small rectangular loop carrying a current I (Problem P.6–19).

P.6–19 For the small rectangular loop with sides a and b that carries a current I, shown in Fig. 6–42:

a) Find the vector magnetic potential **A** at a distant point, $P(x, y, z)$. Show that it can be put in the form of Eq. (6–45).

b) Determine the magnetic flux density **B** from **A**, and show that it is the same as that given in Eq. (6–48).

P.6–20 For a vector field **F** with continuous first derivatives, prove that

$$\int_V (\nabla \times \mathbf{F})\,dv = -\oint_S \mathbf{F} \times d\mathbf{s},$$

where S is the surface enclosing the volume V. (*Hint:* Apply the divergence theorem to $(\mathbf{F} \times \mathbf{C})$, where **C** is a constant vector.)

P.6–21 A very large slab of material of thickness d lies perpendicularly to a uniform magnetic field of intensity $\mathbf{H}_0 = \mathbf{a}_z H_0$. Ignoring edge effect, determine the magnetic field intensity in the slab:

a) if the slab material has a permeability μ,

b) if the slab is a permanent magnet having a magnetization vector $\mathbf{M}_i = \mathbf{a}_z M_i$.

P.6–22 A circular rod of magnetic material with permeability μ is inserted coaxially in the long solenoid of Fig. 6–4. The radius of the rod, a, is less than the inner radius, b, of the solenoid. The solenoid's winding has n turns per unit length and carries a current I.

a) Find the values of **B**, **H**, and **M** inside the solenoid for $r < a$ and for $a < r < b$.

b) What are the equivalent magnetization current densities $\mathbf{J}_m$ and $\mathbf{J}_{ms}$ for the magnetized rod?

P.6–23 The scalar magnetic potential, V_m, due to a current loop can be obtained by first dividing the loop area into many small loops and then summing up the contribution of these small loops (magnetic dipoles); that is,

$$V_m = \int dV_m = \int \frac{d\mathbf{m} \cdot \mathbf{a}_R}{4\pi R^2}, \qquad (6\text{–}223a)$$

where

$$d\mathbf{m} = \mathbf{a}_n I\, ds. \qquad (6\text{–}223b)$$

Prove that

$$V_m = -\frac{I}{4\pi}\Omega, \qquad (6\text{–}224)$$

where Ω is the solid angle subtended by the loop surface at the field point P (see Fig. 6–43).

P.6–24 Do the following by using Eq. (6–224):

a) Determine the scalar magnetic potential at a point on the axis of a circular loop having a radius b and carrying a current I.

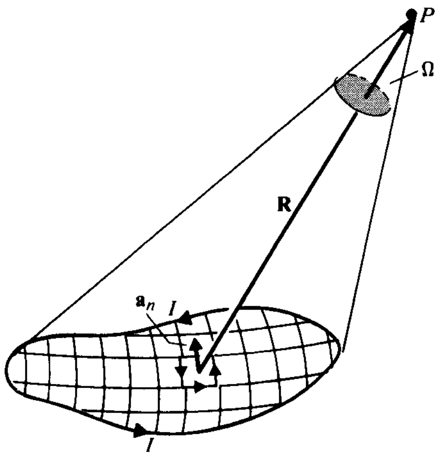


FIGURE 6–43
Subdivided current loop for determination of scalar magnetic potential (Problem P.6–23).

b) Obtain the magnetic flux density **B** from $-\mu_0 \nabla V_m$, and compare the result with Eq. (6–38).

P.6–25 Solve the cylindrical bar magnet problem in Example 6–9, using the equivalent magnetization current density concept.

P.6–26 A ferromagnetic sphere of radius b is magnetized uniformly with a magnetization $\mathbf{M} = \mathbf{a}_z M_0$.

a) Determine the equivalent magnetization current densities $\mathbf{J}_m$ and $\mathbf{J}_{ms}$.
b) Determine the magnetic flux density at the center of the sphere.

P.6–27 A toroidal iron core of relative permeability 3000 has a mean radius $R = 80$ (mm) and a circular cross section with radius $b = 25$ (mm). An air gap $\ell_g = 3$ (mm) exists, and a current I flows in a 500-turn winding to produce a magnetic flux of 10^{-5} (Wb). (See Fig. 6–44.) Neglecting flux leakage and using mean path length, find

a) the reluctances of the air gap and of the iron core,
b) $\mathbf{B}_g$ and $\mathbf{H}_g$ in the air gap, and $\mathbf{B}_c$ and $\mathbf{H}_c$ in the iron core,
c) the required current I.

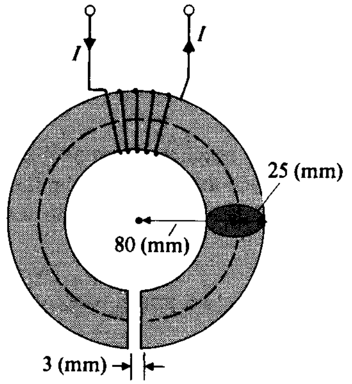


FIGURE 6–44
A toroidal iron core with air gap (Problem P.6–27).

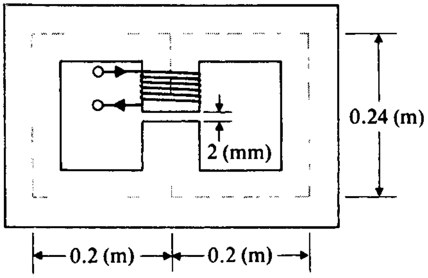


FIGURE 6–45
A magnetic circuit with air gap (Problem P.6–28).

P.6–28 Consider the magnetic circuit in Fig. 6–45. A current of 3 (A) flows through 200 turns of wire on the center leg. Assuming the core to have a constant cross-sectional area of 10^{-3} (m^2) and a relative permeability of 5000:

a) Determine the magnetic flux in each leg.

b) Determine the magnetic field intensity in each leg of the core and in the air gap.

P.6–29 Consider an infinitely long solenoid with n turns per unit length around a ferromagnetic core of cross-sectional area S. When a current is sent through the coil to create a magnetic field, a voltage $v_1 = -n\,d\Phi/dt$ is induced per unit length, which opposes the current change. Power $P_1 = -v_1 I$ per unit length must be supplied to overcome this induced voltage in order to increase the current to I.

a) Prove that the work per unit volume required to produce a final magnetic flux density B_f is

$$W_1 = \int_0^{B_f} H\,dB. \tag{6–225}$$

b) Assuming that the current is changed in a periodic manner such that B is reduced from B_f to $-B_f$ and then is increased again to B_f, prove that the work done per unit volume for such a cycle of change in the ferromagnetic core is represented by the area of the hysteresis loop of the core material.

P.6–30 Prove that the relation $\nabla \times \mathbf{H} = \mathbf{J}$ leads to Eq. (6–111) at an interface between two media.

P.6–31 What boundary conditions must the scalar magnetic potential V_m satisfy at an interface between two different magnetic media?

P.6–32 Consider a plane boundary ($y = 0$) between air (region 1, $\mu_{r1} = 1$) and iron (region 2, $\mu_{r2} = 5000$).

a) Assuming $\mathbf{B}_1 = \mathbf{a}_x 0.5 - \mathbf{a}_y 10$ (mT), find $\mathbf{B}_2$ and the angle that $\mathbf{B}_2$ makes with the interface.

b) Assuming $\mathbf{B}_2 = \mathbf{a}_x 10 + \mathbf{a}_y 0.5$ (mT), find $\mathbf{B}_1$ and the angle that $\mathbf{B}_1$ makes with the normal to the interface.

P.6–33 The *method of images* can also be applied to certain magnetostatic problems. Consider a straight, thin conductor in air parallel to and at a distance d above the plane interface of a magnetic material of relative permeability μ_r. A current I flows in the conductor.

a) Show that all boundary conditions are satisfied if

i) the magnetic field in the air is calculated from I and an image current I_i,

$$I_i = \left(\frac{\mu_r - 1}{\mu_r + 1}\right) I,$$

and these currents are equidistant from the interface and situated in air;

ii) the magnetic field below the boundary plane is calculated from I and $-I_i$, both at the same location. These currents are situated in an infinite magnetic material of relative permeability μ_r.

b) For a long conductor carrying a current I and for $\mu_r \gg 1$, determine the magnetic flux density $\mathbf{B}$ at the point P in Fig. 6–46.

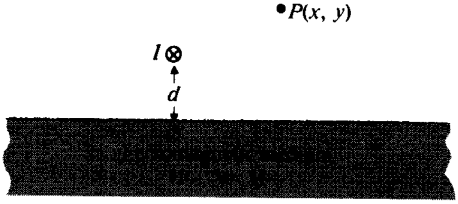


FIGURE 6–46
A current-carrying conductor near a ferromagnetic medium (Problem P.6–33).

P.6–34 A very long conductor in free space carrying a current I is parallel to, and at a distance d from, an infinite plane interface with a medium.

a) Discuss the behavior of the normal and tangential components of $\mathbf{B}$ and $\mathbf{H}$ at the interface:
   i) if the medium is infinitely conducting;
   ii) if the medium is infinitely permeable.
b) Find and compare the magnetic field intensities $\mathbf{H}$ at an arbitrary point in the free space for the two cases in part (a).
c) Determine the surface current densities at the interface, if any, for the two cases.

P.6–35 Determine the self-inductance of a toroidal coil of N turns of wire wound on an air frame with mean radius r_o and a circular cross section of radius b. Obtain an approximate expression assuming $b \ll r_o$.

P.6–36 Refer to Example 6–16. Determine the inductance per unit length of the air coaxial transmission line assuming that its outer conductor is not very thin but is of a thickness d.

P.6–37 Calculate the mutual inductance per unit length between two parallel two-wire transmission lines A–A' and B–B' separated by a distance D, as shown in Fig. 6–47. Assume the wire radius to be much smaller than D and the wire spacing d.

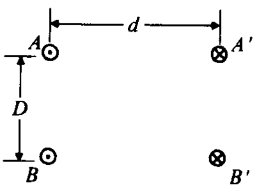


FIGURE 6–47
Coupled two-wire transmission lines (Problem P.6–37).

P.6–38 Determine the mutual inductance between a very long, straight wire and a conducting equilateral triangular loop, as shown in Fig. 6–48.

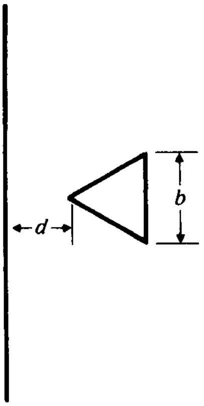


FIGURE 6–48
A long, straight wire and a conducting equilateral triangular loop (Problem P.6–38).

P.6–39 Determine the mutual inductance between a very long, straight wire and a conducting circular loop, as shown in Fig. 6–49.

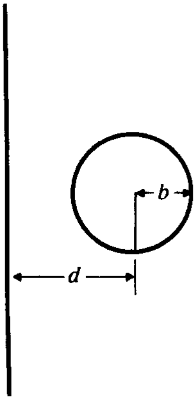


FIGURE 6–49
A long, straight wire and a conducting circular loop (Problem P.6–39).

P.6–40 Find the mutual inductance between two coplanar rectangular loops with parallel sides, as shown in Fig. 6–50. Assume that $h_1 \gg h_2$ $(h_2 > w_2 > d)$.

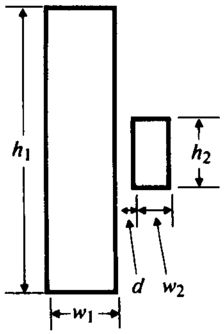


FIGURE 6–50
Two coplanar rectangular loops, $h_1 \gg h_2$ (Problem P.6–40).

P.6–41 Consider two coupled circuits, having self-inductances L_1 and L_2, that carry currents I_1 and I_2, respectively. The mutual inductance between the circuits is M.

a) Using Eq. (6–161), find the ratio I_1/I_2 that makes the stored magnetic energy W_2 a minimum.

b) Show that $M \leq \sqrt{L_1 L_2}$.

P.6–42 Calculate the force per unit length on each of three equidistant, infinitely long, parallel wires 0.15 (m) apart, each carrying a current of 25 (A) in the same direction. Specify the direction of the force.

P.6–43 The cross section of a long thin metal strip and a parallel wire is shown in Fig. 6–51. Equal and opposite currents I flow in the conductors. Find the force per unit length on the conductors.

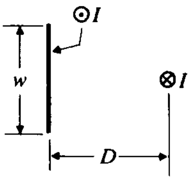


FIGURE 6–51
Cross section of parallel strip and wire conductor (Problem P.6–43).

P.6–44 Determine the force per unit length between two parallel, long, thin conducting strips of equal width w. The strips are at a distance d apart and carry currents I_1 and I_2 in opposite directions as in Fig. 6–52.

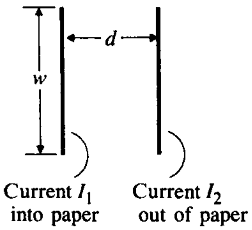


FIGURE 6–52
Cross section of two parallel strips carrying opposite currents (Problem P.6–44).

P.6–45 Refer to Problem 6–39 and Fig. 6–49. Find the force on the circular loop that is exerted by the magnetic field due to an upward current I_1 in the long straight wire. The circular loop carries a current I_2 in the counterclockwise direction.

P.6–46 The bar AA' in Fig. 6–53 serves as a conducting path (such as the blade of a circuit breaker) for the current I in two very long parallel lines. The lines have a radius b and are spaced at a distance d apart. Find the direction and the magnitude of the magnetic force on the bar.

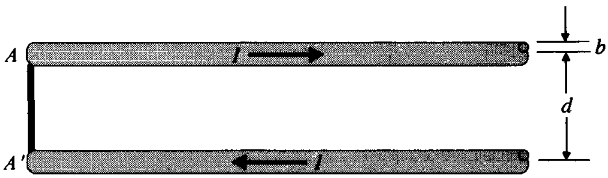


FIGURE 6–53
Force on end conducting bar (Problem P.6–46).

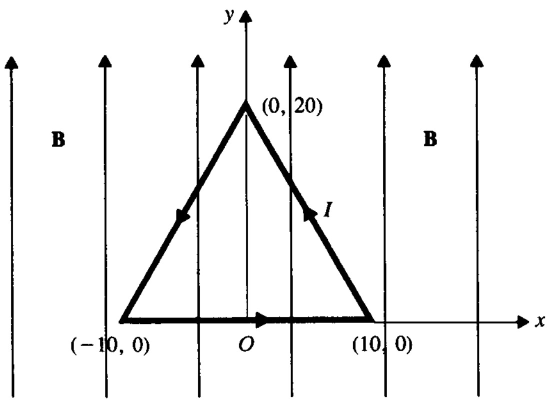


FIGURE 6–54
A triangular loop in a uniform magnetic field (Problem P.6–47).

P.6–47 A d-c current $I = 10$ (A) flows in a triangular loop in the xy-plane as in Fig. 6–54. Assuming a uniform magnetic flux density $\mathbf{B} = \mathbf{a}_y 0.5$ (T) in the region, find the forces and torque on the loop. The dimensions are in (cm).

P.6–48 One end of a long air-core coaxial transmission line having an inner conductor of radius a and an outer conductor of inner radius b is short-circuited by a thin, tight-fitting conducting washer. Find the magnitude and the direction of the magnetic force on the washer when a current I flows in the line.

P.6–49 Assuming that the circular loop in Problem P.6–45 is rotated about its horizontal axis by an angle α, find the torque exerted on the circular loop.

P.6–50 A small circular turn of wire of radius r_1 that carries a steady current I_1 is placed at the center of a much larger turn of wire of radius r_2 ($r_2 \gg r_1$) that carries a steady current I_2 in the same direction. The angle between the normals of the two circuits is θ and the small circular wire is free to turn about its diameter. Determine the magnitude and the direction of the torque on the small circular wire.

P.6–51 A magnetized compass needle will line up with the earth's magnetic field. A small bar magnet (a magnetic dipole) with a magnetic moment 2 (A·m^2) is placed at a distance 0.15 (m) from the center of a compass needle. Assuming the earth's magnetic flux density at the needle to be 0.1 (mT), find the maximum angle at which the bar magnet can cause the needle to deviate from the north–south direction. How should the bar magnet be oriented?

P.6–52 The total mean length of the flux path in iron for the electromagnet in Fig. 6–33 is 3 (m), and the yoke-bar contact areas measure 0.01 (m^2). Assuming the permeability of iron to be $4000\mu_0$ and each of air gaps to be 2 (mm), calculate the mmf needed to lift a total mass of 100 (kg).

P.6–53 A current I flows in a long solenoid with n closely wound coil-turns per unit length. The cross-sectional area of its iron core, which has permeability μ, is S. Determine the force acting on the core if it is withdrawn to the position shown in Fig. 6–55.

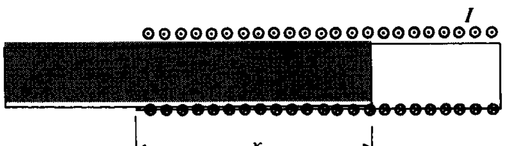


FIGURE 6–55
A long solenoid with iron core partially withdrawn (Problem P.6–53).

7

Time-Varying Fields and Maxwell's Equations

7–1 Introduction

In constructing the electrostatic model we defined an electric field intensity vector, **E**, and an electric flux density (electric displacement) vector, **D**. The fundamental governing differential equations are

$$\nabla \times \mathbf{E} = 0, \tag{3-5}$$

$$\nabla \cdot \mathbf{D} = \rho. \tag{3-98}$$

For linear and isotropic (not necessarily homogeneous) media, **E** and **D** are related by the constitutive relation

$$\mathbf{D} = \epsilon \mathbf{E}. \tag{3-102}$$

For the magnetostatic model we defined a magnetic flux density vector, **B**, and a magnetic field intensity vector, **H**. The fundamental governing differential equations are

$$\nabla \cdot \mathbf{B} = 0, \tag{6-6}$$

$$\nabla \times \mathbf{H} = \mathbf{J}. \tag{6-76}$$

The constitutive relation for **B** and **H** in linear and isotropic media is

$$\mathbf{H} = \frac{1}{\mu}\mathbf{B}. \tag{6-80b}$$

These fundamental relations are summarized in Table 7–1.

We observe that, in the static (non-time-varying) case, electric field vectors **E** and **D** and magnetic field vectors **B** and **H** form separate and independent pairs. In other words, **E** and **D** in the electrostatic model are not related to **B** and **H** in the magnetostatic model. In a conducting medium, static electric and magnetic fields may both exist and form an ***electromagnetostatic field*** (see the statement following Example 5–4 on p. 215). A static electric field in a conducting medium causes a steady current

TABLE 7–1
Fundamental Relations for Electrostatic and Magnetostatic Models

Fundamental Relations	Electrostatic Model	Magnetostatic Model
Governing equations	$\nabla \times \mathbf{E} = 0$ $\nabla \cdot \mathbf{D} = \rho$	$\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \mathbf{J}$
Constitutive relations (linear and isotropic media)	$\mathbf{D} = \epsilon\mathbf{E}$	$\mathbf{H} = \frac{1}{\mu}\mathbf{B}$

to flow that, in turn, gives rise to a static magnetic field. However, the electric field can be completely determined from the static electric charges or potential distributions. The magnetic field is a consequence; it does not enter into the calculation of the electric field.

In this chapter we will see that a changing magnetic field gives rise to an electric field, and vice versa. To explain electromagnetic phenomena under time-varying conditions, it is necessary to construct an electromagnetic model in which the electric field vectors **E** and **D** are properly related to the magnetic field vectors **B** and **H**. The two pairs of the governing equations in Table 7–1 must therefore be modified to show a mutual dependence between the electric and magnetic field vectors in the time-varying case.

We will begin with a fundamental postulate that modifies the $\nabla \times \mathbf{E}$ equation in Table 7–1 and leads to Faraday's law of electromagnetic induction. The concepts of transformer emf and motional emf will be discussed. With the new postulate we will also need to modify the $\nabla \times \mathbf{H}$ equation in order to make the governing equations consistent with the equation of continuity (law of conservation of charge). The two modified curl equations together with the two divergence equations in Table 7–1 are known as Maxwell's equations and form the foundation of electromagnetic theory. The governing equations for electrostatics and magnetostatics are special forms of Maxwell's equations when all quantities are independent of time. Maxwell's equations can be combined to yield wave equations that predict the existence of electromagnetic waves propagating with the velocity of light. The solutions of the wave equations, especially for time-harmonic fields, will be discussed in this chapter.

7–2 Faraday's Law of Electromagnetic Induction

A major advance in electromagnetic theory was made by Michael Faraday, who, in 1831, discovered experimentally that a current was induced in a conducting loop when the magnetic flux linking the loop changed. The quantitative relationship between the induced emf and the rate of change of flux linkage, based on experimental observation, is known as ***Faraday's law***. It is an experimental law and can be considered as a postulate. However, we do not take the experimental relation concerning a finite loop as the starting point for developing the theory of electromagnetic induction. Instead, we follow our approach in Chapter 3 for electrostatics and in Chapter 6 for magnetostatics by putting forth the following fundamental postulate and developing from it the integral forms of Faraday's law.

Fundamental Postulate for Electromagnetic Induction

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.} \tag{7–1}$$

Equation (7–1) expresses a point-function relationship; that is, it applies to every point in space, whether it be in free space or in a material medium. *The electric field intensity in a region of time-varying magnetic flux density is therefore nonconservative and cannot be expressed as the gradient of a scalar potential.*

Taking the surface integral of both sides of Eq. (7–1) over an open surface and applying Stokes's theorem, we obtain

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}. \tag{7–2}$$

Equation (7–2) is valid for any surface S with a bounding contour C, whether or not a physical circuit exists around C. Of course, in a field with no time variation, $\partial \mathbf{B}/\partial t = 0$, Eqs. (7–1) and (7–2) reduce to Eqs. (3–5) and (3–8), respectively, for electrostatics.

In the following subsections we discuss separately the cases of a stationary circuit in a time-varying magnetic field, a moving conductor in a static magnetic field, and a moving circuit in a time-varying magnetic field.

7–2.1 A STATIONARY CIRCUIT IN A TIME-VARYING MAGNETIC FIELD

For a stationary circuit with a contour C and surface S, Eq. (7–2) can be written as

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{s}. \tag{7–3}$$

If we define

$$\mathscr{V} = \oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = \text{emf induced in circuit with contour } C \qquad \text{(V)} \tag{7–4}$$

and

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \text{magnetic flux crossing surface } S \qquad \text{(Wb)}, \tag{7–5}$$

then Eq. (7–3) becomes

$$\boxed{\mathscr{V} = -\frac{d\Phi}{dt} \qquad \text{(V)}.} \tag{7–6}$$

Equation (7–6) states that ***the electromotive force induced in a stationary closed circuit is equal to the negative rate of increase of the magnetic flux linking the circuit.*** This is a statement of ***Faraday's law of electromagnetic induction.*** A time-rate of change of magnetic flux induces an electric field according to Eq. (7–3), even in the absence of a physical closed circuit. The negative sign in Eq. (7–6) is an assertion that the induced emf will cause a current to flow in the closed loop in such a direction as to oppose the change in the linking magnetic flux. This assertion is known as ***Lenz's law.*** The emf induced in a stationary loop caused by a time-varying magnetic field is a ***transformer emf.***

EXAMPLE 7–1 A circular loop of N turns of conducting wire lies in the xy-plane with its center at the origin of a magnetic field specified by $\mathbf{B} = \mathbf{a}_z B_0 \cos(\pi r/2b) \sin \omega t$, where b is the radius of the loop and ω is the angular frequency. Find the emf induced in the loop.

Solution The problem specifies a stationary loop in a time-varying magnetic field; hence Eq. (7–6) can be used directly to find the induced emf, $\mathscr{V}$. The magnetic flux linking each turn of the circular loop is

$$\begin{aligned}\Phi &= \int_S \mathbf{B} \cdot d\mathbf{s} \\ &= \int_0^b \left[\mathbf{a}_z B_0 \cos\frac{\pi r}{2b} \sin \omega t\right] \cdot (\mathbf{a}_z 2\pi r\, dr) \\ &= \frac{8b^2}{\pi}\left(\frac{\pi}{2} - 1\right) B_0 \sin \omega t.\end{aligned}$$

Since there are N turns, the total flux linkage is $N\Phi$, and we obtain

$$\begin{aligned}\mathscr{V} &= -N\frac{d\Phi}{dt} \\ &= -\frac{8N}{\pi} b^2\left(\frac{\pi}{2} - 1\right) B_0 \omega \cos \omega t \qquad \text{(V)}.\end{aligned}$$

The induced emf is seen to be 90° out of time phase with the magnetic flux.

7–2.2 TRANSFORMERS

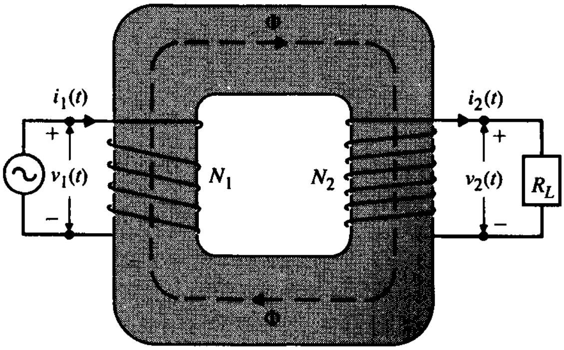
A transformer is an alternating-current (a-c) device that transforms voltages, currents, and impedances. It usually consists of two or more coils coupled magnetically through a common ferromagnetic core, such as that sketched in Fig. 7–1. Faraday's law of electromagnetic induction is the principle of operation of transformers.

For the closed path in the magnetic circuit in Fig. 7–1(a) traced by magnetic flux Φ, we have, from Eq. (6–101),

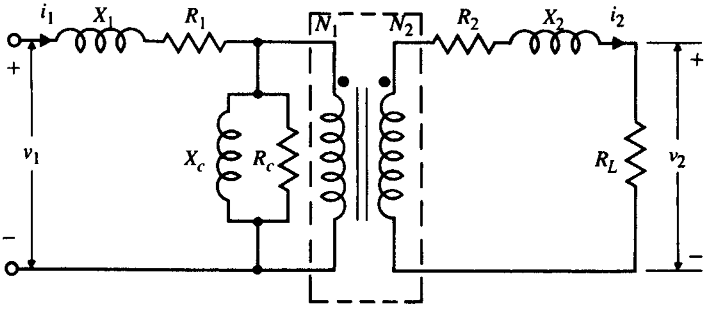
$$N_1 i_1 - N_2 i_2 = \mathscr{R}\Phi, \tag{7–7}$$

where N_1, N_2 and i_1, i_2 are the numbers of turns and the currents in the primary and secondary circuits, respectively, and $\mathscr{R}$ denotes the reluctance of the magnetic circuit. In Eq. (7–7) we have noted, in accordance with Lenz's law, that the induced mmf in the secondary circuit, $N_2 i_2$, opposes the flow of the magnetic flux Φ created by the mmf in the primary circuit, $N_1 i_1$. From Section 6–8 we know that the reluctance of the ferromagnetic core of length ℓ, cross-sectional area S, and permeability μ is

$$\mathscr{R} = \frac{\ell}{\mu S}. \tag{7–8}$$



(a) Schematic diagram of a transformer.



(b) An equivalent circuit.

FIGURE 7–1
Schematic diagram and equivalent circuit of a transformer.

Substituting Eq. (7–8) in Eq. (7–7), we obtain

$$N_1 i_1 - N_2 i_2 = \frac{\ell}{\mu S}\Phi. \tag{7-9}$$

a) *Ideal transformer.* For an ideal transformer we assume that $\mu \to \infty$, and Eq. (7–9) becomes

$$\boxed{\frac{i_1}{i_2} = \frac{N_2}{N_1}.} \tag{7-10}$$

Equation (7–10) states that ***the ratio of the currents in the primary and secondary windings of an ideal transformer is equal to the inverse ratio of the numbers of turns.*** Faraday's law tells us that

$$v_1 = N_1 \frac{d\Phi}{dt} \tag{7-11}$$

and

$$v_2 = N_2 \frac{d\Phi}{dt}, \tag{7-12}$$

the proper signs for v_1 and v_2 having been taken care of by the designated polarities in Fig. 7–1(a). From Eqs. (7–11) and (7–12) we have

$$\boxed{\frac{v_1}{v_2} = \frac{N_1}{N_2}.} \tag{7-13}$$

Thus, ***the ratio of the voltages across the primary and secondary windings of an an ideal transformer is equal to the turns ratio.***

When the secondary winding is terminated in a load resistance R_L, as shown in Fig. 7–1(a), the effective load seen by the source connected to primary winding is

$$(R_1)_{\text{eff}} = \frac{v_1}{i_1} = \frac{(N_1/N_2)v_2}{(N_2/N_1)i_2},$$

or

$$\boxed{(R_1)_{\text{eff}} = \left(\frac{N_1}{N_2}\right)^2 R_L,} \tag{7-14a}$$

which is the load resistance multiplied by the square of the turns ratio. For a sinusoidal source $v_1(t)$ and a load impedance Z_L, it is obvious that the effective load seen by the source is $(N_1/N_2)^2 Z_L$, an impedance transformation. We have

$$\boxed{(Z_1)_{\text{eff}} = \left(\frac{N_1}{N_2}\right)^2 Z_L.} \tag{7-14b}$$

b) *Real transformer.* Referring back to Eq. (7–9), we can write the magnetic flux linkages of the primary and secondary windings as

$$\Lambda_1 = N_1\Phi = \frac{\mu S}{\ell}(N_1^2 i_1 - N_1 N_2 i_2), \tag{7–15}$$

$$\Lambda_2 = N_2\Phi = \frac{\mu S}{\ell}(N_1 N_2 i_1 - N_2^2 i_2). \tag{7–16}$$

Using Eqs. (7–15) and (7–16) in Eqs. (7–11) and (7–12), we obtain

$$v_1 = L_1 \frac{di_1}{dt} - L_{12}\frac{di_2}{dt}, \tag{7–17}$$

$$v_2 = L_{12} \frac{di_1}{dt} - L_2\frac{di_2}{dt}, \tag{7–18}$$

where

$$L_1 = \frac{\mu S}{\ell} N_1^2, \tag{7–19}$$

$$L_2 = \frac{\mu S}{\ell} N_2^2, \tag{7–20}$$

$$L_{12} = \frac{\mu S}{\ell} N_1 N_2 \tag{7–21}$$

are the self-inductance of the primary winding, the self-inductance of the secondary winding, and the mutual inductance between the primary and secondary windings, respectively. For an ideal transformer there is no leakage flux, and $L_{12} = \sqrt{L_1 L_2}$. For real transformers,

$$L_{12} = k\sqrt{L_1 L_2}, \qquad k < 1, \tag{7–22}$$

where k is called the ***coefficient of coupling***. We see that the expressions in Eqs. (7–19), (7–20), and (7–21) are consistent with the inductance per unit length formula, Eq. (6–135), for a long solenoid. In both cases we assume no leakage flux. Note that the assumption of an infinite μ for an ideal transformer also implies infinite inductances.

For real transformers we have the following real-life conditions: the existence of leakage flux ($k < 1$), noninfinite inductances, nonzero winding resistances, and the presence of hysteresis and eddy-current losses. (Eddy-current losses will be discussed presently.) The nonlinear nature of the ferromagnetic core (the dependence of permeability on magnetic field intensity) further compounds the difficulty of an exact analysis of real transformers. Figure 7–1(b) is an approximate equivalent circuit for the transformer in Fig. 7–1(a). In Fig. 7–1(b). R_1 and R_2 are winding resistances, X_1 and X_2 are leakage inductive reactances, R_c represents the power loss due to hysteresis and eddy-current effects, and X_c is a nonlinear inductive reactance representing the nonlinear magnetization behavior of the ferromagnetic core. Analytical determination of these quantities is an exceedingly

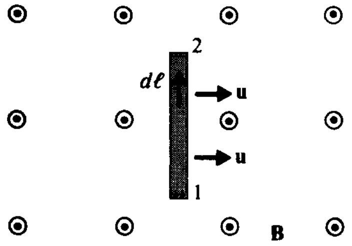


FIGURE 7–2
A conducting bar moving in a magnetic field.

difficult task. The two dots appearing on the winding terminals of the ideal transformer indicate that the potentials of these terminals rise and fall together due to electromagnetic induction. This dot convention is a simple way of showing the relative sense of the windings on the transformer core.†

When time-varying magnetic flux flows in the ferromagnetic core, an induced emf will result in accordance with Faraday's law. This induced emf will produce local currents in the conducting core normal to the magnetic flux. These currents are called ***eddy currents***. Eddy currents produce ohmic power loss and cause local heating. As a matter of fact, this is the principle of induction heating. Induction furnaces have been built to produce high enough temperatures to melt metals. In transformers this eddy-current power loss is undesirable and can be reduced by using core materials that have high permeability but low conductivity (high μ and low σ). Ferrites are such materials. For low-frequency, high-power applications an economical way for reducing eddy-current power loss is to use laminated cores; that is, to make transformer cores out of stacked ferromagnetic (iron) sheets, each electrically insulated from its neighbors by thin varnish or oxide coatings. The insulating coatings are parallel to the direction of the magnetic flux so that eddy currents normal to the flux are restricted to the laminated sheets. It can be proved that the total eddy-current power loss decreases as the number of laminations increases. (See Problem P.7–6.) The amount of power-loss reduction depends on the shape and size of the cross section as well as on the method of lamination. For instance, the circular core in Fig. 7–12(a) could also be laminated into stacked insulated sheets, instead of the filamentary parts shown in Fig. 7–12(b).

7–2.3 A MOVING CONDUCTOR IN A STATIC MAGNETIC FIELD

When a conductor moves with a velocity $\mathbf{u}$ in a static (non-time-varying) magnetic field $\mathbf{B}$ as shown in Fig. 7–2, a force $\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$ will cause the freely movable

† See, for instance, D. K. Cheng, *Analysis of Linear Systems*, Addison-Wesley, Reading, Mass. 1959, p. 50.

electrons in the conductor to drift toward one end of the conductor and leave the other end positively charged. This separation of the positive and negative charges creates a Coulombian force of attraction. The charge-separation process continues until the electric and magnetic forces balance each other and a state of equilibrium is reached. At equilibrium, which is reached very rapidly, the net force on the free charges in the moving conductor is zero.

To an observer moving with the conductor there is no apparent motion, and the magnetic force per unit charge $\mathbf{F}_m/q = \mathbf{u} \times \mathbf{B}$ can be interpreted as an induced electric field acting along the conductor and producing a voltage

$$V_{21} = \int_1^2 (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell}. \tag{7–23}$$

If the moving conductor is a part of a closed circuit C, then the emf generated around the circuit is

$$\boxed{\mathscr{V}' = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell} \qquad \text{(V).}} \tag{7–24}$$

This is referred to as a ***flux cutting emf*** or a ***motional emf***. Obviously, only the part of the circuit that moves in a direction not parallel to (and hence, figuratively, "cutting") the magnetic flux will contribute to $\mathscr{V}'$ in Eq. (7–24).

EXAMPLE 7–2 A metal bar slides over a pair of conducting rails in a uniform magnetic field $\mathbf{B} = \mathbf{a}_z B_0$ with a constant velocity $\mathbf{u}$, as shown in Fig. 7–3.

a) Determine the open-circuit voltage V_0 that appears across terminals 1 and 2.

b) Assuming that a resistance R is connected between the terminals, find the electric power dissipated in R.

c) Show that this electric power is equal to the mechanical power required to move the sliding bar with a velocity $\mathbf{u}$. Neglect the electric resistance of the metal bar and of the conducting rails. Neglect also the mechanical friction at the contact points.

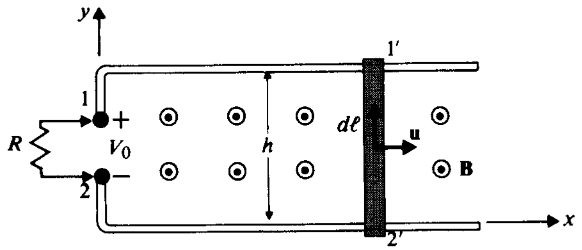


FIGURE 7–3
A metal bar sliding over conducting rails (Example 7–2).

Solution

a) The moving bar generates a flux-cutting emf. We use Eq. (7–24) to find the open-circuit voltage V_0:

$$\begin{aligned} V_0 = V_1 - V_2 &= \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell} \\ &= \int_{2'}^{1'} (\mathbf{a}_x u \times \mathbf{a}_z B_0) \cdot (\mathbf{a}_y\, d\ell) \\ &= -uB_0 h \qquad \text{(V)}. \end{aligned} \tag{7-25}$$

b) When a resistance R is connected between terminals 1 and 2, a current $I = uB_0h/R$ will flow from terminal 2 to terminal 1, so the electric power, P_e, dissipated in R is

$$P_e = I^2R = \frac{(uB_0h)^2}{R} \qquad \text{(W)}. \tag{7-26}$$

c) The mechanical power, P_m, required to move the sliding bar is

$$P_m = \mathbf{F} \cdot \mathbf{u} \qquad \text{(W)}, \tag{7-27}$$

where $\mathbf{F}$ is the mechanical force required to counteract the magnetic force, $\mathbf{F}_m$, which the magnetic field exerts on the current-carrying metal bar. From Eq. (6–184) we have

$$\mathbf{F}_m = I \int_{2'}^{1'} d\boldsymbol{\ell} \times \mathbf{B} = -\mathbf{a}_x IB_0h \qquad \text{(N)}. \tag{7-28}$$

The negative sign in Eq. (7–28) arises because current I flows in a direction opposite to that of $d\boldsymbol{\ell}$. Hence,

$$\mathbf{F} = -\mathbf{F}_m = \mathbf{a}_x IB_0h = \mathbf{a}_x uB_0^2h^2/R \qquad \text{(N)}. \tag{7-29}$$

Substitution of Eq. (7–29) in Eq. (7–27) proves that $P_m = P_e$, which upholds the principle of conservation of energy. ■

EXAMPLE 7–3 The ***Faraday disk generator*** consists of a circular metal disk rotating with a constant angular velocity ω in a uniform and constant magnetic field of

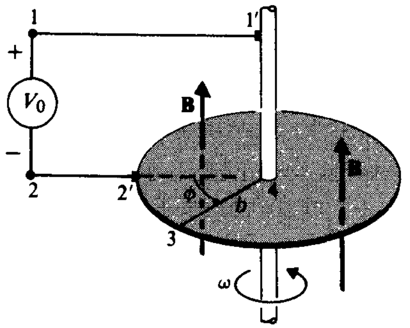


FIGURE 7–4
Faraday disk generator (Example 7–3).

flux density $\mathbf{B} = \mathbf{a}_z B_0$ that is parallel to the axis of rotation. Brush contacts are provided at the axis and on the rim of the disk, as depicted in Fig. 7–4. Determine the open-circuit voltage of the generator if the radius of the disk is b.

Solution Let us consider the circuit 122′341′1. Of the part 2′34 that moves with the disk, only the straight portion 34 "cuts" the magnetic flux. We have, from Eq. (7–24),

$$\begin{aligned} V_0 &= \oint (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell} \\ &= \int_3^4 [(\mathbf{a}_\phi r\omega) \times \mathbf{a}_z B_0] \cdot (\mathbf{a}_r\, dr) \qquad (7\text{–}30) \\ &= \omega B_0 \int_b^0 r\, dr = -\frac{\omega B_0 b^2}{2} \quad \text{(V)}, \end{aligned}$$

which is the emf of the Faraday disk generator. To measure V_0, we must use a voltmeter of a very high resistance so that no appreciable current flows in the circuit to modify the externally applied magnetic field. ■

7–2.4 A MOVING CIRCUIT IN A TIME-VARYING MAGNETIC FIELD

When a charge q moves with a velocity $\mathbf{u}$ in a region where both an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ exist, the electromagnetic force $\mathbf{F}$ on q, as measured by a laboratory observer, is given by Lorentz's force equation, Eq. (6–5), which is repeated below:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}). \qquad (7\text{–}31)$$

To an observer moving with q, there is no apparent motion, and the force on q can be interpreted as caused by an electric field $\mathbf{E}'$, where

$$\mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B} \qquad (7\text{–}32)$$

or

$$\mathbf{E} = \mathbf{E}' - \mathbf{u} \times \mathbf{B}. \qquad (7\text{–}33)$$

Hence, when a conducting circuit with contour C and surface S moves with a velocity $\mathbf{u}$ in a field $(\mathbf{E}, \mathbf{B})$, we use Eq. (7–33) in Eq. (7–2) to obtain

$$\boxed{\oint_C \mathbf{E}' \cdot d\boldsymbol{\ell} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} + \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell} \quad \text{(V).}} \qquad (7\text{–}34)$$

Equation (7–34) is the general form of ***Faraday's law*** for a moving circuit in a time-varying magnetic field. The line integral on the left side is the emf induced in the moving frame of reference. The first term on the right side represents the transformer emf due to the time variation of $\mathbf{B}$; and the second term represents the motional emf due to the motion of the circuit in $\mathbf{B}$. The division of the induced emf between the transformer and the motional parts depends on the chosen frame of reference.

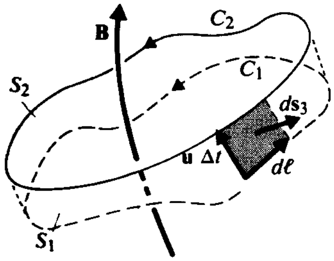


FIGURE 7–5
A moving circuit in a time-varying magnetic field.

Let us consider a circuit with contour C that moves from C_1 at time t to C_2 at time $t + \Delta t$ in a changing magnetic field **B**. The motion may include translation, rotation, and distortion in an arbitrary manner. Figure 7–5 illustrates the situation. The time-rate of change of magnetic flux through the contour is

$$\begin{aligned}\frac{d\Phi}{dt} &= \frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} \\ &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{S_2}\mathbf{B}(t+\Delta t)\cdot d\mathbf{s}_2 - \int_{S_1}\mathbf{B}(t)\cdot d\mathbf{s}_1\right].\end{aligned} \tag{7–35}$$

$\mathbf{B}(t + \Delta t)$ in Eq. (7–35) can be expanded as a Taylor's series:

$$\mathbf{B}(t+\Delta t) = \mathbf{B}(t) + \frac{\partial \mathbf{B}(t)}{\partial t}\Delta t + \text{H.O.T.}, \tag{7–36}$$

where the higher-order terms (H.O.T.) contain the second and higher powers of (Δt). Substitution of Eq. (7–36) in Eq. (7–35) yields

$$\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s} + \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\int_{S_2}\mathbf{B}\cdot d\mathbf{s}_2 - \int_{S_1}\mathbf{B}\cdot d\mathbf{s}_1 + \text{H.O.T.}\right], \tag{7–37}$$

where **B** has been written for $\mathbf{B}(t)$ for simplicity. In going from C_1 to C_2 the circuit covers a region that is bounded by S_1, S_2, and S_3. Side surface S_3 is the area swept out by the contour in time Δt. An element of the side surface is

$$d\mathbf{s}_3 = d\boldsymbol{\ell} \times \mathbf{u}\,\Delta t. \tag{7–38}$$

We now apply the divergence theorem for **B** at time t to the region sketched in Fig. 7–5:

$$\int_V \nabla\cdot\mathbf{B}\,dv = \int_{S_2}\mathbf{B}\cdot d\mathbf{s}_2 - \int_{S_1}\mathbf{B}\cdot d\mathbf{s}_1 + \int_{S_3}\mathbf{B}\cdot d\mathbf{s}_3, \tag{7–39}$$

where a negative sign is included in the term involving $d\mathbf{s}_1$ because *outward* normals must be used in the divergence theorem. Using Eq. (7–38) in Eq. (7–39) and noting that $\nabla\cdot\mathbf{B} = 0$, we have

$$\int_{S_2}\mathbf{B}\cdot d\mathbf{s}_2 - \int_{S_1}\mathbf{B}\cdot d\mathbf{s}_1 = -\Delta t\oint_C (\mathbf{u}\times\mathbf{B})\cdot d\boldsymbol{\ell}. \tag{7–40}$$

Combining Eqs. (7–37) and (7–40), we obtain

$$\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s} - \oint_C (\mathbf{u}\times\mathbf{B})\cdot d\boldsymbol{\ell}, \tag{7–41}$$

which can be identified as the negative of the right side of Eq. (7–34).

If we designate

$$\begin{aligned}\mathscr{V}' &= \oint_C \mathbf{E}'\cdot d\boldsymbol{\ell}\\ &= \text{emf induced in circuit } C \text{ measured in the moving frame,}\end{aligned} \tag{7–42}$$

Eq. (7–34) can be written simply as

$$\boxed{\begin{aligned}\mathscr{V}' &= -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s}\\ &= -\frac{d\Phi}{dt} \qquad \text{(V)},\end{aligned}} \tag{7–43}$$

which is of the same form as Eq. (7–6). Of course, if a circuit is not in motion, $\mathscr{V}'$ reduces to $\mathscr{V}$, and Eqs. (7–43) and (7–6) are exactly the same. Hence, Faraday's law that the emf induced in a closed circuit equals the negative time-rate of increase of the magnetic flux linking a circuit applies to a stationary circuit as well as a moving one. Either Eq. (7–34) or Eq. (7–43) can be used to evaluate the induced emf in the general case. If a high-impedance voltmeter is inserted in a conducting circuit, it will read the open-circuit voltage due to electromagnetic induction whether the circuit is stationary or moving. We have mentioned that the division of the induced emf in Eq. (7–34) into transformer and motional emf's is not unique, but their sum is always equal to that computed by using Eq. (7–43).

In Example 7–2 (Fig. 7–3) we determined the open-circuit voltage V_0 by using Eq. (7–24). If we use Eq. (7–43), we have

$$\Phi = \int_S \mathbf{B}\cdot d\mathbf{s} = B_0(hut)$$

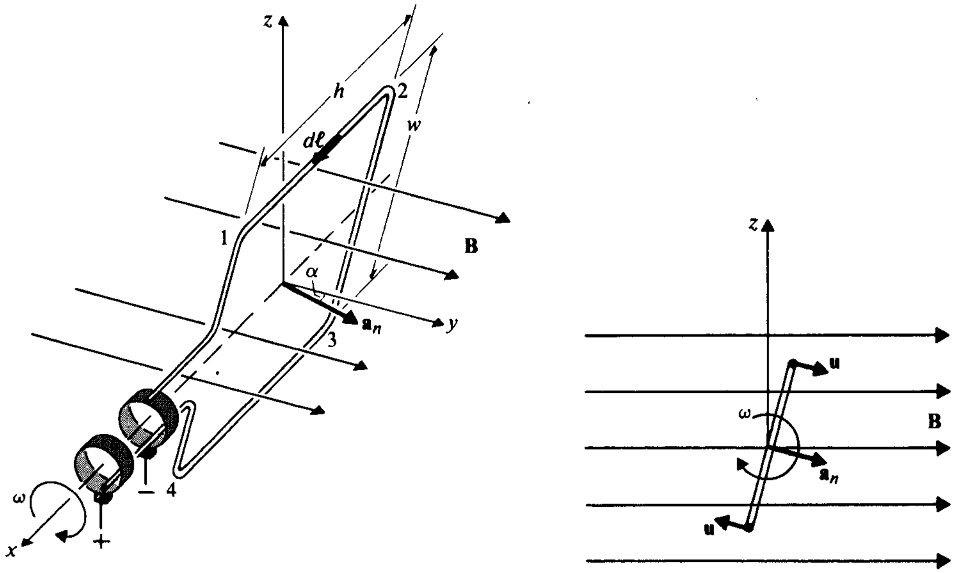
and

$$V_0 = -\frac{d\Phi}{dt} = -uB_0h \qquad \text{(V)},$$

which is the same as Eq. (7–25).

Similarly, for the Faraday disk generator in Example 7–3 the magnetic flux linking the circuit 122′341′1 is that which passes through the wedge-shaped area 2′342′:

$$\begin{aligned}\Phi &= \int_S \mathbf{B}\cdot d\mathbf{s} = B_0\int_0^b\int_0^{\omega t} r\,d\phi\,dr\\ &= B_0(\omega t)\frac{b^2}{2}\end{aligned}$$



(a) Perspective view. (b) View from $+x$ direction.

FIGURE 7–6
A rectangular conducting loop rotating in a changing magnetic field (Example 7–4).

and

$$V_0 = -\frac{d\Phi}{dt} = -\frac{\omega B_0 b^2}{2},$$

which is the same as Eq. (7–30).

EXAMPLE 7–4 An h by w rectangular conducting loop is situated in a changing magnetic field $\mathbf{B} = \mathbf{a}_y B_0 \sin \omega t$. The normal of the loop initially makes an angle α with $\mathbf{a}_y$, as shown in Fig. 7–6. Find the induced emf in the loop: (a) when the loop is at rest, and (b) when the loop rotates with an angular velocity ω about the x-axis.

Solution

a) When the loop is at rest, we use Eq. (7–6):

$$\begin{aligned}\Phi &= \int \mathbf{B} \cdot d\mathbf{s} \\ &= (\mathbf{a}_y B_0 \sin \omega t) \cdot (\mathbf{a}_n hw) \\ &= B_0 hw \sin \omega t \cos \alpha.\end{aligned}$$

Therefore,

$$\mathscr{V}_a = -\frac{d\Phi}{dt} = -B_0 S\omega \cos \omega t \cos \alpha, \tag{7–44}$$

where $S = hw$ is the area of the loop. The relative polarities of the terminals are as indicated. If the circuit is completed through an external load, $\mathscr{V}_a$ will produce a current that will oppose the change in Φ.

b) When the loop rotates about the x-axis, both terms in Eq. (7–34) contribute: the first term contributes the transformer emf $\mathscr{V}_a$ in Eq. (7–44), and the second term contributes a motional emf $\mathscr{V}'_a$ where

$$\begin{aligned}\mathscr{V}'_a &= \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\boldsymbol{\ell} \\ &= \int_2^1 \left[\left(\mathbf{a}_n \frac{w}{2}\omega\right) \times (\mathbf{a}_y B_0 \sin \omega t)\right] \cdot (\mathbf{a}_x\, dx) \\ &\quad + \int_4^3 \left[\left(-\mathbf{a}_n \frac{w}{2}\omega\right) \times (\mathbf{a}_y B_0 \sin \omega t)\right] \cdot (\mathbf{a}_x\, dx) \\ &= 2\left(\frac{w}{2}\omega B_0 \sin \omega t \sin \alpha\right) h.\end{aligned}$$

Note that the sides 23 and 41 do not contribute to $\mathscr{V}'_a$ and that the contributions of sides 12 and 34 are of equal magnitude and in the same direction. If $\alpha = 0$ at $t = 0$, then $\alpha = \omega t$, and we can write

$$\mathscr{V}'_a = B_0 S\omega \sin \omega t \sin \omega t. \tag{7–45}$$

The total emf induced or generated in the rotating loop is the sum of $\mathscr{V}_a$ in Eq. (7–44) and $\mathscr{V}'_a$ in Eq. (7–45):

$$\mathscr{V}'_t = -B_0 S\omega(\cos^2 \omega t - \sin^2 \omega t) = -B_0 S\omega \cos 2\omega t, \tag{7–46}$$

which has an angular frequency 2ω.

We can determine the total induced emf $\mathscr{V}'_t$ by applying Eq. (7–43) directly. At any time t, the magnetic flux linking the loop is

$$\begin{aligned}\Phi(t) &= \mathbf{B}(t) \cdot [\mathbf{a}_n(t)S] = B_0 S \sin \omega t \cos \alpha \\ &= B_0 S \sin \omega t \cos \omega t = \tfrac{1}{2} B_0 S \sin 2\omega t.\end{aligned}$$

Hence,

$$\begin{aligned}\mathscr{V}'_t &= -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(\frac{1}{2} B_0 S \sin 2\omega t\right) \\ &= -B_0 S\omega \cos 2\omega t\end{aligned}$$

as before. ■

7–3 Maxwell's Equations

The fundamental postulate for electromagnetic induction assures us that a time-varying magnetic field gives rise to an electric field. This assurance has been amply verified by numerous experiments. The $\nabla \times \mathbf{E} = 0$ equation in Table 7–1 must therefore be replaced by Eq. (7–1) in the time-varying case. Following are the revised set of two curl and two divergence equations from Table 7–1:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{7-47a}$$

$$\nabla \times \mathbf{H} = \mathbf{J}, \tag{7-47b}$$

$$\nabla \cdot \mathbf{D} = \rho, \tag{7-47c}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{7-47d}$$

In addition, we know that the principle of conservation of charge must be satisfied at all times. The mathematical expression of charge conservation is the equation of continuity, Eq. (5–44), which is repeated below:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}. \tag{7-48}$$

The crucial question here is whether the set of four equations in (7–47a, b, c, and d) are now consistent with the requirement specified by Eq. (7–48) in a time-varying situation. That the answer is in the negative is immediately obvious by simply taking the divergence of Eq. (7–47b),

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J}, \tag{7-49}$$

which follows from the null identity, Eq. (2–149). We are reminded that the divergence of the curl of any well-behaved vector field is zero. Since Eq. (7–48) asserts that $\nabla \cdot \mathbf{J}$ does not vanish in a time-varying situation, Eq. (7–49) is, in general, not true.

How should Eqs. (7–47a, b, c, and d) be modified so that they are consistent with Eq. (7–48)? First of all, a term $\partial\rho/\partial t$ must be added to the right side of Eq. (7–49):

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}. \tag{7-50}$$

Using Eq. (7–47c) in Eq. (7–50), we have

$$\nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right), \tag{7-51}$$

which implies that

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.} \qquad (7\text{-}52)^\dagger$$

Equation (7–52) indicates that a time-varying electric field will give rise to a magnetic field, even in the absence of a current flow. The additional term $\partial\mathbf{D}/\partial t$ is necessary to make Eq. (7–52) consistent with the principle of conservation of charge.

† An integration constant could be added to Eq. (7–52) without violating Eq. (7–51), but this constant must be zero in order that Eq. (7–52) reduces to Eq. (7–47b) in the static case.

It is easy to verify that $\partial \mathbf{D}/\partial t$ has the dimension of a current density (SI unit: A/m^2). The term $\partial \mathbf{D}/\partial t$ is called ***displacement current density***, and its introduction in the $\nabla \times \mathbf{H}$ equation was one of the major contributions of James Clerk Maxwell (1831–1879). In order to be consistent with the equation of continuity in a time-varying situation, both of the curl equations in Table 7–1 must be generalized. The set of four consistent equations to replace the inconsistent equations, Eqs. (7–47a, b, c, and d), are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad (7\text{–}53a)$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \qquad (7\text{–}53b)$$

$$\nabla \cdot \mathbf{D} = \rho, \qquad (7\text{–}53c)$$

$$\nabla \cdot \mathbf{B} = 0. \qquad (7\text{–}53d)$$

They are known as ***Maxwell's equations***. Note that ρ in Eq. (7–53c) is the volume density of *free charges*, and $\mathbf{J}$ in Eq. (7–53b) is the density of *free currents*, which may comprise both convection current ($\rho \mathbf{u}$) and conduction current ($\sigma \mathbf{E}$). These four equations, together with the equation of continuity in Eq. (7–48) and Lorentz's force equation in Eq. (6–5), form the foundation of electromagnetic theory. These equations can be used to explain and predict *all* macroscopic electromagnetic phenomena.

Although the four Maxwell's equations in Eqs. (7–53a, b, c, and d) are consistent, they are not all independent. As a matter of fact, the two divergence equations, Eqs. (5–53c and d), can be derived from the two curl equations, Eqs. (7–53a and b), by making use of the equation of continuity, Eq. (7–48) (see Problem P.7–11). The four fundamental field vectors **E, D, B, H** (each having three components) represent twelve unknowns. Twelve scalar equations are required for the determination of these twelve unknowns. The required equations are supplied by the two vector curl equations and the two vector constitutive relations $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{H} = \mathbf{B}/\mu$, each vector equation being equivalent to three scalar equations.

7–3.1 INTEGRAL FORM OF MAXWELL'S EQUATIONS

The four Maxwell's equations in (7–53a, b, c, and d) are differential equations that are valid at every point in space. In explaining electromagnetic phenomena in a physical environment we must deal with finite objects of specified shapes and boundaries. It is convenient to convert the differential forms into their integral-form equivalents. We take the surface integral of both sides of the curl equations in Eqs. (7–53a) and (7–53b) over an open surface S with a contour C and apply Stokes's theorem to obtain

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} \qquad (7\text{–}54a)$$

and

$$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s}. \tag{7-54b}$$

Taking the volume integral of both sides of the divergence equations in Eqs. (7–53c) and (7–53d) over a volume V with a *closed* surface S and using divergence theorem, we have

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_V \rho \, dv \tag{7-54c}$$

and

$$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0. \tag{7-54d}$$

The set of four equations in (7–54a, b, c, and d) are the integral form of Maxwell's equations. We see that Eq. (7–54a) is the same as Eq. (7–2), which is an expression of Faraday's law of electromagnetic induction. Equation (7–54b) is a generalization of Ampère's circuital law given in Eq. (6–78), the latter applying only to static magnetic fields. Note that the current density $\mathbf{J}$ may consist of a convection current density $\rho\mathbf{u}$ due to the motion of a free-charge distribution, as well as a conduction current density $\sigma\mathbf{E}$ caused by the presence of an electric field in a conducting medium. The surface integral of $\mathbf{J}$ is the current I flowing through the open surface S.

Equation (7–54c) can be recognized as Gauss's law, which we used extensively in electrostatics and which remains the same in the time-varying case. The volume integral of ρ equals the total charge Q that is enclosed in surface S. No particular law is associated with Eq. (7–54d); but, in comparing it with Eq. (7–54c) we conclude that there are no isolated magnetic charges and that the total outward magnetic flux through any closed surface is zero. Both the differential and the integral forms

TABLE 7–2
Maxwell's Equations

Differential Form	Integral Form	Significance
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\frac{d\Phi}{dt}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}$	Ampère's circuital law
$\nabla \cdot \mathbf{D} = \rho$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$	No isolated magnetic charge

of Maxwell's equations are collected in Table 7–2 for easy reference. It is obvious that in non-time-varying cases these equations simplify to the fundamental relations in Table 7–1 for electrostatic and magnetostatic models.

EXAMPLE 7–5 An a-c voltage source of amplitude V_0 and angular frequency ω, $v_c = V_0 \sin \omega t$, is connected across a parallel-plate capacitor C_1, as shown in Fig. 7–7. (a) Verify that the displacement current in the capacitor is the same as the conduction current in the wires. (b) Determine the magnetic field intensity at a distance r from the wire.

Solution

a) The conduction current in the connecting wire is

$$i_C = C_1 \frac{dv_C}{dt} = C_1 V_0 \omega \cos \omega t \qquad \text{(A)}.$$

For a parallel-plate capacitor with an area A, plate separation d, and a dielectric medium of permittivity ϵ the capacitance is

$$C_1 = \epsilon \frac{A}{d}.$$

With a voltage v_C appearing between the plates, the uniform electric field intensity E in the dielectric is equal to (neglecting fringing effects) $E = v_C/d$, whence

$$D = \epsilon E = \epsilon \frac{V_0}{d} \sin \omega t.$$

The displacement current is then

$$i_D = \int_A \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} = \left(\epsilon \frac{A}{d}\right) V_0 \omega \cos \omega t$$
$$= C_1 V_0 \omega \cos \omega t = i_C. \qquad \text{Q.E.D.}$$

b) The magnetic field intensity at a distance r from the conducting wire can be found by applying the generalized Ampère's circuital law, Eq. (7–54b), to contour

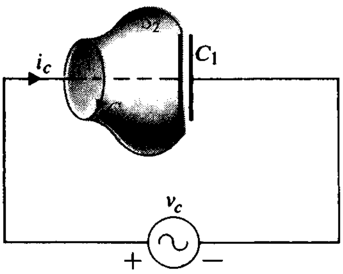


FIGURE 7–7
A parallel-plate capacitor connected to an a-c voltage source (Example 7–5).

C in Fig. 7–7. Two typical open surfaces with rim C may be chosen: (1) a planar disk surface S_1, or (2) a curved surface S_2 passing through the dielectric medium. Symmetry around the wire ensures a constant H_ϕ along the contour C. The line integral on the left side of Eq. (7–54b) is

$$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = 2\pi r H_\phi.$$

For the surface S_1, only the first term on the right side of Eq. (7–54b) is nonzero because no charges are deposited along the wire and, consequently, $\mathbf{D} = 0$.

$$\int_{S_1} \mathbf{J} \cdot d\mathbf{s} = i_C = C_1 V_0 \omega \cos \omega t.$$

Since the surface S_2 passes through the dielectric medium, no conduction current flows through S_2. If the second surface integral were not there, the right side of Eq. (7–54b) would be zero. This would result in a contradiction. The inclusion of the displacement-current term by Maxwell eliminates this contradiction. As we have shown in part (a), $i_D = i_C$. Hence we obtain the same result whether surface S_1 or surface S_2 is chosen. Equating the two previous integrals, we find that

$$H_\phi = \frac{C_1 V_0}{2\pi r} \omega \cos \omega t \qquad \text{(A/m)}.$$

7–4 Potential Functions

In Section 6–3 the concept of the vector magnetic potential $\mathbf{A}$ was introduced because of the solenoidal nature of $\mathbf{B}$ ($\nabla \cdot \mathbf{B} = 0$):

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A} \qquad \text{(T)}.} \tag{7–55}$$

If Eq. (7–55) is substituted in the differential form of Faraday's law, Eq. (7–1), we get

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A})$$

or

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0. \tag{7–56}$$

Since the sum of the two vector quantities in the parentheses of Eq. (7–56) is curl-free, it can be expressed as the gradient of a scalar. To be consistent with the definition of the scalar electric potential V in Eq. (3–43) for electrostatics, we write

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V,$$

from which we obtain

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \qquad (\text{V/m}).} \tag{7–57}$$

In the static case, $\partial\mathbf{A}/\partial t = 0$, and Eq. (7–57) reduces to $\mathbf{E} = -\nabla V$. Hence $\mathbf{E}$ can be determined from V alone, and $\mathbf{B}$ from $\mathbf{A}$ by Eq. (7–55). For time-varying fields, $\mathbf{E}$ depends on both V and $\mathbf{A}$; that is, an electric field intensity can result both from accumulations of charge through the $-\nabla V$ term and from time-varying magnetic fields through the $-\partial\mathbf{A}/\partial t$ term. Inasmuch as $\mathbf{B}$ also depends on $\mathbf{A}$, $\mathbf{E}$ and $\mathbf{B}$ are coupled.

The electric field in Eq. (7–57) can be viewed as composed of two parts: the first part, $-\nabla V$, is due to charge distribution ρ; and the second part, $-\partial\mathbf{A}/\partial t$, is due to time-varying current $\mathbf{J}$. We are tempted to find V from ρ by Eq. (3–61):

$$V = \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{\rho}{R}\, dv', \tag{7–58}$$

and to find $\mathbf{A}$ by Eq. (6–23):

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\mathbf{J}}{R}\, dv'. \tag{7–59}$$

However, the preceding two equations were obtained under static conditions, and V and $\mathbf{A}$ as given were, in fact, solutions of Poisson's equations, Eqs. (4–6) and (6–21), respectively. These solutions may themselves be time-dependent because ρ and $\mathbf{J}$ may be functions of time, but they neglect the time-retardation effects associated with the finite velocity of propagation of time-varying electromagnetic fields. When ρ and $\mathbf{J}$ vary slowly with time (at a very low frequency) and the range of interest R is small in comparison with the wavelength, it is allowable to use Eqs. (7–58) and (7–59) in Eqs. (7–55) and (7–57) to find ***quasi-static fields***. We will discuss this again in Subsection 7–7.2.

Quasi-static fields are approximations. Their consideration leads from field theory to circuit theory. However, when the source frequency is high and the range of interest is no longer small in comparison to the wavelength, quasi-static solutions will not suffice. Time-retardation effects must then be included, as in the case of electromagnetic radiation from antennas. These points will be discussed more fully when we study solutions to wave equations.

Let us substitute Eqs. (7–55) and (7–57) into Eq. (7–53b) and make use of the constitutive relations $\mathbf{H} = \mathbf{B}/\mu$ and $\mathbf{D} = \epsilon\mathbf{E}$. We have

$$\nabla \times \nabla \times \mathbf{A} = \mu\mathbf{J} + \mu\epsilon\frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t}\right), \tag{7–60}$$

where a homogeneous medium has been assumed. Recalling the vector identity for $\nabla \times \nabla \times \mathbf{A}$ in Eq. (6–17a), we can write Eq. (7–60) as

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} - \nabla\left(\mu\epsilon \frac{\partial V}{\partial t}\right) - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

or

$$\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} + \nabla\left(\nabla \cdot \mathbf{A} + \mu\epsilon \frac{\partial V}{\partial t}\right). \tag{7–61}$$

Now, the definition of a vector requires the specification of both its curl and its divergence. Although the curl of $\mathbf{A}$ is designated $\mathbf{B}$ in Eq. (7–55), we are still at liberty to choose the divergence of $\mathbf{A}$. We let

$$\boxed{\nabla \cdot \mathbf{A} + \mu\epsilon \frac{\partial V}{\partial t} = 0,} \tag{7–62}$$

which makes the second term on the right side of Eq. (7–61) vanish, so we obtain

$$\boxed{\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}.} \tag{7–63}$$

Equation (7–63) is the ***nonhomogeneous wave equation for vector potential*** $\mathbf{A}$. It is called a wave equation because its solutions represent waves traveling with a velocity equal to $1/\sqrt{\mu\epsilon}$. This will become clear in Section 7–6 when the solution of wave equations is discussed. The relation between $\mathbf{A}$ and V in Eq. (7–62) is called the ***Lorentz condition (or Lorentz gauge) for potentials.*** It reduces to the condition $\nabla \cdot \mathbf{A} = 0$ in Eq. (6–20) for static fields. The Lorentz condition can be shown to be consistent with the equation of continuity (Problem P.7–12).

A corresponding wave equation for the scalar potential V can be obtained by substituting Eq. (7–57) in Eq. (7–53c). We have

$$-\nabla \cdot \epsilon\left(\nabla V + \frac{\partial \mathbf{A}}{\partial t}\right) = \rho,$$

which, for a constant ϵ, leads to

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon}. \tag{7–64}$$

Using Eq. (7–62), we get

$$\boxed{\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon},} \tag{7–65}$$

which is the ***nonhomogeneous wave equation for scalar potential*** V. Hence the Lorentz condition in Eq. (7–62) uncouples the wave equations for $\mathbf{A}$ and for V. The nonhomogeneous wave equations in (7–63) and (7–65) reduce to Poisson's equations in static cases. Since the potential functions given in Eqs. (7–58) and (7–59) are solutions of Poisson's equations, they cannot be expected to be the solutions of nonhomogeneous wave equations in time-varying situations without modification.

7–5 Electromagnetic Boundary Conditions

In order to solve electromagnetic problems involving contiguous regions of different constitutive parameters, it is necessary to know the boundary conditions that the field vectors **E**, **D**, **B**, and **H** must satisfy at the interfaces. Boundary conditions are derived by applying the integral form of Maxwell's equations (7–54a, b, c, and d) to a small region at an interface of two media in manners similar to those used in obtaining the boundary conditions for static electric and magnetic fields. The integral equations are assumed to hold for regions containing discontinuous media. The reader should review the procedures followed in Sections 3–9 and 6–10. In general, the application of the integral form of a curl equation to a flat closed path at a boundary with top and bottom sides in the two touching media yields the boundary condition for the tangential components; and the application of the integral form of a divergence equation to a shallow pillbox at an interface with top and bottom faces in the two contiguous media gives the boundary condition for the normal components.

The boundary conditions for the tangential components of **E** and **H** are obtained from Eqs. (7–54a) and (7–54b), respectively:

$$\boxed{E_{1t} = E_{2t} \qquad \text{(V/m)};} \tag{7–66a}$$

$$\boxed{\mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \qquad \text{(A/m)}.} \tag{7–66b}$$

We note that Eqs. (7–66a) and (7–66b) for the time-varying case are exactly the same as Eq. (3–118) for static electric fields and Eq. (6–111) for static magnetic fields, respectively, in spite of the existence of the time-varying terms in Eqs. (7–54a) and (7–54b). The reason is that, in letting the height of the flat closed path (*abcda* in Figs. 3–23 and 6–19) approach zero, the area bounded by the path approaches zero, causing the surface integrals of $\partial\mathbf{B}/\partial t$ and $\partial\mathbf{D}/\partial t$ to vanish.

Similarly, the boundary conditions for the normal components of **D** and **B** are obtained from Eqs. (7–54c) and (7–54d):

$$\boxed{\mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \qquad \text{(C/m}^2\text{)};} \tag{7–66c}$$

$$\boxed{B_{1n} = B_{2n} \qquad \text{(T)}.} \tag{7–66d}$$

These are the same as, respectively, Eq. (3–121a) for static electric fields and Eq. (6–107) for static magnetic fields because we start from the same divergence equations.

We can make the following general statements about electromagnetic boundary conditions:

1. ***The tangential component of an* E *field is continuous across an interface.***
2. ***The tangential component of an* H *field is discontinuous across an interface where a surface current exists, the amount of discontinuity being determined by Eq. (7–66b).***
3. ***The normal component of a* D *field is discontinuous across an interface where a surface charge exists, the amount of discontinuity being determined by Eq. (7–66c).***
4. ***The normal component of a* B *field is continuous across an interface.***

As we have noted previously, the two divergence equations can be derived from the two curl equations and the equation of continuity; hence, the boundary conditions in Eqs. (7–66c) and (7–66d), which are obtained from the divergence equations, cannot be independent from those in Eqs. (7–66a) and (7–66b), which are obtained from the curl equations. As a matter of fact, in the time-varying case the boundary condition for the tangential component of **E** in Eq. (7–66a) is equivalent to that for the normal component of **B** in Eq. (7–66d), and the boundary condition for the tangential component of **H** in Eq. (7–66b) is equivalent to that for the normal component of **D** in Eq. (7–66c). The simultaneous specification of the tangential component of **E** and the normal component of **B** at a boundary surface in a time-varying situation, for example, would be redundant and, if we are not careful, could result in contradictions.

We now examine the important special cases of (1) a boundary between two lossless linear media, and (2) a boundary between a good dielectric and a good conductor.

7–5.1 INTERFACE BETWEEN TWO LOSSLESS LINEAR MEDIA

A lossless linear medium can be specified by a permittivity ϵ and a permeability μ, with $\sigma = 0$. There are usually no free charges and no surface currents at the interface between two lossless media. We set $\rho_s = 0$ and $\mathbf{J}_s = 0$ in Eqs. (7–66a, b, c, and d) and obtain the boundary conditions listed in Table 7–3.

TABLE 7–3
Boundary Conditions between Two Lossless Media

$$E_{1t} = E_{2t} \rightarrow \frac{D_{1t}}{D_{2t}} = \frac{\epsilon_1}{\epsilon_2} \tag{7–67a}$$

$$H_{1t} = H_{2t} \rightarrow \frac{B_{1t}}{B_{2t}} = \frac{\mu_1}{\mu_2} \tag{7–67b}$$

$$D_{1n} = D_{2n} \rightarrow \epsilon_1 E_{1n} = \epsilon_2 E_{2n} \tag{7–67c}$$

$$B_{1n} = B_{2n} \rightarrow \mu_1 H_{1n} = \mu_2 H_{2n} \tag{7–67d}$$

TABLE 7–4
Boundary Conditions between a Dielectric (Medium 1) and a Perfect Conductor (Medium 2) (Time-Varying Case)

On the Side of Medium 1	On the Side of Medium 2	
$E_{1t} = 0$	$E_{2t} = 0$	(7–68a)
$\mathbf{a}_{n2} \times \mathbf{H}_1 = \mathbf{J}_s$	$H_{2t} = 0$	(7–68b)
$\mathbf{a}_{n2} \cdot \mathbf{D}_1 = \rho_s$	$D_{2n} = 0$	(7–68c)
$B_{1n} = 0$	$B_{2n} = 0$	(7–68d)

7–5.2 INTERFACE BETWEEN A DIELECTRIC AND A PERFECT CONDUCTOR

A perfect conductor is one with an infinite conductivity. In the physical world we have an abundance of "good conductors" such as silver, copper, gold, and aluminum, whose conductivities are of the order of 10^7 (S/m). (See the table in Appendix B–4). There are superconducting materials whose conductivities are essentially infinite (in excess of 10^{20} S/m) at cryogenic temperatures. They are called ***superconductors***. Because of the requirement of extremely low temperatures, they have not found much practical use. (The apparent upper limit for transition temperature in 1973 was 23 K. Cooling by expensive liquid helium was required.) However, this situation is expected to change in the near future, since scientists have recently found ceramic materials that show superconducting properties at much higher transition temperatures (20–30 degrees above the 77 K boiling point of nitrogen, raising the possibility of using inexpensive liquid nitrogen as a coolant). At the present time the brittleness of the ceramic materials and limitations on usable current density and magnetic field intensity remain obstacles to industrial applications.† Room-temperature superconductivity is still a dream.

In order to simplify the analytical solution of field problems, good conductors are often considered perfect conductors in regard to boundary conditions. In the *interior* of a perfect conductor the electric field is zero (otherwise, it would produce an infinite current density), and any charges the conductor will have will reside on the surface only. The interrelationship between (**E**, **D**) and (**B**, **H**) through Maxwell's equations ensures that **B** and **H** are also zero in the *interior* of a conductor in a *time-varying situation.*‡ Consider an interface between a lossless dielectric (medium 1) and a perfect conductor (medium 2). In medium 2, $\mathbf{E}_2 = 0$, $\mathbf{H}_2 = 0$, $\mathbf{D}_2 = 0$, and $\mathbf{B}_2 = 0$. The general boundary conditions in Eqs. (7–66a, b, c, and d) reduce to those listed in Table 7–4. When we apply Eqs. (7–68b) and (7–68c), it is important to note that the reference unit normal is an *outward normal from medium 2* in order to avoid an error in sign. As mentioned in Section 6–10, currents in media with finite conductivities are expressed in terms of volume current densities, and surface current densities defined for currents flowing through an infinitesimal thickness is zero. In this case, Eq.

† R. K. Jurgen, "Technology '88—The main event," *IEEE Spectrum*, vol. 25, pp. 27–28, January 1988.

‡ In the *static* case a steady current in a conductor produces a static magnetic field that does not affect the electric field. Hence, **E** and **D** within a good conductor may be zero, but **B** and **H** may not be zero.

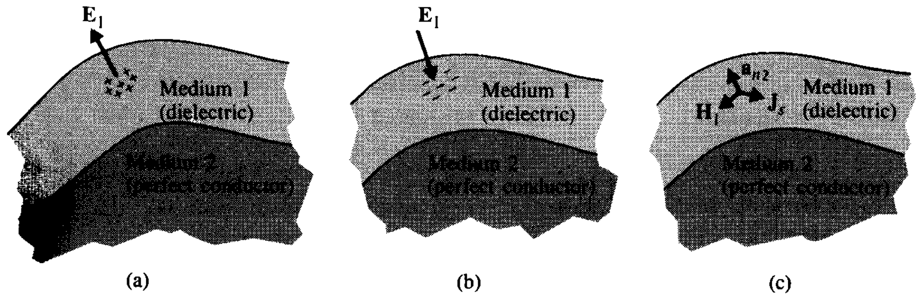


FIGURE 7–8
Boundary conditions at an interface between a dielectric (medium 1) and a perfect conductor (medium 2).

(7–68b) leads to the condition that the tangential component of **H** is continuous across an interface with a conductor having a finite conductivity.

At an interface between a dielectric and a perfect conductor, it is possible to conclude from Eqs. (7–68a) and (7–68c) that the electric field intensity **E** is normal to and points away from (into) the conductor surface when the surface charges are positive (negative), as illustrated in Figs. 7–8(a) and 7–8(b). The magnitude E_{1n} of $\mathbf{E}_1$ at the interface is related to ρ_s by the equation

$$|\mathbf{E}_1| = E_{1n} = \frac{\rho_s}{\epsilon_1}. \tag{7-69}$$

Similarly, Eqs. (7–68b) and (7–68d) show that the magnetic field intensity $\mathbf{H}_1$ is tangential to the interface with a magnitude equal to that of the surface current density:

$$|\mathbf{H}_1| = |\mathbf{H}_{1t}| = |\mathbf{J}_s|. \tag{7-70}$$

The direction of $\mathbf{H}_{1t}$ is determined from Eq. (7–68b). This is illustrated in Fig. 7–8(c). Equations (7–69) and (7–70) are analytically quite simple relations.

In this section we have discussed the relations that field vectors must satisfy at an interface between different media. Boundary conditions are of basic importance in the solution of electromagnetic problems because general solutions of Maxwell's equations carry little meaning until they are adapted to physical problems each with a given region and associated boundary conditions. Maxwell's equations are partial differential equations. Their solutions will contain integration constants that are determined from the additional information supplied by boundary conditions so that each solution will be unique for each given problem.

7–6 Wave Equations and Their Solutions

At this point we are in possession of the essentials of the fundamental structure of electromagnetic theory. Maxwell's equations give a complete description of the relation between electromagnetic fields and charge and current distributions. Their solutions provide the answers to all electromagnetic problems, albeit in some cases the solutions are difficult to obtain. Special analytical and numerical techniques may be devised to aid in the solution procedure; but they do not add to or refine the fundamental structure. Such is the importance of Maxwell's equations.

For given charge and current distributions, ρ and $\mathbf{J}$, we first solve the nonhomogeneous wave equations, Eqs. (7–63) and (7–65), for potentials $\mathbf{A}$ and V. With $\mathbf{A}$ and V determined, $\mathbf{E}$ and $\mathbf{B}$ can be found from Eqs. (7–57) and (7–55), respectively, by differentiation.

7–6.1 SOLUTION OF WAVE EQUATIONS FOR POTENTIALS

We now consider the solution of the nonhomogeneous wave equation, Eq. (7–65), for scalar electric potential V. We can do this by first finding the solution for an elemental point charge at time t, $\rho(t)\,\Delta v'$, located at the origin of the coordinates and then by summing the effects of all the charge elements in a given region. For a point charge at the origin it is most convenient to use spherical coordinates. Because of spherical symmetry, V depends only on R and t (not on θ or ϕ). Except at the origin, V satisfies the following homogeneous equation:

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) - \mu\epsilon\frac{\partial^2 V}{\partial t^2} = 0. \tag{7–71}$$

We introduce a new variable

$$V(R, t) = \frac{1}{R}\,U(R, t), \tag{7–72}$$

which converts Eq. (7–71) to

$$\frac{\partial^2 U}{\partial R^2} - \mu\epsilon\frac{\partial^2 U}{\partial t^2} = 0. \tag{7–73}$$

Equation (7–73) is a one-dimensional homogeneous wave equation. It can be verified by direct substitution (see Problem P.7–20) that *any* twice-differentiable function of $(t - R\sqrt{\mu\epsilon})$ or of $(t + R\sqrt{\mu\epsilon})$ is a solution of Eq. (7–73). Later in this section we will see that a function of $(t + R\sqrt{\mu\epsilon})$ does not correspond to a physically useful solution. Hence we have

$$U(R, t) = f(t - R\sqrt{\mu\epsilon}). \tag{7–74}$$

Equation (7–74) represents a wave traveling in the positive R direction with a velocity $1/\sqrt{\mu\epsilon}$. As we see, the function at $R + \Delta R$ at a later time $t + \Delta t$ is

$$U(R + \Delta R, t + \Delta t) = f[t + \Delta t - (R + \Delta R)\sqrt{\mu\epsilon}] = f(t - R\sqrt{\mu\epsilon}).$$

Thus the function retains its form if $\Delta t = \Delta R\sqrt{\mu\epsilon} = \Delta R/u$, where $u = 1/\sqrt{\mu\epsilon}$ is the ***velocity of propagation***, a characteristic of the medium. From Eq. (7–72) we get

$$V(R, t) = \frac{1}{R}f(t - R/u). \tag{7–75}$$

To determine what the specific function $f(t - R/u)$ must be, we note from Eq. (3–47) that for a static point charge $\rho(t)\,\Delta v'$ at the origin,

$$\Delta V(R) = \frac{\rho(t)\,\Delta v'}{4\pi\epsilon R}. \tag{7–76}$$

Comparison of Eqs. (7–75) and (7–76) enables us to identify

$$\Delta f(t - R/u) = \frac{\rho(t - R/u)\,\Delta v'}{4\pi\epsilon}.$$

The potential due to a charge distribution over a volume V' is then

$$\boxed{V(R, t) = \frac{1}{4\pi\epsilon}\int_{V'} \frac{\rho(t - R/u)}{R}\,dv' \qquad \text{(V)}.} \tag{7–77}$$

Equation (7–77) indicates that the scalar potential at a distance R from the source at time t depends on the value of the charge density at an *earlier* time $(t - R/u)$. It takes time R/u for the effect of ρ to be felt at distance R. For this reason, $V(R, t)$ in Eq. (7–77) is called the ***retarded scalar potential.*** It is now clear that a function of $(t + R/u)$ cannot be a physically useful solution, since it would lead to the impossible situation that the effect of ρ would be felt at a distant point before it occurs at the source.

The solution of the nonhomogeneous wave equation, Eq. (7–63), for vector magnetic potential $\mathbf{A}$ can proceed in exactly the same way as that for V. The vector equation, Eq. (7–63), can be decomposed into three scalar equations, each similar to Eq. (7–65) for V. The ***retarded vector potential*** is thus given by

$$\boxed{\mathbf{A}(R, t) = \frac{\mu}{4\pi}\int_{V'} \frac{\mathbf{J}(t - R/u)}{R}\,dv' \qquad \text{(Wb/m)}.} \tag{7–78}$$

The electric and magnetic fields derived from $\mathbf{A}$ and V by differentiation will obviously also be functions of $(t - R/u)$ and therefore retarded in time. It takes time for electromagnetic waves to travel and for the effects of time-varying charges and currents to be felt at distant points. In the quasi-static approximation we ignore this time-retardation effect and assume instant response. This assumption is implicit in dealing with circuit problems.

7–6.2 SOURCE-FREE WAVE EQUATIONS

In problems of wave propagation we are concerned with the behavior of an electromagnetic wave in a source-free region where ρ and $\mathbf{J}$ are both zero. In other words, we are often interested not so much in how an electromagnetic wave is originated, but in how it propagates. If the wave is in a simple (linear, isotropic, and homogeneous) nonconducting medium characterized by ϵ and μ ($\sigma = 0$), Maxwell's equations (7–53a, b, c, and d) reduce to

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \tag{7–79a}$$

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}, \tag{7–79b}$$

$$\nabla \cdot \mathbf{E} = 0, \tag{7–79c}$$

$$\nabla \cdot \mathbf{H} = 0. \tag{7–79d}$$

Equations (7–79a, b, c, and d) are first-order differential equations in the two variables **E** and **H**. They can be combined to give a second-order equation in **E** or **H** alone. To do this, we take the curl of Eq. (7–79a) and use Eq. (7–79b):

$$\nabla \times \nabla \times \mathbf{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \mathbf{H}) = -\mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

Now $\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}$ because of Eq. (7–79c). Hence we have

$$\nabla^2 \mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0; \tag{7–80}$$

or, since $u = 1/\sqrt{\mu\epsilon}$,

$$\boxed{\nabla^2 \mathbf{E} - \frac{1}{u^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0.} \tag{7–81}$$

In an entirely similar way we can also obtain an equation in **H**:

$$\boxed{\nabla^2 \mathbf{H} - \frac{1}{u^2} \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0.} \tag{7–82}$$

Equations (7–81) and (7–82) are ***homogeneous vector wave equations.***

We can see that in Cartesian coordinates Eqs. (7–81) and (7–82) can each be decomposed into three one-dimensional, homogeneous, scalar wave equations. Each component of **E** and of **H** will satisfy an equation exactly like Eq. (7–73), whose solutions represent waves. We will extensively discuss wave behavior in various environments in the next two chapters.

7–7 Time-Harmonic Fields

Maxwell's equations and all the equations derived from them so far in this chapter hold for electromagnetic quantities with an arbitrary time-dependence. The actual type of time functions that the field quantities assume depends on the source functions

ρ and **J**. In engineering, sinusoidal time functions occupy a unique position. They are easy to generate; arbitrary periodic time functions can be expanded into Fourier series of harmonic sinusoidal components; and transient nonperiodic functions can be expressed as Fourier integrals.[†] Since Maxwell's equations are *linear* differential equations, sinusoidal time variations of source functions of a given frequency will produce sinusoidal variations of **E** and **H** with the *same frequency* in the steady state. For source functions with an arbitrary time dependence, electrodynamic fields can be determined in terms of those caused by the various frequency components of the source functions. The application of the principle of superposition will give us the total fields. In this section we examine *time-harmonic* (steady-state sinusoidal) field relationships.

7-7.1 THE USE OF PHASORS—A REVIEW

For time-harmonic fields it is convenient to use a phasor notation. At this time we digress briefly to review the use of phasors. Conceptually, it is simpler to discuss a scalar phasor. The instantaneous (time-dependent) expression of a sinusoidal scalar quantity, such as a current i, can be written as either a cosine or a sine function. *If* we choose a cosine function as the *reference* (which is usually dictated by the functional form of the excitation), then all derived results will refer to the cosine function. The specification of a sinusoidal quantity requires the knowledge of three parameters: amplitude, frequency, and phase. For example,

$$i(t) = I \cos(\omega t + \phi), \tag{7-83}$$

where I is the amplitude; ω is the angular frequency (rad/s)—ω is always equal to $2\pi f$, f being the frequency in hertz; and ϕ is the phase referred to the cosine function. We could write $i(t)$ in Eq. (7–83) as a sine function if we wish: $i(t) = I \sin(\omega t + \phi')$, with $\phi' = \phi + \pi/2$. Thus it is important to decide at the outset whether our reference is a cosine or a sine function, then to stick to that decision throughout a problem.

To work directly with an instantaneous expression such as the cosine function is inconvenient when differentiations or integrations of $i(t)$ are involved because they lead to both sine (first-order differentiation or integration) and cosine (second-order differentiation or integration) functions and because it is tedious to combine sine and cosine functions. For instance, the loop equation for a series RLC circuit with an applied voltage $e(t) = E \cos \omega t$ is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = e(t). \tag{7-84}$$

If we write $i(t)$ as in Eq. (7–83), Eq. (7–84) yields

$$I\left[-\omega L \sin(\omega t + \phi) + R\cos(\omega t + \phi) + \frac{1}{\omega C}\sin(\omega t + \phi)\right] = E\cos\omega t. \tag{7-85}$$

[†] D. K. Cheng, *op. cit.*, Chapter 5.

Complicated mathematical manipulations are required in order to determine the unknown I and ϕ from Eq. (7–85).

It is much simpler to use exponential functions by writing the applied voltage as

$$\begin{aligned} e(t) &= E \cos \omega t = \mathcal{R}e[(Ee^{j0})e^{j\omega t}] \\ &= \mathcal{R}e(E_s e^{j\omega t}) \end{aligned} \tag{7–86}$$

and $i(t)$ in Eq. (7–83) as

$$\begin{aligned} i(t) &= \mathcal{R}e[(Ie^{j\phi})e^{j\omega t}] \\ &= \mathcal{R}e(I_s e^{j\omega t}), \end{aligned} \tag{7–87}$$

where $\mathcal{R}e$ means "the real part of." In Eqs. (7–86) and (7–87),

$$E_s = Ee^{j0} = E \tag{7–88a}$$

$$I_s = Ie^{j\phi} \tag{7–88b}$$

are (scalar) ***phasors*** that contain amplitude and phase information but are independent of t. The phasor E_s in Eq. (7–88a) with zero phase angle is the reference phasor. Now,

$$\frac{di}{dt} = \mathcal{R}e(j\omega I_s e^{j\omega t}), \tag{7–89}$$

$$\int i\,dt = \mathcal{R}e\left(\frac{I_s}{j\omega} e^{j\omega t}\right). \tag{7–90}$$

Substitution of Eqs. (7–86) through (7–90) in Eq. (7–84) yields

$$\left[R + j\left(\omega L - \frac{1}{\omega C}\right)\right] I_s = E_s, \tag{7–91}$$

from which the current phasor I_s can be solved easily. Note that the time-dependent factor $e^{j\omega t}$ disappears from Eq. (7–91) because it is present in every term in Eq. (7–84) after the substitution and is therefore canceled. This is the essence of the usefulness of phasors in the analysis of linear systems with time-harmonic excitations. After I_s has been determined, the instantaneous current response $i(t)$ can be found from Eq. (7–87) by (1) multiplying I_s by $e^{j\omega t}$, and (2) taking the real part of the product.

If the applied voltage had been given as a *sine function* such as $e(t) = E \sin \omega t$, the series RLC-circuit problem would be solved in terms of phasors in exactly the same way; only the instantaneous expressions would be obtained by taking the *imaginary part* of the product of the phasors with $e^{j\omega t}$. The complex phasors represent the magnitudes and the phase shifts of the quantities in the solution of time-harmonic problems.

EXAMPLE 7–6 Express $3 \cos \omega t - 4 \sin \omega t$ as first (a) $A_1 \cos (\omega t + \theta_1)$, and then (b) $A_2 \sin (\omega t + \theta_2)$. Determine A_1, θ_1, A_2, and θ_2.

Solution We can conveniently use phasors to solve this problem.

a) To express $3\cos\omega t - 4\sin\omega t$ as $A_1\cos(\omega t + \theta_1)$, we *use* $\cos\omega t$ *as the reference* and consider the sum of the two phasors 3 and $-4e^{-j\pi/2}$ $(=j4)$, since $\sin\omega t = \cos(\omega t - \pi/2)$ lags behind $\cos\omega t$ by $\pi/2$ rad:

$$3 + j4 = 5e^{j\tan^{-1}(4/3)} = 5e^{j53.1^\circ}.$$

Taking the *real part* of the product of this phasor and $e^{j\omega t}$, we have

$$\begin{aligned} 3\cos\omega t - 4\sin\omega t &= \mathscr{R}e[(5e^{j53.1^\circ})e^{j\omega t}] \\ &= 5\cos(\omega t + 53.1^\circ). \end{aligned} \tag{7-92a}$$

So, $A_1 = 5$, and $\theta_1 = 53.1^\circ = 0.927$ (rad).

b) To express $3\cos\omega t - 4\sin\omega t$ as $A_2\sin(\omega t + \theta_2)$, we *use* $\sin\omega t$ *as the reference* and consider the sum of the two phasors $3e^{j\pi/2}$ $(=j3)$ and -4:

$$j3 - 4 = 5e^{j\tan^{-1}3/(-4)} = 5e^{j143.1^\circ}.$$

(The reader should note that the angle above is 143.1°, *not* -36.9°.) Now we take the *imaginary part* of the product of the phasor above and $e^{j\omega t}$ to obtain the desired answer:

$$\begin{aligned} 3\cos\omega t - 4\sin\omega t &= \mathscr{I}m[(5e^{j143.1^\circ})e^{j\omega t}] \\ &= 5\sin(\omega t + 143.1^\circ). \end{aligned} \tag{7-92b}$$

Hence, $A_2 = 5$ and $\theta_2 = 143.1^\circ = 2.50$ (rad).

The reader should recognize that the results in Eqs. (7–92a) and (7–92b) are identical. ■

7–7.2 TIME-HARMONIC ELECTROMAGNETICS

Field vectors that vary with space coordinates and are sinusoidal functions of time can similarly be represented by vector phasors that depend on space coordinates but not on time. As an example, we can write a time-harmonic E field *referring to* $\cos\omega t$[†] *as*

$$\mathbf{E}(x, y, z, t) = \mathscr{R}e[\mathbf{E}(x, y, z)e^{j\omega t}], \tag{7-93}$$

where $\mathbf{E}(x, y, z)$ is a ***vector phasor*** that contains information on direction, magnitude, and phase. Phasors are, in general, complex quantities. From Eqs. (7–93), (7–87), (7–89), and (7–90) we see that, if $\mathbf{E}(x, y, z, t)$ is to be represented by the vector phasor $\mathbf{E}(x, y, z)$, then $\partial\mathbf{E}(x, y, z, t)/\partial t$ and $\int \mathbf{E}(x, y, z, t)\,dt$ would be represented by vector phasors $j\omega\mathbf{E}(x, y, z)$ and $\mathbf{E}(x, y, z)/j\omega$, respectively. Higher-order differentiations and integrations with respect to t would be represented by multiplications and divisions, respectively, of the phasor $\mathbf{E}(x, y, z)$ by higher powers of $j\omega$.

We now write time-harmonic Maxwell's equations (7–53a, b, c, and d) in terms of vector field phasors ($\mathbf{E}$, $\mathbf{H}$) and source phasors (ρ, $\mathbf{J}$) in a simple (linear, isotropic, and homogeneous) medium as follows.

[†] If the time reference is not explicitly specified, it is customarily taken as $\cos\omega t$.

$$\boxed{\begin{aligned} \nabla \times \mathbf{E} &= -j\omega\mu\mathbf{H}, \\ \nabla \times \mathbf{H} &= \mathbf{J} + j\omega\epsilon\mathbf{E}, \\ \nabla \cdot \mathbf{E} &= \rho/\epsilon, \\ \nabla \cdot \mathbf{H} &= 0. \end{aligned}} \quad \begin{aligned} &(7\text{-}94a) \\ &(7\text{-}94b) \\ &(7\text{-}94c) \\ &(7\text{-}94d) \end{aligned}$$

The space-coordinate arguments have been omitted for simplicity. The fact that the same notations are used for the phasors as are used for their corresponding time-dependent quantities should create little confusion because we will deal almost exclusively with time-harmonic fields (and therefore with phasors) in the rest of this book. When there is a need to distinguish an instantaneous quantity from a phasor, the time dependence of the instantaneous quantity will be indicated explicitly by the inclusion of a t in its argument. Phasor quantities are not functions of t. It is useful to note that any quantity containing j must necessarily be a phasor.

The time-harmonic wave equations for scalar potential V and vector potential $\mathbf{A}$—Eqs. (7–65) and (7–63)—become, respectively,

$$\nabla^2 V + k^2 V = -\frac{\rho}{\epsilon} \qquad (7\text{-}95)$$

and

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu\mathbf{J}, \qquad (7\text{-}96)$$

where

$$k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{u} \qquad (7\text{-}97)$$

is called the ***wavenumber***. Equations (7–95) and (7–96) are referred to as ***nonhomogeneous Helmholtz's equations***. The Lorentz condition for potentials, Eq. (7–62), is now

$$\nabla \cdot \mathbf{A} + j\omega\mu\epsilon V = 0. \qquad (7\text{-}98)$$

The phasor solutions of Eqs. (7–95) and (7–96) are obtained from Eqs. (7–77) and (7–78), respectively:

$$\boxed{V(R) = \frac{1}{4\pi\epsilon}\int_{V'} \frac{\rho e^{-jkR}}{R}\,dv' \qquad (\text{V}),} \qquad (7\text{-}99)$$

$$\boxed{\mathbf{A}(R) = \frac{\mu}{4\pi}\int_{V'} \frac{\mathbf{J} e^{-jkR}}{R}\,dv' \qquad (\text{Wb/m}).} \qquad (7\text{-}100)$$

These are the expressions for the retarded scalar and vector potentials due to time-harmonic sources. Now the Taylor-series expansion for the exponential factor e^{-jkR} is

$$e^{-jkR} = 1 - jkR + \frac{k^2R^2}{2} + \cdots, \qquad (7\text{-}101)$$

where k, defined in Eq. (7–97), can be expressed in terms of the wavelength $\lambda = u/f$ in the medium. We have

$$k = \frac{2\pi f}{u} = \frac{2\pi}{\lambda}. \tag{7–102}$$

Thus, if

$$kR = 2\pi \frac{R}{\lambda} \ll 1, \tag{7–103}$$

or if the distance R is very small in comparison to the wavelength λ, e^{-jkR} can be approximated by 1. Equations (7–99) and (7–100) then simplify to the static expressions in Eqs. (7–58) and (7–59), which are used in Eqs. (7–55) and (7–57) to find quasi-static fields.

The formal procedure for determining the electric and magnetic fields due to time-harmonic charge and current distributions is as follows:[†]

1. Find phasors $V(R)$ and $\mathbf{A}(R)$ from Eqs. (7–99) and (7–100).
2. Find phasors $\mathbf{E}(R) = -\nabla V - j\omega\mathbf{A}$ and $\mathbf{B}(R) = \nabla \times \mathbf{A}$.
3. Find instantaneous $\mathbf{E}(R, t) = \mathcal{R}e[\mathbf{E}(R)e^{j\omega t}]$ and $\mathbf{B}(R, t) = \mathcal{R}e[\mathbf{B}(R)e^{j\omega t}]$ for a cosine reference.

The degree of difficulty of a problem depends on how difficult it is to perform the integrations in Step 1.

7–7.3 SOURCE-FREE FIELDS IN SIMPLE MEDIA

In a simple, nonconducting source-free medium characterized by $\rho = 0$, $\mathbf{J} = 0$, $\sigma = 0$, the time-harmonic Maxwell's equations (7–94a, b, c, and d) become

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}, \tag{7–104a}$$

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E}, \tag{7–104b}$$

$$\nabla \cdot \mathbf{E} = 0, \tag{7–104c}$$

$$\nabla \cdot \mathbf{H} = 0. \tag{7–104d}$$

Equations (7–104a, b, c, and d) can be combined to yield second-order partial differential equations in **E** and **H**. From Eqs. (7–81) and (7–82) we obtain

$$\boxed{\nabla^2\mathbf{E} + k^2\mathbf{E} = 0} \tag{7–105}$$

and

$$\boxed{\nabla^2\mathbf{H} + k^2\mathbf{H} = 0,} \tag{7–106}$$

[†] Alternatively, Steps 1 and 2 can be replaced by the following: (1′) Find phasor $A(R)$ from Eq. (7–100). (2′) Find $\mathbf{H}(R) = \frac{1}{\mu}\nabla \times \mathbf{A}$, and $\mathbf{E}(R) = \frac{1}{j\omega\epsilon}(\nabla \times \mathbf{H} - \mathbf{J})$ from Eq. (7–94b).

which are ***homogeneous vector Helmholtz's equations.*** Solutions of homogeneous Helmholtz's equations with various boundary conditions is the main concern of Chapters 8 and 10.

EXAMPLE 7-7 Show that if (**E**, **H**) are solutions of source-free Maxwell's equations in a simple medium characterized by ϵ and μ, then so also are (**E**′, **H**′), where

$$\mathbf{E}' = \eta \mathbf{H} \tag{7-107a}$$

$$\mathbf{H}' = -\frac{\mathbf{E}}{\eta}. \tag{7-107b}$$

In the above equations, $\eta = \sqrt{\mu/\epsilon}$ is called the ***intrinsic impedance*** of the medium.

Solution We prove the statement by taking the curl and the divergence of **E**′ and **H**′ and using Eqs. (7-104a, b, c, and d):

$$\begin{aligned}\nabla \times \mathbf{E}' &= \eta(\nabla \times \mathbf{H}) = \eta(j\omega\epsilon\mathbf{E}) \\ &= -j\omega\epsilon\eta^2\left(-\frac{\mathbf{E}}{\eta}\right) = -j\omega\mu\mathbf{H}'\end{aligned} \tag{7-108a}$$

$$\begin{aligned}\nabla \times \mathbf{H}' &= -\frac{1}{\eta}(\nabla \times \mathbf{E}) = -\frac{1}{\eta}(-j\omega\mu\mathbf{H}) \\ &= j\omega\mu\frac{1}{\eta^2}(\eta\mathbf{H}) = j\omega\epsilon\mathbf{E}'\end{aligned} \tag{7-108b}$$

$$\nabla \cdot \mathbf{E}' = \eta(\nabla \cdot \mathbf{H}) = 0 \tag{7-108c}$$

$$\nabla \cdot \mathbf{H}' = -\frac{1}{\eta}(\nabla \cdot \mathbf{E}) = 0. \tag{7-108d}$$

Equations (7-108a, b, c, and d) are source-free Maxwell's equations in **E**′ and **H**′. (Q.E.D.)

This example shows that source-free Maxwell's equations in a simple medium are invariant under the linear transformation specified by Eqs. (7-107a) and (7-107b). This is a statment of the ***principle of duality.*** This principle is a consequence of the symmetry of source-free Maxwell's equations. An illustration of the principle of duality and dual devices can be found in Subsection 11-2.2.

If the simple medium is conducting ($\sigma \neq 0$), a current $\mathbf{J} = \sigma\mathbf{E}$ will flow, and Eq. (7-104b) should be changed to

$$\begin{aligned}\nabla \times \mathbf{H} &= (\sigma + j\omega\epsilon)\mathbf{E} = j\omega\left(\epsilon + \frac{\sigma}{j\omega}\right)\mathbf{E} \\ &= j\omega\epsilon_c\mathbf{E}\end{aligned} \tag{7-109}$$

with

$$\epsilon_c = \epsilon - j\frac{\sigma}{\omega} \quad \text{(F/m)}. \tag{7-110}$$

The other three equations, Eqs. (7–104a, c, and d), are unchanged. Hence, all the previous equations for nonconducting media will apply to conducting media if ϵ is replaced by the ***complex permittivity*** ϵ_c.

As we discussed in Section 3–7, when an external time-varying electric field is applied to material bodies, small displacements of bound charges result, giving rise to a volume density of polarization. This polarization vector will vary with the same frequency as that of the applied field. As the frequency increases, the inertia of the charged particles tends to prevent the particle displacements from keeping in phase with the field changes, leading to a frictional damping mechanism that causes power loss because work must be done to overcome the damping forces. This phenomenon of out-of-phase polarization can be characterized by a complex electric susceptibility and hence a complex permittivity. If, in addition, the material body or medium has an appreciable amount of free charge carriers such as the electrons in a conductor, the electrons and holes in a semiconductor, or the ions in an electrolyte, there will also be ohmic losses. In treating such media it is customary to include the effects of both the damping and the ohmic losses in the imaginary part of a complex permittvity ϵ_c:

$$\epsilon_c = \epsilon' - j\epsilon'' \qquad \text{(F/m)}, \tag{7–111}$$

where both ϵ' and ϵ'' may be functions of frequency. Alternatively, we may define an equivalent conductivity representing all losses and write

$$\sigma = \omega\epsilon'' \qquad \text{(S/m)}. \tag{7–112}$$

Combination of Eqs. (7–111) and (7–112) gives Eq. (7–110). In low-loss media, damping losses are very small, and the real part of ϵ_c in Eq. (7–110) is usually written as ϵ without a prime.

Similar loss arguments apply to the existence of an out-of-phase component of magnetization under the influence of an external time-varying magnetic field. We expect the permeability also to be complex at high frequencies:

$$\mu = \mu' - j\mu''. \tag{7–113}$$

For ferromagnetic materials the real part, μ', is many orders of magnitude larger than the imaginary part, μ'', and the effect of the latter is normally neglected. In view of the above, the real wavenumber k in the Helmholtz's equations, Eqs. (7–105) and (7–106), should be changed to a complex wavenumber:

$$\begin{aligned} k_c &= \omega\sqrt{\mu\epsilon_c} \\ &= \omega\sqrt{\mu(\epsilon' - j\epsilon'')} \end{aligned} \tag{7–114}$$

in a lossy dielectric medium.

The ratio ϵ''/ϵ' is called a ***loss tangent*** because it is a measure of the power loss in the medium:

$$\tan\delta_c = \frac{\epsilon''}{\epsilon'} \cong \frac{\sigma}{\omega\epsilon}. \tag{7–115}$$

The quantity δ_c in Eq. (7–115) may be called the ***loss angle***.

On the basis of Eq. (7–110) a medium is said to be a ***good conductor*** if $\sigma \gg \omega\epsilon$, and a ***good insulator*** if $\omega\epsilon \gg \sigma$. Thus, a material may be a good conductor at low frequencies but may have the properties of a lossy dielectric at very high frequencies. For example, a moist ground has a dielectric constant ϵ_r and a conductivity σ that are in the neighborhood of 10 and 10^{-2} (S/m), respectively. The loss tangent $\sigma/\omega\epsilon$ of the moist ground then equals 1.8×10^4 at 1 (kHz), making it a relatively good conductor. At 10 (GHz), $\sigma/\omega\epsilon$ becomes 1.8×10^{-3}, and the moist ground behaves more like an insulator.†

EXAMPLE 7–8 A sinusoidal electric intensity of amplitude 250 (V/m) and frequency 1 (GHz) exists in a lossy dielectric medium that has a relative permittivity of 2.5 and a loss tangent of 0.001. Find the average power dissipated in the medium per cubic meter.

Solution First we must find the effective conductivity of the lossy medium:

$$\tan\delta_c = 0.001 = \frac{\sigma}{\omega\epsilon_0\epsilon_r},$$

$$\sigma = 0.001(2\pi 10^9)\left(\frac{10^{-9}}{36\pi}\right)(2.5)$$

$$= 1.39 \times 10^{-4} \text{ (S/m)}.$$

The average power dissipated per unit volume is

$$p = \tfrac{1}{2}JE = \tfrac{1}{2}\sigma E^2$$

$$= \tfrac{1}{2} \times (1.39 \times 10^{-4}) \times 250^2 = 4.34 \quad (\text{W/m}^3).$$

A microwave oven cooks food by irradiating the food with microwave power generated by a magnetron. The operating frequency is usually set at 2.45 GHz (2.45×10^9 Hz). For a beef steak that has approximately a dielectric constant of 40 and a loss tangent of 0.35 at 2.45 (GHz), calculations following those in Example 7–8 will yield $\sigma = 1.91$ (S/m) and $p = 59.6$ (kW/m^3). However, since high-frequency currents in a conducting body tend to concentrate near the surface layer (due to ***skin effect***—see Subsection 8–3.2), the value of p obtained here is only a rough estimate.

7–7.4 THE ELECTROMAGNETIC SPECTRUM

We have seen that **E** and **H** in source-free regions satisfy homogeneous wave equations (7–81) and (7–82), respectively. If the sources of the fields are time-harmonic, these equations reduce to homogeneous Helmholtz's equations (7–105) and (7–106). That the solutions of Eqs. (7–105) and (7–106) represent propagating waves will become clear in the beginning of the next chapter. For the moment we note two

† Actually, the loss mechanism of a dielectric material is a very complicated process, and the assumption of a constant conductivity is only a rough approximation.

important points. First, Maxwell's equations, and therefore the wave and Helmholtz's equations, impose *no limit* on the frequency of the waves. The electromagnetic spectrum that has been investigated experimentally extends from very low power frequencies through radio, television, microwave, infrared, visible light, ultraviolet, X-ray, and gamma (γ)-ray frequencies exceeding 10^{24} (Hz). Second, all electromagnetic

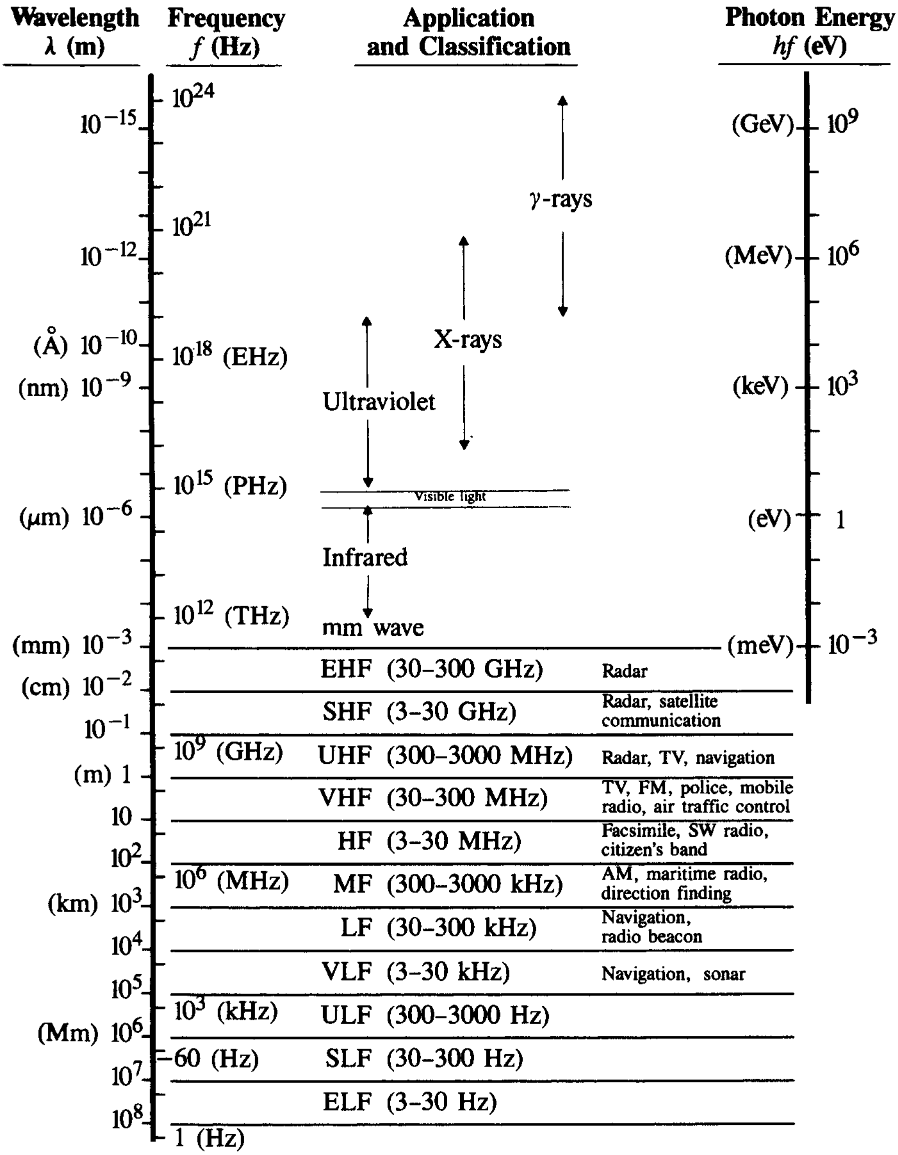


FIGURE 7–9
Spectrum of electromagnetic waves.

TABLE 7–5
Band Designations for Microwave Frequency Ranges

Old†	New	Frequency Ranges (GHz)
Ka	K	26.5–40
K	K	20–26.5
K	J	18–20
Ku	J	12.4–18
X	J	10–12.4
X	I	8–10
C	H	6–8
C	G	4–6
S	F	3–4
S	E	2–3
L	D	1–2
UHF	C	0.5–1

† Because the old band designations have been in wide use since the early days of radar, they are still in common use because of habit.

waves in whatever frequency range propagate in a medium with the *same velocity*, $u = 1/\sqrt{\mu\epsilon}$ ($c \cong 3 \times 10^8$ m/s in air).

Figure 7–9 shows the electromagnetic spectrum divided into frequency and wavelength ranges on logarithmic scales according to application and natural occurrence. The term "microwave" is somewhat nebulous and imprecise; it could mean electromagnetic waves above a frequency of 1 (GHz) and all the way up to the lower limit of the infrared band, encompassing UHF, SHF, EHF, and mm-wave regions. Beyond the frequency range of visible light it is also customary to show the energy level of a photon (quantum of radiation), hf in electron-volts (eV), where h = Planck's constant = 6.63×10^{-34} (J·s). This is included in Fig. 7–9.† The wavelength range of visible light is from deep red at 720 (nm) to violet at 380 (nm), or from 0.72 (μm) to 0.38 (μm), corresponding to a frequency range of from 4.2×10^{14} (Hz) to 7.9×10^{14} (Hz). The bands used for radar, satellite communication, navigation aids, television (TV), FM and AM radio, citizen's band radio (CB), sonar, and others are also noted. Frequencies below the VLF range are seldom used for wireless transmission because huge antennas would be needed for efficient radiation of electromagnetic waves and because of the very low data rate at these low frequencies. There have been proposals to use these frequencies for strategic global communication with submarines submerged in conducting seawater. In radar work it has been found convenient to assign alphabet names to the different microwave frequency bands. They are listed in Table 7–5.

In the next chapter we shall discuss the characteristics of plane electromagnetic waves and examine their behavior as they propagate across discontinuous boundaries.

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† The conversion relations are: 1 (Hz) ↔ 4.14×10^{-15} (eV) ↔ 3×10^8 (m), or 2.42×10^{14} (Hz) ↔ 1 (eV) ↔ 1.24×10^{-6} (m).

Review Questions

R.7–1 What constitutes an *electromagnetostatic field*? In what ways are **E** and **B** related in a conducting medium under static conditions?

R.7–2 Write the fundamental postulate for electromagnetic induction, and explain how it leads to Faraday's law.

R.7–3 State Lenz's law.

R.7–4 Write the expression for transformer emf.

R.7–5 What are the characteristics of an ideal transformer?

R.7–6 What is the definition of *coefficient of coupling* in inductive circuits?

R.7–7 What are *eddy currents*?

R.7–8 What are *superconductors*?

R.7–9 Why are materials having high permeability and low conductivity preferred as transformer cores?

R.7–10 Why are the cores of power transformers laminated?

R.7–11 Write the expression for flux-cutting emf.

R.7–12 Write the expression for the induced emf in a closed circuit that moves in a changing magnetic field.

R.7–13 What is a Faraday disk generator?

R.7–14 Write the differential form of Maxwell's equations.

R.7–15 Are all four Maxwell's equations independent? Explain.

R.7–16 Write the integral form of Maxwell's equations, and identify each equation with the proper experimental law.

R.7–17 Explain the significance of *displacement current.*

R.7–18 Why are potential functions used in electromagnetics?

R.7–19 Express **E** and **B** in terms of potential functions V and **A**.

R.7–20 What do we mean by *quasi-static fields*? Are they exact solutions of Maxwell's equations? Explain.

R.7–21 What is the Lorentz condition for potentials? What is its physical significance?

R.7–22 Write the nonhomogeneous wave equation for scalar potential V and for vector potential **A**.

R.7–23 State the boundary conditions for the tangential component of **E** and for the normal component of **B**.

R.7–24 Write the boundary conditions for the tangential component of **H** and for the normal component of **D**.

R.7–25 Why is the **E** field immediately outside of a perfect conductor perpendicular to the conductor surface?

R.7–26 Why is the **H** field immediately outside of a perfect conductor tangential to the conductor surface?

R.7–27 Can a static magnetic field exist in the interior of a perfect conductor? Explain. Can a time-varying magnetic field? Explain.

R.7–28 What do we mean by a *retarded potential*?

R.7–29 In what ways do the retardation time and the velocity of wave propagation depend on the constitutive parameters of the medium?

R.7–30 Write the source-free wave equations for **E** and **H** in free space.

R.7–31 What is a *phasor*? Is a phasor a function of t? A function of ω?

R.7–32 What is the difference between a phasor and a vector?

R.7–33 Discuss the advantages of using phasors in electromagnetics.

R.7–34 Are conduction and displacement currents in phase for time-harmonic fields? Explain.

R.7–35 Write in terms of phasors the time-harmonic Maxwell's equations for a simple medium.

R.7–36 Define *wavenumber*.

R.7–37 Write the expressions for time-harmonic retarded scalar and vector potentials in terms of charge and current distributions.

R.7–38 Write the homogeneous vector Helmholtz's equation for **E** in a simple, nonconducting, source-free medium.

R.7–39 Write the expression for the wavenumber of a lossy medium in terms of its permittivity and permeability.

R.7–40 What is meant by the *loss tangent* of a medium?

R.7–41 In a time-varying situation how do we define a *good conductor*? A *lossy dielectric*?

R.7–42 What is the velocity of propagation of electromagnetic waves? Is it the same in air as in vacuum? Explain.

R.7–43 What is the wavelength range of visible light?

R.7–44 Why are frequencies below the VLF range rarely used for wireless transmission?

Problems

P.7–1 Express the transformer emf induced in a stationary loop in terms of time-varying vector potential **A**.

P.7–2 The circuit in Fig. 7–10 is situated in a magnetic field

$$\mathbf{B} = \mathbf{a}_z 3 \cos(5\pi 10^7 t - \tfrac{2}{3}\pi x) \qquad (\mu\text{T}).$$

Assuming $R = 15\ (\Omega)$, find the current i.

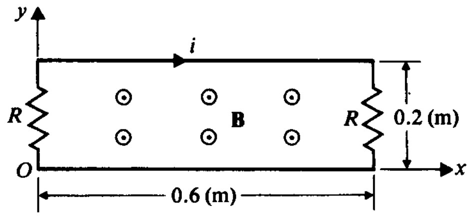

FIGURE 7–10
A circuit in a time-varying magnetic field (Problem P.7–2).

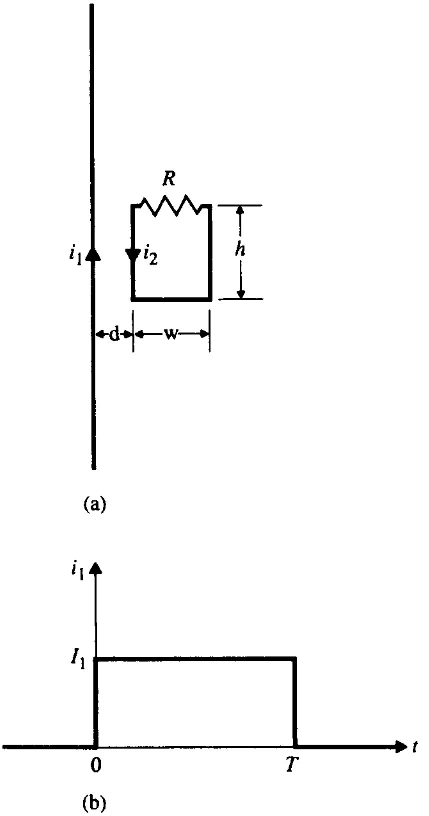

FIGURE 7–11
A rectangular loop near a long current-carrying wire (Problem P.7–3).

P.7–3 A rectangular loop of width w and height h is situated near a very long wire carrying a current i_1 as in Fig. 7–11(a). Assume i_1 to be a rectangular pulse as shown in Fig. 7–11(b).

a) Find the induced current i_2 in the rectangular loop whose self-inductance is L.

b) Find the energy dissipated in the resistance R if $T \gg L/R$.

P.7–4 A conducting equilateral triangular loop is placed near a very long straight wire, shown in Fig. 6–48, with $d = b/2$. A current $i(t) = I \sin \omega t$ flows in the straight wire.

a) Determine the voltage registered by a high-impedance rms voltmeter inserted in the loop.

b) Determine the voltmeter reading when the triangular loop is rotated by 60° about a perpendicular axis through its center.

P.7–5 A conducting circular loop of a radius 0.1 (m) is situated in the neighborhood of a very long power line carrying a 60-(Hz) current, as shown in Fig. 6–49, with $d = 0.15$ (m). An a-c milliammeter inserted in the loop reads 0.3 (mA). Assume the total impedance of the loop including the milliammeter to be 0.01 (Ω).

a) Find the magnitude of the current in the power line.

b) To what angle about the horizontal axis should the circular loop be rotated in order to reduce the milliammeter reading to 0.2 (mA)?

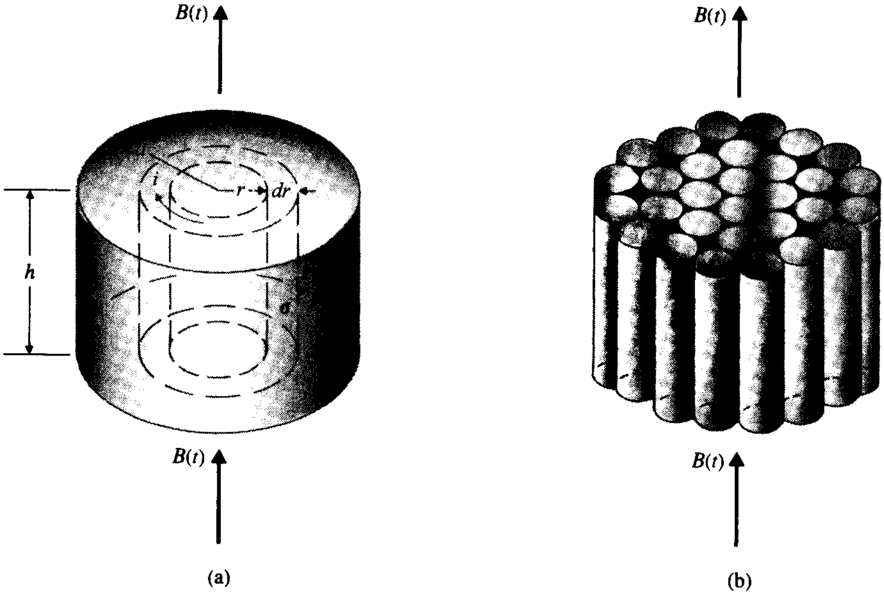

FIGURE 7–12
Suggested eddy-current power-loss reduction scheme (Problem P.7–6).

P.7–6 A suggested scheme for reducing eddy-current power loss in transformer cores with a circular cross section is to divide the cores into a large number of small insulated filamentary parts. As illustrated in Fig. 7–12, the section shown in part (a) is replaced by that in part (b). Assuming that $B(t) = B_0 \sin \omega t$ and that N filamentary areas fill 95% of the original cross-sectional area, find

a) the average eddy-current power loss in the section of core of height h in Fig. 7–12(a),
b) the total average eddy-current power loss in the N filamentary sections in Fig. 7–12(b).

The magnetic field due to eddy currents is assumed to be negligible. (*Hint:* First find the current and power dissipated in the differential circular ring section of height h and width dr at radius r.)

P.7–7 A conducting sliding bar oscillates over two parallel conducting rails in a sinusoidally varying magnetic field

$$\mathbf{B} = \mathbf{a}_z 5 \cos \omega t \qquad \text{(mT)},$$

as shown in Fig. 7–13. The position of the sliding bar is given by $x = 0.35(1 - \cos \omega t)$ (m), and the rails are terminated in a resistance $R = 0.2$ (Ω). Find i.

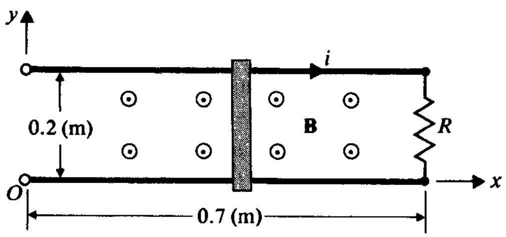

FIGURE 7–13
A conducting bar sliding over parallel rails in a time-varying magnetic field (Problem P.7–7).

P.7–8 In the d-c motor illustrated in Fig. 6–32 we noted that a current I sent through the loop in a magnetic field **B** produces a torque that makes the loop rotate. As the loop rotates, the amount of the magnetic flux linking with the loop changes, giving rise to an induced emf. Energy must be expended by an external electric source to counter this emf and establish the current in the loop. Prove that this electric energy is equal to the mechanical work done by the rotating loop. (*Hint:* Consider the normal of the loop at an arbitary angle α with **B**, and let it rotate by an angle $\Delta\alpha$.)

P.7–9 Assuming that a resistance R is connected across the slip rings of the rectangular conducting loop that rotates in a constant magnetic field $\mathbf{B} = \mathbf{a}_y B_0$, shown in Fig. 7–6, prove that the power dissipated in R is equal to the power required to rotate the loop at an angular frequency ω.

P.7–10 A hollow cylindrical magnet with inner radius a and outer radius b rotates about its axis at an angular frequency ω. The magnet has a uniform axial magnetization $\mathbf{M} = \mathbf{a}_z M_0$. Sliding brush contacts are provided at the inner and outer surfaces as shown in Fig. 7–14. Assuming that $\mu_r = 5000$ and $\sigma = 10^7$ (S/m) for the magnet, find

a) **H** and **B** in the magnet,
b) open-circuit voltage V_0,
c) short-circuit current.

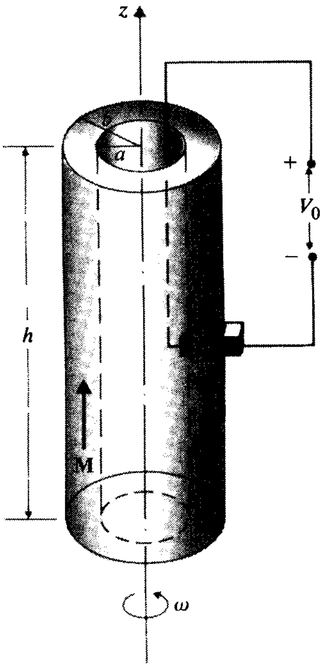

FIGURE 7–14
A rotating hollow cylindrical magnet (Problem P.7–10).

P.7–11 Derive the two divergence equations, Eqs. (7–53c) and (7–53d), from the two curl equations, Eqs. (7–53a) and (7–53b), and the equation of continuity, Eq. (7–48).

P.7–12 Prove that the Lorentz condition for potentials as expressed in Eq. (7–62) is consistent with the equation of continuity.

P.7–13 The vector magnetic potential **A** and scalar electric potential V defined in Section 7–4 are not unique in that it is possible to add to **A** the gradient of a scalar ψ, $\nabla\psi$, with no change in **B** from Eq. (7–55).

$$\mathbf{A}' = \mathbf{A} + \nabla\psi. \tag{7-116}$$

In order not to change **E** in using Eq. (7–57), V must be modified to V'.

a) Find the relation between V' and V.

b) Discuss the condition that ψ must satisfy so that the new potentials $\mathbf{A}'$ and V' remain governed by the uncoupled wave equations (7–63) and (7–65).

P.7–14 Substitute Eqs. (7–55) and (7–57) in Maxwell's equations to obtain wave equations for scalar potential V and vector potential **A** for a linear, isotropic but inhomogeneous medium. Show that these wave equations reduce to Eqs. (7–65) and (7–63) for simple media. (*Hint:* Use the following gauge condition for potentials in an inhomogeneous medium:

$$\nabla\cdot(\epsilon\mathbf{A}) + \mu\epsilon^2\frac{\partial V}{\partial t} = 0.) \tag{7-117}$$

P.7–15 Write the set of four Maxwell's equations, Eqs. (7–53a, b, c and d), as eight scalar equations

a) in Cartesian coordinates,

b) in cylindrical coordinates,

c) in spherical coordinates.

P.7–16 Supply the detailed steps for the derivation of the electromagnetic boundary conditions, Eqs. (7–66a, b, c, and d).

P.7–17 Discuss the relations

a) between the boundary conditions for the tangential components of **E** and those for the normal components of **B**,

b) between the boundary conditions for the normal components of **D** and those for the tangential components of **H**.

P.7–18 In Eqs. (3–88) and (3–89) it was shown that for field calculations a polarized dielectric may be replaced by an equivalent polarization surface charge density ρ_{ps} and an equivalent polarization volume charge density ρ_p. Find the boundary conditions at the interface of two different media for

a) the normal component of **P**,

b) the normal components of **E**

in terms of free and equivalent polarization surface charge densities ρ_s and ρ_{ps}.

P.7–19 Write the boundary conditions that exist at the interface of free space and a magnetic material of infinite (an approximation) permeability.

P.7–20 Prove by direct substitution that any twice differentiable function of $(t - R\sqrt{\mu\epsilon})$ or of $(t + R\sqrt{\mu\epsilon})$ is a solution of the homogeneous wave equation, Eq. (7–73).

P.7–21 Prove that the retarded potential in Eq. (7–77) satisfies the nonhomogeneous wave equation, Eq. (7–65).

P.7–22 For the assumed $f(t)$ at $R = 0$ in Fig. 7–15, sketch

a) $f(t - R/u)$ versus t,

b) $f(t - R/u)$ versus R for $t > T$.

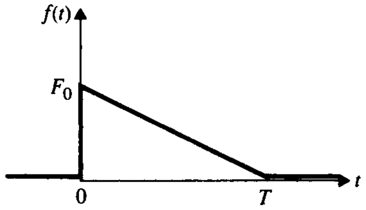

FIGURE 7–15
A triangular time function (Problem P.7–22).

P.7–23 The electric field of an electromagnetic wave

$$\mathbf{E} = \mathbf{a}_x E_0 \cos\left[10^8\pi\left(t - \frac{z}{c}\right) + \theta\right]$$

is the sum of

$$\mathbf{E}_1 = \mathbf{a}_x 0.03 \sin 10^8\pi\left(t - \frac{z}{c}\right)$$

and

$$\mathbf{E}_2 = \mathbf{a}_x 0.04 \cos\left[10^8\pi\left(t - \frac{z}{c}\right) - \frac{\pi}{3}\right].$$

Find E_0 and θ.

P.7–24 Derive the general wave equations for **E** and **H** in a nonconducting simple medium where a charge distribution ρ and a current distribution **J** exist. Convert the wave equations to Helmholtz's equations for sinusoidal time dependence. Write the general solutions for **E**(R, t) and **H**(R, t) in terms of ρ and **J**.

P.7–25 Given that

$$\mathbf{E} = \mathbf{a}_y 0.1 \sin(10\pi x)\cos(6\pi 10^9 t - \beta z) \qquad \text{(V/m)}$$

in air, find **H** and β.

P.7–26 Given that

$$\mathbf{H} = \mathbf{a}_y 2 \cos(15\pi x)\sin(6\pi 10^9 t - \beta z) \qquad \text{(A/m)}$$

in air, find **E** and β.

P.7–27 It is known that the electric field intensity of a spherical wave in free space is

$$\mathbf{E} = \mathbf{a}_\theta \frac{E_0}{R} \sin\theta \cos(\omega t - kR).$$

Determine the magnetic field intensity **H** and the value of k.

P.7–28 In Section 7–4 we indicated that **E** and **B** can be determined from the potentials V and **A**, which are related by the Lorentz condition, Eq. (7–98), in the time-harmonic case. The vector potential **A** was introduced through the relation $\mathbf{B} = \nabla \times \mathbf{A}$ because of the solenoidal nature of **B**. In a source-free region, $\nabla \cdot \mathbf{E} = 0$, we can define another type of vector potential $\mathbf{A}_e$, such that $\mathbf{E} = \nabla \times \mathbf{A}_e$. Assuming harmonic time dependence:

a) Express **H** in terms of $\mathbf{A}_e$.
b) Show that $\mathbf{A}_e$ is a solution of a homogeneous Helmholtz's equation.

P.7–29 For a source-free polarized medium where $\rho = 0$, $\mathbf{J} = 0$, $\mu = \mu_0$, but where there is a volume density of polarization **P**, a single vector potential $\boldsymbol{\pi}_e$ may be defined such that

$$\mathbf{H} = j\omega\epsilon_0 \nabla \times \boldsymbol{\pi}_e. \qquad (7\text{–}118)$$

a) Express electric field intensity **E** in terms of $\boldsymbol{\pi}_e$ and **P**.

b) Show that $\boldsymbol{\pi}_e$ satisfies the nonhomogeneous Helmholtz's equation

$$\nabla^2 \boldsymbol{\pi}_e + k_0^2 \boldsymbol{\pi}_e = -\frac{\mathbf{P}}{\epsilon_0}. \tag{7-119}$$

The quantity $\boldsymbol{\pi}_e$ is known as the ***electric Hertz potential.***

P.7–30 Calculations concerning the electromagnetic effect of currents in a good conductor usually neglect the displacement current even at microwave frequencies.

a) Assuming $\epsilon_r = 1$ and $\sigma = 5.70 \times 10^7$ (S/m) for copper, compare the magnitude of the displacement current density with that of the conduction current density at 100 (GHz).

b) Write the governing differential equation for magnetic field intensity **H** in a source-free good conductor.